LECTURE NOTES
GENERAL EQUILIBRIUM THEORY: SS205

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0. Disclaimer

The ideas and organization in these lecture notes owe a lot to Chris Shannon's Econ 201(a) class at UC Berkeley, as well as various textbooks: notably Mas-Colell et al. (1995), and to a lesser extent Bewley (2009). Over the years, when I've had time in class, I've added some more advanced material. Usually in the last week of the quarter. I've never covered all the material here in a single quarter.

The current write-up started from Alejandro Robinson's notes from the class I taught in the Winter of 2015. I have added material, and rewritten the notes each time I have taught the class since. They are a work in progress, so please let me know of any problems you find.

1. Preliminary definitions

1.1. **Binary relations.** Let X be a set. A ***binary relation*** on X is a subset of $X \times X$. If B is a binary relation on X, and $x, y \in X$ we write $x\ B\ y$ to denote that $(x, y) \in B$. A binary relation B is ***complete*** if $x\ B\ y$ or $y\ B\ x$ (or both) for any $x, y \in X$. It is ***transitive*** if for any $x, y, z \in X$

$$x\ B\ y \text{ and } y\ B\ z \text{ imply } x\ B\ z.$$

A binary relation is a ***weak order*** if it is complete and transitive. In economics we use the term ***preference relation*** (or ***rational preference relation***) for weak orders.

A preference relation is denoted by $\succeq$. Associated to any preference relation $\succeq$ are two more binary relation. The first captures indifference: $x \sim y$ is $x \succeq y$ and $y \succeq x$. The second captures strict preference: $x \succ y$ if $x \succeq y$ and it is not the case that $x \sim y$.

1.2. **Preferences in Euclidean space.** A subset $A \subseteq \mathbf{R}^n$ is ***convex*** if $\lambda x + (1 - \lambda)y \in A$ for any $x, y \in A$ and any $\lambda \in (0, 1)$.

We use the following notational conventions: For vectors $x, y \in \mathbf{R}^n$, $x \leq y$ means that $x_i \leq y_i$ for all $i = 1, \dots, n$; $x < y$ means that $x \leq y$ and $x \neq y$; and $x \ll y$ means that $x_i < y_i$ for all $i = 1, \dots, n$.

When $X \subseteq \mathbf{R}^n$ and $\succeq$ is a preference relation over X we have the following definitions: $\succeq$ is

- ***locally nonsatiated*** if for any $x \in X$ and $\varepsilon > 0$ there is $y \in X$ such that $\|x - y\| < \varepsilon$ and $y \succ x$.
- ***weakly monotone*** if $x \leq y$ implies $y \succeq x$ and $x \ll y$ implies $y \succ x$
- ***strongly monotone*** if $x < y$ implies $y \succ x$.

Define the following sets: Let $U(x) = \{y \in X : y \succeq x\}$ be the ***upper contour set*** of $\succeq$ at x and let $L(x) = \{y \in X : x \succeq y\}$ be the ***lower contour set*** of $\succeq$ at x.

The preference relation $\succeq$ is

- ***convex*** if $U(x)$ is a convex set for all $x \in X$;
- ***strictly convex*** if

$$\lambda y + (1 - \lambda)y' \succ x$$

for any $y, y' \in U(x)$ and $\lambda \in (0,1)$, for any $x \in X$;
- and ***continuous*** if $U(x)$ and $L(x)$ are closed sets (in X) for all $x \in X$.

The following theorem is due to Debreu.

Theorem 1. *If a preference relation $\succeq$ is continuous on $X \subseteq \mathbf{R}^n$ then there exists $u : X \to \mathbf{R}$ such that $x \succeq y$ iff $u(x) \geq u(y)$.*

The function u is a ***utility representation*** for $\succeq$.

2. Consumer Theory

A ***consumer*** is a pair $(X, \succeq)$, where X is a set termed ***consumption space***, and $\succeq$ is a preference relation on X. As we shall see, we shall require a bit more information when we place the consumer in an economy. The set X represents all the possible consumption bundles that the consumer can choose. The preference $\succeq$ is harder to interpret, but the standard view in economics is that $\succeq$ is simply a description of the consumer's choice behavior: $\succeq$ specifies what the consumer chooses from each pair of alternative bundles in X.

We assume throughout that $X = \mathbf{R}^L_+$. This means that there are L goods, and that the consumer can choose to consume these goods in any (continuous) quantity. As we shall see, the notion of "good" is quite flexible, and accommodates goods that differ in the time when they are consumed, or the uncertain events upon which they are delivered (Debreu, 1987).

The consumer chooses from a budget set. Let $p \in \mathbf{R}^L_+$ and $W \geq 0$. The set

$$B(p, W) = \{y \in \mathbf{R}^L_+ : p \cdot y \leq W\}$$

is the ***budget set*** for a consumer when prices are p and income is W.

Given a preference relation $\succeq$, the optimal choices of the consumer are:

$$x^*(p, W) = \{x \in B(p, W) : x \succeq y \text{ for all } y \in B(p, W)\}.$$

The mapping from (p, W) into $x^*(p, W)$ is the consumer's ***demand correspondence.***

The theory of the consumer predicts that demand (choices made from budget sets) are optimal choices according to an underlying (rational) preference relation.

The following results are simple observations that you should prove on your own (or find in MWG).

Observation 2. $x \in x^*(p, W)$ if and only if $y \succ x$ implies that $p \cdot y > W$.

2.1. **Digression: upper hemi continuity.**

a ***correspondence*** is a function ϕ with domain A and range 2^B for some set B, such that $\phi(a)$ is a nonempty subset of B for each a. We denote a correspondence by $\phi : A \to B$.

A correspondence $\phi : A \subseteq \mathbf{R}^n \to B \subseteq \mathbf{R}^m$ has ***closed graph*** if $\{(x, y) \in A \times B : y \in \phi(x)\}$ is a closed subset of $A \times B$.

Let $\phi : A \subseteq \mathbf{R}^n \to B \subseteq \mathbf{R}^m$, where B is closed. We say that ϕ is ***upper hemicontinuous*** (uhc) if it has closed graph and the image of compact sets are bounded.

Note: this is a practical way of understanding uhc. It is how we use it in this class. But you can find a general definition, for example, in Aliprantis and Border (2006).

2.2. **Properties of demand.**

Proposition 3. *If $\succeq$ is locally nonsatiated and continuous then $x^*(p, W)$ is nonempty and satisfies that $p \cdot x = W$ for all $x \in x^*(p, W)$, for all p and W. Moreover, the demand correspondence $(p, W) \mapsto x^*(p, W)$ is upper hemicontinuous.*

Proposition 4. *If $\succeq$ is locally nonsatiated, continuous and convex, then $x^*(p, W)$ is nonempty, compact, and convex, for all p and W. Moreover the demand correspondence $(p, W) \mapsto x^*(p, W)$ is upper hemicontinuous.*

Proposition 5. *If $\succeq$ is locally nonsatiated, continuous and strictly convex then $x^*(p, W)$ is a singleton for all p and W. Moreover the demand correspondence $(p, W) \mapsto x^*(p, W)$ is continuous as a function.*

Consider a consumer with convex, continuous, and monotone preferences $\succeq$. Fix a bundle x in $\mathbf{R}^n$. What do we need to do if we want this consumer to demand x? Consider the upper contour set $U(x)$. Given our assumptions on preferences, this set is in the hypothesis of the supporting hyperplane theorem.[1] Consider the picture in Figure 1.

[1] You should be familiar with the supporting hyperplane theorem, but you will in any case prove it as part of your homework.

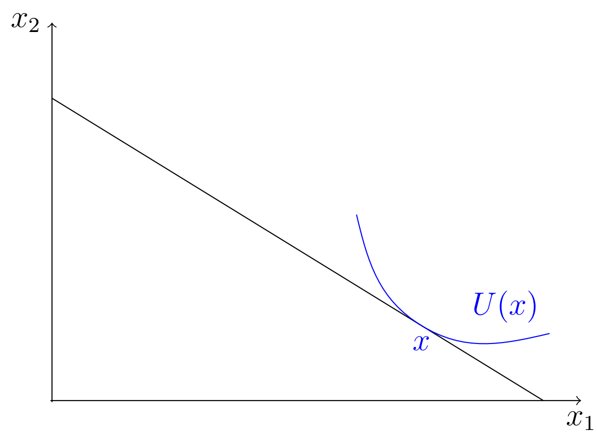

FIGURE 1. The "second welfare theorem for a single consumer."

A supporting hyperplane at $U(x)$ gives us prices p and income W for which x is a demanded bundle. The reason is that any bundle that is strictly preferred to x must be in the interior of $U(x)$, and therefore not affordable to a consumer with such income, and facing such prices. Observe that we can use the monotonicity of preferences to ensure that the prices obtained are positive.

The observation in Figure 1 can be called the "second welfare theorem for an individual consumer," for reasons that will be clear in a few lectures.

3. Economies

3.1. **Exchange economies.** Our first model of an economy is meant to capture the motivations behind pure economic exchange. I desire oranges but own only apples. You own oranges but no apples. By exchanging goods, we satisfy our needs for the fruits that we lack: a "coincidence of wants" (or "mutual needs"). Prices capture the rate of exchange between goods; how many oranges should I receive for each apple that I give you.

The model primitive is a description of a collection of consumers. Each consumer is described by a preference relation and a consumption space, as in the theory of the consumer. We take consumption space to be $\mathbf{R}^L_+$. An important feature of the theory is to account for who owns and sells the goods that the consumers buy. We also need to account for where consumers' incomes come from. Each consumer will be ***endowed*** with non-negative quantities of each good. These are the goods that are sold in market, at the prevailing market prices, and consumers' incomes come from selling the goods that they are endowed with at market prices. Thus, a consumer will be described (fixing consumption space to be $\mathbf{R}^L_+$, by a preference and an ***endowment vector*** in $\mathbf{R}^L_+$.

An economy then is a description of a collection of consumers. Formally, an ***exchange economy*** is a tuple

$$\mathcal{E} = ((\succeq_1, \omega_1), (\succeq_2, \omega_2), \ldots, (\succeq_I, \omega_I)),$$

denoted by $(\succeq_i, \omega_i)_{i=1}^I$, in which each $\succeq_i$ is a preference relation over $\mathbf{R}^L_+$ and each ω_i is a vector in $\mathbf{R}^L_+$.

Let $\mathcal{E} = (\succsim_i, \omega_i)_{i=1}^I$ be an exchange economy.

- the *aggregate endowment* in $\mathcal{E}$ is
$$\bar{\omega} = \sum_{i=1}^I \omega_i;$$
- An *allocation*$_\leq$ in $\mathcal{E}$ is a collection of consumption bundles $(x_i)_{i=1}^I$, where $x_i \in \mathbf{R}^L_+$, $i = 1, \ldots, I$, such that
$$\sum_{i=1}^I x_i \leq \bar{\omega}$$
- an *allocation*$_=$ in $\mathcal{E}$ is an allocation$_\leq (x_i)_{i=1}^I$ such that $\sum_{i=1}^I x_i = \bar{\omega}$.

Definition 6 (Pareto Optimality)**.** An allocation$_\leq (x_i)_{i=1}^I$ is ***Pareto optimal*** in $\mathcal{E}$ if there is no allocation$_\leq (y_i)_{i=1}^I$ such that $y_i \succsim_i x_i$ for every $i = 1, \ldots, I$, and $y_h \succ_h x_h$ for some $h \in \{1, \ldots, I\}$.

Pareto optimality is a basic notion of economic efficiency. In a Pareto optimal allocation, one cannot make some agent better off without making some other agent worse off.

Definition 7 (Walrasian Equilibrium)**.** A ***Walrasian equilibrium*** in $\mathcal{E}$ is a pair (x, p) such that $x = (x_i)_{i=1}^I \in \mathbf{R}^{IL}_+$, and $p \in \mathbb{R}^L_+$ (a ***price vector***), s.t.:

(i) for every $i = 1, \ldots, I$, $x_i \in B(p, p \cdot \omega_i)$, and $x'_i \in B(p, p \cdot \omega_i) \Rightarrow x_i \succsim_i x'_i$ (all consumers optimize when choosing x_i at prices p);
(ii) $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i$ (demand equals supply).

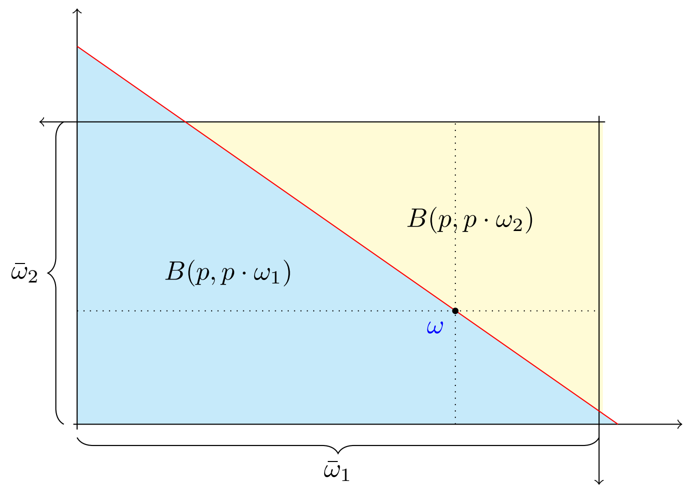

FIGURE 2. The Edgeworth box

When there are two goods ($L = 2$), an exchange economy with two consumers can be conveniently represented using an ***Edgeworth box***. Consider the following figure. The two agents' preferences are represented by indifference curves in the figure. The allocation corresponding to the red dot is not Pareto optimal: any point in the orange set describes an allocation that is better for both consumers. Note that each point in the box describes an allocation$_=$,

The Pareto optimal allocations are those for which, if we fix an upper contour set for consumer 2 (given some given point, and his preference relation) and choose one of the best points on that set from the perspective of 1's preferences. The resulting set of points, as we vary the upper contour set for consumer 2, is the set of all Pareto optimal allocations. In the figure it is represented with a thick blue line.

3.1.1. *A characterization of Pareto optimal allocations.* Let $\mathcal{E} = (\succeq_i, \omega_i)$ be an exchange economy in which each preference $\succeq_i$ is represented by a utility u_i. Consider the maximization problem $P(\bar{v})$:

$$\text{(PO-e)} \qquad \begin{array}{ll} \max_{x \in \mathbf{R}^{LI}_+} & u_1(x_1) \\ \text{s.t.} & u_i(x_i) \geq \bar{v}_i \quad \forall i = 2, 3, ..., I, \\ & \sum_{i=1}^{I} x_i \leq \bar{\omega}, \end{array}$$

where $\bar{v} \in \mathbf{R}^I$ is a vector of utility values.

Proposition 8. *Let each $\succeq_i$ be continuous and strictly monotone. An allocation is Pareto optimal iff there is $\bar{v} \in \mathbf{R}^I$ for which x solves $P(\bar{v})$.*

3.2. **Economies with production.** The second model allows for production. We introduce firms, and consider an economy in which firms are described by their production possibility sets. Consumers are described by their preferences and endowments as before, but now they also have shares in the profits of the firms. These shares are fixed.

We include production by assuming that there are J firms in the economy, each described by a production possibility set Y_j, $j = 1, \ldots, J$. For a production vector $y_j \in Y_j$, and price vector $p \in \mathbf{R}^L_+$, firms j's profits are $p \cdot y_j$. Firms profits need to go to some agent in the economy: this is often called "closing" the model, which roughly means that one accounts for all the endogenous quantities in the model. Each consumer i will have a share $\theta_{i,j} \geq 0$ in the profits of firm j. We may have$\theta_{i,j} = 0$, but all profits go to some agent, so $\sum_{i=1}^I \theta_{i,j} = 1$ for all j.

An ***private ownership economy*** is a tuple

$$\mathcal{E} = \left((Y_j)_{j=1}^J, (\succeq_i, \omega_i, \theta_i)_{i=1}^I\right)$$

in which,

- for each $j = 1, \ldots J$, $Y_j \subseteq \mathbf{R}^L$ is a ***production possibility set***;
- for each $i = 1, \ldots, I$, $\succeq_i$ is a preference relation over $\mathbf{R}^L_+$, and ω_i is a vector in $\mathbf{R}^L_+$;
- for each $j = 1, \ldots J$ $\sum_{i=1}^I \theta_{i,j} = 1$, and for each $i = 1, \ldots, I$ $\theta_{i,j} \geq 0$.

Let $\mathcal{E} = ((Y_j)_{j=1}^J, (\succsim_i, \omega_i, \theta_i)_{i=1}^I)$ be a private ownership economy.

- An *allocation*$_\leq$ in $\mathcal{E}$ is a pair (x, y), where $x = (x_i)_{i=1}^I \in \mathbf{R}^{IL}_+$, $y = (y_j)_{j=1}^J \in \mathbf{R}^{IL}$, $y_j \in Y_j$ $\forall$ $j = 1, ..., J$, and

$$\sum_{i=1}^I x_i \leq \bar{\omega} + \sum_{j=1}^J y_j.$$

- An *allocation*$_=$ in $\mathcal{E}$ is an allocation$_\leq(x, y)$, where

$$\sum_{i=1}^I x_i = \bar{\omega} + \sum_{j=1}^J y_j.$$

Let $Y = \sum_{j=1}^{J} Y_j$ be the *aggregate production set* of a private ownership economy.[2] Consider the set:

$$Y + \{\bar{\omega}\} = \{x \in \mathbb{R}^L : \exists\ y \in Y \text{ s.th. } x = y + \bar{\omega}\}.$$

Define $PPS = (Y + \{w\}) \cap \mathbb{R}^L_+$ as the *production possibility set* of the economy. Its boundary is referred to as the economy's *production possibility frontier*.

Definition 9 (Pareto Optimality)**.** An allocation$_\leq(x, y)$ in $\mathcal{E}$ is ***Pareto optimal*** if there is no allocation$_\leq(x', y')$ such that $x'_i \succsim_i x_i$ for every $i = 1, \ldots, I$, and $= x_h \succ_h x_h$ for some $h \in \{1, \ldots, I\}$.

Definition 10 (Walrasian Equilibrium)**.** Let $\mathcal{E}$ be a private ownership economy. A ***Walrasian Equilibrium*** is a pair $(x, y) \in \mathbf{R}^{IL}_+ \times \mathbf{R}^{JL}$, together with a price vector $p \in \mathbb{R}^L_+$ s.t.

(i) for every $i = 1, ..., I$, $x_i \in B(p, M_i)$, and $x'_i \in B(p, M_i) \Rightarrow x_i \succsim x'_i$, where $M_i = p \cdot \omega_i + \sum_{j=1}^{J} \theta_{i,j} p \cdot y_j$ (consumers optimize by choosing x_i in their budget sets);
(ii) for every $j = 1, ..., J$, $y_j \in Y_j$, and $p \cdot y_j \geq p \cdot y'_j\ \forall\ y'_j \in Y_j$ (firms optimize profits by choosing y_j in Y_j);
(iii) $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j$ (demand equals supply).

There is a simple and useful characterization of Pareto optimal allocations. Let $((Y_j)_{j=1}^J, (\succsim_i, \omega_i, \theta_i)_{i=1}^I)$ be a private ownership economy in which each preference relation $\succeq_i$ has a continuous and strictly monotone utility representation $u_i : \mathbf{R}^L_+ \to \mathbf{R}$.

$$\text{(PO)} \qquad \begin{aligned} \max_{(x,y)\in \mathbf{R}^{LI}_+ \times \mathbf{R}^{LJ}} \quad & u_1(x_1) \\ \text{subject to} \quad & u_i(x_i) \geq \bar{u}_i \quad \forall i = 2, 3, ..., I, \\ & \sum_{i=1}^{I} x_i \leq \bar{\omega} + \sum_{j=1}^{J} y_j, \\ & y_j \in Y_j \quad \forall j = 1, \ldots, J. \end{aligned}$$

Proposition 11. *Suppose that each preference $\succeq_i$ is continuous and strictly monotone. An assignment (x, y) solves the maximization problem PO if and only if it is Pareto Optimal.*

[2] Define the sum of two sets $A, B \subseteq \mathbb{R}^n$ according to *Minkowski addition*: $A + B = \{c \in \mathbb{R}^n : \exists\ a \in A,\ b \in B \text{ s.th. } c = a + b\}$.

As an illustration, let us analyze the above result for the case of exchange economies and differentiable utility functions. Let $(\succsim_i, \omega_i)$ be an exchange economy. According to Proposition 11, an assignment $x \in \mathbf{R}_+^{IL}$ is Pareto Optimal if and only if it solves the following problem:

(PO')
$$\max_{x \in \mathbf{R}_+^{LI}} \; u_1(x_1)$$
$$\text{subject to} \quad u_i(x_i) \geq \bar{u}_i \quad \forall i = 2, 3, ..., I,$$
$$\sum_{i=1}^{I} x_i \leq \bar{\omega},$$
(1)

Assume that u_i is a differentiable function, and consider an interior solution to the above problem.[3]

4. Welfare Theorems

4.1. First Welfare Theorem.

Theorem 12 (First Welfare Theorem)**.** *Let $((Y_j)_{j=1}^J, (\succsim_i, \omega_i, \theta_i)_{i=1}^I)$ be a private ownership economy in which the preference relation of every consumer is locally nonsatiated. If (x, y) and p constitute a Walrasian Equilibrium, then (x, y) is Pareto Optimal.*

Proof of First Welfare Theorem. Let $((x, y), p)$ be a Walrasian equilibrium. Let $(x', y') \in \mathbf{R}_+^{IL} \times \mathbf{R}^{JL}$ with $y' = (y'_j)$ and $y'_j \in Y_j$ for all j. Suppose that $x'_i \succsim_i x_i$ for every $i = 1, \ldots, I$, and there exists $h \in \{1, \ldots, I\}$ such that $x'_h \succ_h x_h$. We shall prove that (x', y') cannot be an allocation$_\leq$.

First, let us prove that

$$(2) \qquad x'_i \succsim_i x_i \implies p \cdot x'_i \geq W_i = p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j.$$

Proceed by contradiction: Assume that $x'_i \succsim_i x_i$, but $p \cdot x'_i < W_i$. Then, there exists $\varepsilon > 0$ such that $||z - x'_i|| < \varepsilon \Rightarrow p \cdot z < W_i$. By local non-satiation, $\exists\, z$ such that $z \succ_i x'_i$ and $||z - x'_i|| < \varepsilon$. Therefore,

[3]This may be achieved by assuming that each individual has strictly convex utility

$p \cdot z < W_i$ and $z \succ_i x_i$. This contradicts that consumer i is optimizing when choosing x_i in $B(p, p \cdot \omega_h + \sum_j \theta_{i,j} p \cdot y_j)$.

Second, note that $x'_h \succ_h x_h$ implies $p \cdot x'_h > W_h$, again because consumer h is optimizing when choosing x_h in $B(p, p \cdot \omega_h + \sum_j \theta_{h,j} p \cdot y_j)$.

Third, note that, since firms are maximizing profits in equilibrium and $y'_j \in Y_j$ for every firm, then $p \cdot y_j \geq p \cdot y'_j$ for every $j = 1, \ldots, J$.

Thus, if we put together these inequalities and sum over consumers we obtain that Therefore,

$$\begin{aligned}
\sum_{i=1}^{I} p \cdot x'_i &> \sum_{i=1}^{I} (p \cdot \omega_i + \sum_{j=1}^{J} \theta_{i,j} p \cdot y_j) \\
&= p \cdot \bar{\omega} + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{i,j} p \cdot y_j \\
&= p \cdot \bar{\omega} + \sum_{j=1}^{J} (p \cdot y_j) \sum_{i=1}^{I} I \theta_{i,j} \\
&= p \cdot \bar{\omega} + \sum_{j=1}^{J} (p \cdot y_j) \\
&\geq p \cdot \bar{\omega} + \sum_{j=1}^{J} (p \cdot y'_j) \\
&= p \cdot \left(\bar{\omega} + \sum_{j=1}^{J} y'_j \right)
\end{aligned}$$

The first (strict) inequality follows from $p \cdot x'_i \geq W_i = p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j$ and $p \cdot x'_h > W_h$. Then we use that $\sum_j \theta_{i,j} = 1$. The second (weak) inequality follows from $p \cdot y_j \geq p \cdot y'_j$.

Thus

$$p \cdot \sum_{i=1}^{I} x'_i > p \cdot \left(\bar{\omega} + \sum_{j=1}^{J} y'_j \right),$$

and therefore (x', y') could not be an allocation$_{\leq}$. □

4.2. **Second Welfare Theorem.** Let $\mathcal{E} = ((Y_j)_{j=1}^{J}, (\succsim_i, \omega_i, \theta_i)_{i=1}^{I})$ be a private ownership economy (POE).

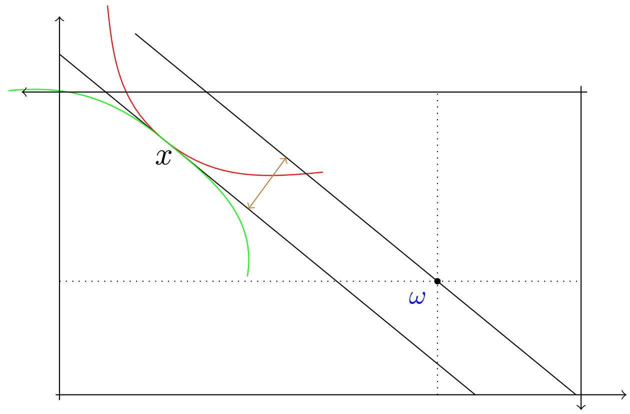

FIGURE 3. The second welfare theorem in the Edgeworth box

Definition 13. A ***Walrasian equilibrium with transfers*** is a tuple (x, y, p, T), where $(x, y) \in \mathbf{R}_+^{IL} \times \mathbf{R}^{JL}$, $p \in \mathbb{R}_+^L$ (a ***price vector***) and $T = (T_i)_{i=1}^I \in \mathbb{R}^I$ (a vector of ***net transfers***), s.t.

(i) for every $i = 1, \dots, I$, $x_i \in \mathbb{B}(p, M_i)$, and

$$x_i' \in \mathbb{B}(p, M_i) \Rightarrow x_i \succsim x_i',$$

where $M_i = p \cdot \omega_i + \sum_{j=1}^J \theta_{i,j} p \cdot y_j + T_i$ (consumers optimize by choosing x_i in their budget sets);

(ii) for every $j = 1, \dots, J$, $y_j \in Y_j$, and $p \cdot y_j \geq p \cdot y_j'$ $\forall$ $y_j' \in Y_j$ (firms optimize profits by choosing y_j in Y_j);

(iii) $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i + \sum_{j=1}^J y_j$ (demand equals supply).

(iv) $\sum_{i=1}^I T_i = 0$ (net transfers are “budget balanced”).

Theorem 14 (Second Welfare Theorem)**.** *Let*

$$\mathcal{E} = \left((\succeq_i, \omega_i, \theta_i)_{i=1}^I, (Y_j)_{j=1}^J\right)$$

be a P.O.E. in which each Y_j is closed and convex, and each preference $\succeq_i$ is strongly monotone, convex, and continuous. If (x^*, y^*) is a Pareto optimal allocation$_\leq$ in which $\sum_{i=1}^I x_i^* \gg 0$, then there is a price vector $p^* \in \mathbf{R}_+^L$ and transfers $T = (T_i)_{i=1}^I$ such that (x^*, y^*, p^*, T) is a Walrasian equilibrium with transfers.

Digression: The separating hyperplane theorem. Let $p \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$. The hyperplane defined by (p, α) is:

$$H(p, \alpha) = \{x \in \mathbf{R}^n : p \cdot x = \alpha\}.$$

A hyperplane defines two ***half-spaces***:

$H_+(p,\alpha) = \{x \in \mathbf{R}^n : p \cdot x \geq \alpha\}$ and $H_-(p,\alpha) = \{x \in \mathbf{R}^n : p \cdot x \leq \alpha\}$.

Let $X, Y \subseteq \mathbf{R}^n$. We say that the hyperplane $H(p,\alpha)$ ***properly separates*** X and Y if $X \subseteq H_+(p,\alpha)$ and $Y \subseteq H_-(p,\alpha)$, or $Y \subseteq H_+(p,\alpha)$ and $X \subseteq H_-(p,\alpha)$, while $p \cdot x \neq p \cdot y$, for some $x \in X$ and $y \in Y$.

Lemma 15 (Separating-hyperplane Theorem)**.** *Let X and Y be disjoint and nonempty convex sets in $\mathbb{R}^n$. Then there exists a hyperplane that properly separates X and Y.*

Proof of Second Welfare Theorem. Let (x^*, y^*) be a Pareto optimal allocation$_\leq$in the hypotheses of the theorem. The first observation is that condition (iii) of the definition of Walrasian equilibrium with transfers (WET) holds, which is equivalent to (x^*, y^*) being an allocation$_=$. Indeed, if we were to have $\sum_{i=1}^I x_i^* < \bar{\omega} + \sum_{j=1}^J y_j^*$. Then we could modify x' by giving some consumer the difference $\bar{\omega} + \sum_{j=1}^J y_j^* - \sum_{i=1}^I x_i^*$ and (using the strict monotonicity of preferences), contradict the Pareto optimality of (x', y').

We need to prove that there exist a price vector $p^* \in \mathbb{R}_+^L$ and a transference schedule $T \in \mathbb{R}^L$ such that conditions (i) to (iv) of the definition are satisfied. Let

$$P_i = \{z_i \in \mathbf{R}_+^L : z_i \succ_i x_i^*\}$$

for $i = 1, \ldots, I$.

Observe that P_i is a convex set as a direct consequence of the convexity of $\succeq_i$.[4]

Define $P = \sum_{i=1}^I P_i$, and note that, being the sum of convex sets, P is also convex. The set P contains all the "aggregate bundles" that can be disaggregated into individual consumption bundles for which every individual is strictly better off than under the individual consumption bundle x_i^*.

Let $Y = \sum_{j=1}^J Y_j$ be the aggregate production set, and note that it is convex, as every firm's production set is convex. Hence,

$$PPS = (\{\bar{\omega}\} + Y) \cap \mathbf{R}_+^L$$

[4]Let $z, z' \in P_i$. Say wlog that $z \succeq_i z'$. Then for any $\lambda \in (0,1)$ the convexity of $\succeq_i$ means that $\lambda z + (1-\lambda)z' \succeq_i z' \succ_i x_i^*$. Hence, by transitivity of preferences, $\lambda z + (1-\lambda)z' \in P_i$.

is also a convex set. Since (x^*, y^*) is Pareto Optimal,

$$P \cap PPS = \emptyset.$$

Otherwise we could produce an allocation$_{\leq}$in which all consumers are better off than in x^* by finding $z_i \in P_i$ with $\sum_i z_i \in \{\bar{\omega}\} + Y$.

Now, by the Separating-hyperplane Theorem (Lemma 15), there exists $p^* \in \mathbb{R}^L$ such that

$$(3) \qquad p^* \cdot z \geq p^* \cdot q \quad \text{for all } z \in P, \text{ and } q \in (\{\bar{\omega}\} + Y).$$

We shall see that p^* is going to be the price vector in our Walrasian equilibrium with transfers. Observe that the definition of proper separation implies that $p^* \neq 0$. We could, however, in principle have $p_l^* < 0$. We'll need to do some work to rule out negative prices.

Firstly, we shall prove the following:

$$(4) \qquad x_i' \succ_i x_i^* \implies p^* \cdot x_i' \geq p^* \cdot x_i^*, \quad \text{for every } i.$$

Let $x_i' \succ_i x_i^*$. Note that we must have $x_i' > 0$, as $\succeq_i$ is strictly monotone. By continuity of $\succeq_i$, $\exists\ \delta \in (0,1)$ such that $(1-\delta)x_i' \succ_i x_i^*$. By strict monotonicity and the fact that $x_i' > 0$,

$$x_h^* + \left(\frac{\delta}{I-1}\right) x_i' \succ_h x_h^*, \quad \text{for every } h \neq i.$$

Therefore, $(1-\delta)x_i' \in P_i$, and $x_h^* + \left(\frac{\delta}{I-1}\right) x_i' \in P_h$ for every $h \neq i$. Then,

$$x_i' + \sum_{h \neq i} x_h^* = (1-\delta)x_i' + \sum_{h \neq i} \left(x_h^* + \frac{\delta x_i'}{I-1}\right) \in P.$$

Since (x^*, y^*) is an allocation$_=$,

$$\sum_{h=1}^{I} x_h^* \in (\{\bar{\omega}\} + Y).$$

By (3), then, we obtain that

$$p^* \cdot \left(x_i' + \sum_{h \neq i} x_h^*\right) \geq p^* \cdot \left(x_i^* + \sum_{h \neq i} x_h^*\right).$$

Simplifying, $p^* \cdot x_i' \geq p^* \cdot x_i^*$.

Secondly, let us prove that $p^* > 0$. Fix $l \in \{1, \ldots, L\}$. Let $e_l \in \mathbf{R}^L$ be the vector that equals 0 in all its entries except for the lth entry, in which it equals 1 (that is, $e_{ls} = 0\ \forall\ s \neq l$ and $e_{ll} = 1$). By strict monotonicity, for any $i \in 1, \ldots, I$, $x_i^* + e_l \succ_i x_i^*$. Then, $p^* \cdot (x_i^* + e_l) \geq$

$p^* \cdot x_i^*$ applying (4). The latter yields $p^* \cdot e_l = p_l \geq 0$. As l is arbitrary, and $p^* \neq 0$, we conclude that $p^* > 0$.

Knowing that $p^* > 0$ we are in a position to strengthen property (4). We shall prove that

$$x_i' \succ_i x_i^* \implies p^* \cdot x_i' > p^* \cdot x_i^*, \quad \text{for every } i. \tag{5}$$

Note that $\sum_i x_i^* \gg 0$ and $p^* > 0$ imply that $p^* \cdot \sum_i x_i^* > 0$, and so there exists at least one consumer i with $p^* x_i^* > 0$. For such a consumer, we show that (5) must hold. To that end, suppose (towards a contradiction) that $x_i' \succ_i x_i^*$ but $p^* \cdot x_i' \leq p^* \cdot x_i^*$. Given that we have established (4), it must be the case that $p^* \cdot x_i' = p^* \cdot x_i^*$. Then, by continuity of $\succeq_i$, $\exists\, \delta \in (0,1)$ such that $(1-\delta)x_i' \succ_i x_i^*$. Then

$$p^* \cdot (1-\delta)x_i' = (1-\delta)p^* \cdot x_i' < p^* \cdot x_i' = p^* \cdot x_i^*,$$

observe that the strict inequality is due to $p^* \cdot x_i' > 0$. Note that the latter contradicts (4), yielding our desired result (5) for the specific consumer i that we know has $p^* \cdot x_i^* > 0$.

Now, reasoning as in the proof that $p > 0$, we see that, in fact, $p \gg 0$: Since (5) holds for consumer i, $x_i^* + e_l \succ_i x_i^*$ implies that $p_l > 0$. So $p \gg 0$.

Finally, to show that (5) holds for all consumers, note that $p \gg 0$ implies that $p^* \cdot x_i^* > 0$ for all i with $x_i > 0$. Then the proof of (5) applies to all consumers with $x_i > 0$. For a consumer i with $x_i = 0$ we know that if $x_i' \succ_i x_i$ then $x_i' > 0$ so that $p \cdot x_i' > 0 = p \cdot x_i$.

At this point we turn to the definition of transfers. For any i, define the transfer to i by

$$T_i = p^* \cdot x_i^* - p^* \cdot \omega_i - \sum_{j=1}^{J} \theta_{ij} p^* \cdot y_j^*. \tag{6}$$

Now we have all the elements to establish condition (i) of the definition of Walrasian equilibrium with transfers. Let

$$M_i = p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j + T_i,$$

and note that for every consumer i: $p^* \cdot x_i^* = M_i$. Thus $x_i^* \in B(p, M_i)$. By (5), condition (i) is satisfied.

Using the latter result, it is straightforward to verify condition (iv). Note the following:

$$\begin{aligned}\sum_{i=1}^{I} T_i &= \sum_{i=1}^{I}\left(p^* \cdot x_i^* - p^* \cdot \omega_i - \sum_{j=1}^{J}\theta_{ij} p^* \cdot y_j^*\right)\\ &= p^* \cdot \left(\sum_{i=1}^{I} x_i^* - \bar{\omega} - \sum_{i=1}^{I}\sum_{j=1}^{J}\theta_{ij} y_j^*\right)\\ &= p^* \cdot \left(\sum_{i=1}^{I} x_i^* - \bar{\omega} - \sum_{j=1}^{J} y_j^*\right)\\ &= 0.\end{aligned}$$

We now proceed to condition (ii). First we shall prove that

$$p^* \cdot \sum_{i=1}^{I} x_i^* \geq p^* \cdot q \quad \forall\; q \in \{\bar{\omega}\} + Y. \tag{7}$$

Property (7) says that the aggregate bundle $\sum_i x_i^*$ "maximizes value" among all the elements of $\{\bar{\omega}\} + Y$.

Let $q \in \{\bar{\omega}\} + Y$. For each n, let

$$z_i^n = x_I^* + (1/n, \ldots, 1/n).$$

By monotonicity of $\succeq_i$, $z_i^n \in P_i$ for all i. So $z^n = \sum_{i=1}^{I} z_i^n \in P$. By (3),

$$p^* \cdot z^n \geq p^* \cdot q \quad \text{for every } n \in \mathbf{N}. \tag{8}$$

Note that $z^n \to \sum_i x_i^*$. Since (8) holds for each n, in the limit as $n \to \infty$ we obtain that $p^* \cdot \sum_i x_i^* \geq q$. [5]

Now, let us prove condition (ii) of the definition of WET. We must show that for every firm j, $p^* \cdot y_j^* \geq p^* \cdot y_j' \;\forall\; y_j \in Y_j$. Let $y_j' \in Y_j$, and note that by (7):

$$p^* \cdot \sum_{i=1}^{I} x_i^* \geq p^* \cdot \bar{\omega} + p^* \cdot y_j' + \sum_{k \neq j} p^* \cdot y_k^*.$$

---

[5] The proof of statement 7 relies on showing that aggregate consumption $\sum_i x_i^*$ lies on the boundary of the set P. In fact, it lies at the intersection of the boundary of P and the production possibility set of the economy, $PPS = (\{\bar{\omega}\} + Y) \cap \mathbb{R}_+^L$. Statement 7 says that aggregate consumption maximizes "value" in the PPS.

As condition (iii) states $\sum_{i=1}^{I} x_i^* = \bar{\omega} + \sum_{j=1}^{J} y_j^*$, we have:

$$p^* \cdot \bar{\omega} + \sum_{j=1}^{J} p^* \cdot y_j^* \geq p^* \cdot \bar{\omega} + p^* \cdot y_j' + \sum_{k \neq j} p^* \cdot y_k^* \implies p^* \cdot y_j^* \geq p^* \cdot y_j'.$$

As we have proved the existence of a price vector $p^* \in \mathbb{R}_+^L$ and a transfer schedule T that satisfy conditions (i) through (iv) of a Walrasian Equilibrium with transfers. □

Terminology: "Aggregation". Economy-wide variables are called "aggregate." If consumers get the bundles in $(x_1, \ldots, x_I)$, then $\sum_{i=1}^{n} x_i$ is an aggregate consumption bundle. Similarly, when we add up (excess) demand functions we'll talk about aggregate (excess) demand.

5. Scitovsky Contours and cost-benefit analysis

Let $(\succeq_i)_{i=1}^{I}$ be a collection of preferences. The ***Scitovsky contour*** at $x = (x_i)_{i=1}^{I}$ is

$$S(x_1, \ldots, x_I) = \{\sum_{i=1}^{I} \tilde{x}_i : \tilde{x}_i \succeq_i x_i \forall i \in \{1, \ldots, I\}\}$$

Recall the definition of upper contour set: let $U_i(x_i) = \{y_i \in \mathbf{R}_+^L : y_i \succeq_i x_i\}$. Then

$$S(x_1, \ldots, x_I) = \sum_{i=1}^{I} U_i(x_i).$$

Consider a vector $(x_1, \ldots, x_I)$. Think of an exchange economy with these I, agent i having preference $\succeq_i$, and endowments being such that $\bar{\omega} = \sum_{i=1}^{I} x_i$. Then we can consider the possibility that $(x_1, \ldots, x_I)$ is a Pareto optimal allocation$_=$of the aggregate bundle $\sum_i x_i$.

The set $S(x_1, \ldots, x_I)$ is the set of aggregate bundles that can be disaggregated in a way that makes all consumers weakly better than they are at $(x_1, \ldots, x_I)$. If $(x_1, \ldots, x_I)$ is a Pareto optimal allocation$_=$of the aggregate bundle $\sum_i x_i$, then the Scitovsky contour $S(x_1, \ldots, x_I)$ must be disjoint from the set $\{z \in \mathbf{R}_+^L : z < \sum_i x_i\}$.

Scitovsky contours can be used to evaluate policy decisions. The ***Kaldor criterion*** says that a policy change is desirable if everyone can be made better off after the policy change than before. Scitovsky contours embody the Kaldor criterion. In particular, suppose that the group of agents $\{1, \ldots, I\}$ are considering a collective change from the aggregate bundle $\bar{x} = \sum_i x_i$ to $\hat{x} \in \mathbf{R}^L_+$. If $\bar{x}$ is allocated as in $(x_1, \ldots, x_I)$ and $\hat{x} \in S(x_1, \ldots, x_I)$ then the change is desirable because the resulting aggregate bundle can be disaggregated in a way that makes no consumer worse off than in $\bar{x}$. Moreover, if $\hat{x}$ is in the interior of the contour, the consumers can be made strictly better off.

When each preference $\succeq_i$ is convex, continuous and strictly monotonic, we can essentially use the the second welfare theorem to describe Scitovsky contours and to operationalize the Kaldor criterion via ***cost-benefit analysis***. The idea is that if (x_i) is a Pareto optimal allocation$_{=}$of $\sum_i x_i$, then we obtain prices p^* such that $p^* \cdot z \geq p^* \cdot \sum_i x_i$ for all $z \in S(x_1, \ldots, x_I)$.[6] See Figure 4.

This means that prices p^* support $S(x_1, \ldots, x_I)$ at $\sum_i x_i$: $p \cdot z \geq p \cdot \sum_i x_i$ for all $z \in S(x_1, \ldots, x_I)$. By observing market prices p^* we obtain information about the shape of $S(x_1, \ldots, x_I)$. In other words, market prices convey enough information about agents' preferences to be useful policy tools.

When the group of agents $\{1, \ldots, I\}$ are considering a collective change from the aggregate bundle $\bar{x} = \sum_i x_i$ to $\hat{x} \in \mathbf{R}^L_+$. If $p \cdot \hat{x} < p \cdot \bar{x}$ then we know that $\hat{x} \notin S(x_1, \ldots, x_I)$, and there is therefore no way to distribute $\hat{x}$ in a way that will make all the agents better off than they are currently in $(x_1, \ldots, x_I)$. Put differently, *even without knowing agents' preferences*, if we "price" the difference $\hat{x} - \bar{x}$ using prevailing market prices p, and the value is negative, we know that the change is from $\bar{x}$ to $\hat{x}$ is undesirable.

On the other hand if the change is positive then it has some chance of belonging to the Scitovsky contour at $\bar{x}$. Moreover, if preferences are smooth then the supporting hyperplane defined by p will be a good local description of $S(x_1, \ldots, x_I)$ around $\bar{x}$. So when the change $\hat{x} - \bar{x}$ is relatively small, we can be relatively confident that $\hat{x} \in S(x_1, \ldots, x_I)$ when $p \cdot (\hat{x} - \bar{x}) > 0$.

This reasoning is known as ***cost-benefit analysis***. We use prevailing prices to evaluate the change from $\bar{x}$ to $\hat{x}$, and decide based on the value of the two aggregate bundles at the prevailing prices.

One problem with the Kaldor criterion is that it is possible that $\hat{x} \in S(x_1, \ldots, x_I)$ and that there is $(x'_i) \in \mathbf{R}^{IL}_+$ with $\hat{x} = \sum_i x'_i$, and $\bar{x} \in$

[6] The reason is that $x'_i \in U_i(x_i)$ implies that $p^* \cdot x'_i \geq p^* \cdot x_i$. So the prices support each of the individual upper contour sets at the consumption bundle x_i.

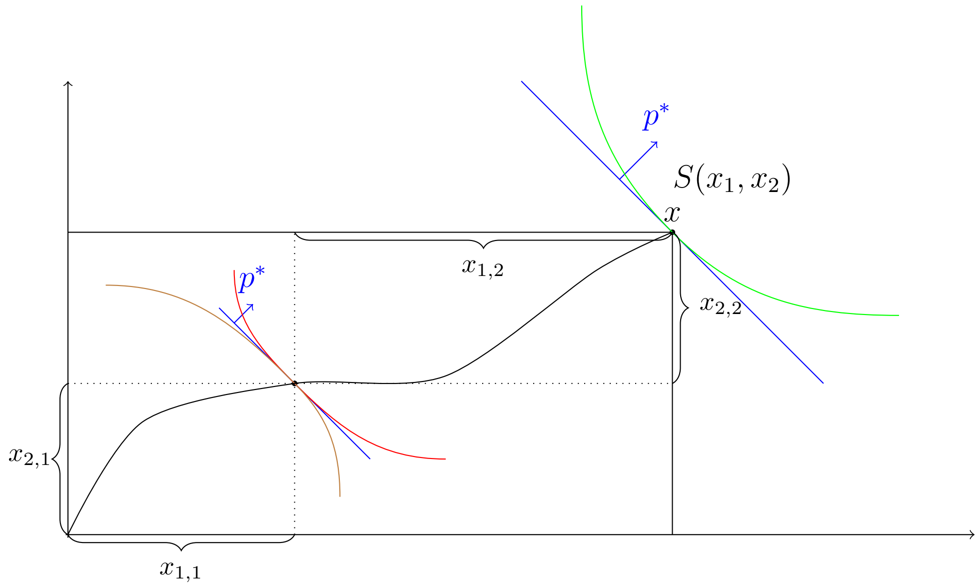

FIGURE 4. Scitovsky contour at $x = x_1 + x_2$.

$S(x'_1, \ldots, x'_I)$. Draw such a case for yourself. Think about what it means.

6. Excess demand functions

6.1. **Notation.** Let $\Delta = \{p \in \mathbf{R}^L_+ : \sum_{l=1}^L p_l = 1\}$ denote the simplex in $\mathbf{R}^L$. The interior of the simplex is denoted by $\Delta^o = \{p \in \Delta : p \gg 0\}$, and the boundary by $\partial\Delta = \Delta \setminus \Delta^o$.

For $\varepsilon > 0$, we also use the notation $\Delta^\varepsilon = \{p \in \Delta : p_l \geq \varepsilon, l = 1, \ldots, L\}$, and

$$\partial\Delta^\varepsilon = \{p \in \Delta^\varepsilon : p_l = \varepsilon, \text{ for some } l = 1, \ldots, L\}.$$

6.2. **Aggregate excess demand in an exchange economy.** Suppose that $(\succsim_i, \omega_i)_{i=1}^I$ is an exchange economy in which each $\succsim_i$ is continuous, strictly convex, and strictly monotonic. Then we can define a demand function $p \mapsto x^*_i(p, p \cdot \omega_i)$, and an excess demand function $p \mapsto z_i(p) = x^*_i(p, p \cdot \omega_i) - \omega_i$ for all consumer i. The aggregate excess demand function is then

$$z(p) = \sum_{i=1}^I z_i(p) = \sum_{i=1}^I x^*_i(p, p \cdot \omega_i) - \bar{\omega}$$

This function $z : \mathbf{R}^L_{++} \to \mathbf{R}^L$ satisfies a number of basic properties that will be essential in our study of equilibrium.

Proposition 16. *If $(\succsim_i, \omega_i)_{i=1}^I$ is an exchange economy in which $\bar{\omega} = \sum_{i=1}^I \omega_i \gg 0$, and each $\succsim_i$ is continuous, strictly convex, and strictly monotonic; then, the aggregate excess demand function satisfies:*

(i) *(continuity) z is continuous;*
(ii) *(homogeneity) z is homogeneous degree zero;*
(iii) *(Walras Law) $\forall\ p \in \mathbf{R}^L_{++}$, $p \cdot z(p) = 0$;*
(iv) *(bounded below) $\exists\ M > 0$ s.th. $\forall\ l$, $p \in \mathbf{R}^L_{++}$, $z_l(p) > -M$;*
(v) *(boundary condition) If $\{p^n\}$ is a sequence in $\mathbf{R}^L_{++}$, and $\bar{p} = \lim_{n\to\infty} p^n$, where $\bar{p} \in \mathbf{R}^L_0 \setminus \mathbf{R}^L_{++}$ and $\bar{p} \neq 0$, then there is $l \in \{1, \ldots, L\}$ such that $\{z_l(p^n)\}$ is unbounded.*

Proof. Let us proceed in order:

(i) If $\succsim_i$ is strictly monotonic, then it is locally non-satitated. By the maximum theorem, every demand function $x_i^*(p, p \cdot \omega_i)$ is continuous. As $z(p)$ is a linear combination of continuous functions, it is continuous.
(ii) The budget set of every consumer i is unchanged if prices are multiplied by a constant $\alpha > 0$, i.e, $\mathbf{B}(p, p \cdot \omega_i) = \mathbf{B}(\alpha p, \alpha p \cdot \omega_i)$ for every $\alpha \in \mathbf{R}_{++}$ with $p \in \mathbf{R}^L_+$. Then, the maximization problem of consumer i is unchanged if the prices are multiplied by $\alpha > 0$. This shows that the demand function of every consumer is homogeneous of degree zero: $x_i^*(p, p \cdot \omega_i) = x_i^*(\alpha p, \alpha p \cdot \omega_i)$ for every $\alpha > 0$. Then, by definition, $z(\alpha p) = z(p)$ for every $\alpha > 0$.
(iii) $\succsim_i$ strictly monotonic implies $p \cdot x_i^*(p, p \cdot \omega_i) = p \cdot \omega_i$ for every i, i.e., every consumer's expenditure level is equal to her income. (Note that otherwise, the consumer would be able to increase her utility by consuming more from any good). Summing over

all consumers, we obtain:

$$\begin{aligned} & \sum_{i=1}^{I} p \cdot x_i^*(p, p \cdot \omega_i) = \sum_{i=1}^{I} p \cdot \omega_i \\ \Longrightarrow \quad & p \cdot \sum_{i=1}^{I} x_i^*(p, p \cdot \omega_i) = p \cdot \sum_{i=1}^{I} \omega_i \\ \Longrightarrow \quad & p \cdot \left(\sum_{i=1}^{I} x_i^*(p, p \cdot \omega_i) - \bar{\omega} \right) = 0 \\ \Longrightarrow \quad & p \cdot z(p) = 0 \end{aligned}$$

(iv) Note that $X = \mathbf{R}_+^L$ implies $x_{li}^*(p, p \cdot \omega_i) \geq 0$ for every consumer i and good l. Hence, $z_l(p) \geq -\bar{\omega}_l$ for every good l. Let $M \in \mathbf{R}$ be such that $M > \max_{l \in \{1,\ldots,L\}}\{\bar{\omega}_l\}$, and note that $z_l(p) > -M$ $\forall\, l$, $p \in \mathbf{R}_{++}^L$.

(v) Towards contradiction, assume $z_l(p^n)$ is bounded above for all $l \in \{1, \ldots, L\}$. Let $\bar{p} \in \mathbf{R}_+^L \setminus \mathbf{R}_{++}^L$ be s.th. $\bar{p} \neq 0$. Then, there are $m, k \in \{1, \ldots, L\}$ s.th. $\bar{p}_k = 0$ and $\bar{p}_m > 0$. As the aggregate endowment is strictly positive, $\bar{\omega} >> 0$, then there exists a consumer j such that $\omega_{mj} > 0$ and thus $\bar{p} \cdot \omega_j > 0$. Furthermore, as $\succsim_j$ is strictly monotonic, j strictly prefers bundles that have more of good k, all else equal. Note that this implies that the demand function $x_j^*(p, p \cdot \omega_j)$ is not well defined at $p = \bar{p}$ since j has positive income and good k, which she likes, is free. To see this formally, assume by contradiction $\exists\, x_j^* \in \mathbf{R}_+^L$ such that

$$x_j^* \in \mathbf{B}(\bar{p}, \bar{p} \cdot \omega_j) \quad \text{and} \quad x_j^* \succsim_j x_j \,\forall\, x_j \in \mathbf{B}(\bar{p}, \bar{p} \cdot \omega_j).$$

Let $\hat{x}_j = x_i^* + \varepsilon e_k$ with $\varepsilon > 0$, where $e_k \in \mathbf{R}^L$ is the vector of zeroes, except for its k-th entry which equals one. As $\bar{p}_k = 0$, then $\bar{p} \cdot x_j^* = \bar{p} \cdot \hat{x}_j \Rightarrow \hat{x}_j \in \mathbf{B}(\bar{p}, \bar{p} \cdot \omega_j)$. However, by strict monotonicity, $\hat{x}_j \succ_j x_j^*$, and we obtain the desired contradiction.

Let $\{p^n\}$ be a sequence in $\mathbf{R}_{++}^L$ s.th. $p^n \to \bar{p}$. Let $z_{li}(p^n)$ be the sequence of the excess demand of consumer i for good l. We can write the aggregate excess demand for any good l as:

$$z_l(p^n) = z_{lj}(p^n) + \sum_{i \neq j} z_{li}(p^n).$$

Note that the excess demand of every consumer for good l is bounded below by $M_{li} > \omega_{li}$. As $z_l(p^n)$ is bounded above by hypothesis for every good l, then it must be that $z_{li}(p^n)$ is also bounded above for every consumer i, j included. Hence, as

$z_{lj}(p^n)$ is bounded above and below, it has a convergent subsequence, say $z_{lj}(p^{n_k})$. Let z^*_{lj} be the limit of this subsequence.[7]

Let $z^*_j = (z^*_{lj})_{l=1}^L$ and $x^*_j = z^*_j + \omega_j$. As $p^n >> 0$ for every n, then by strict monotonicity, for every n, $p^n \cdot x^*_j(p^n, p^n \cdot \omega_j) = p^n \cdot \omega_j$, i.e., consumer j spends all her income. Therefore, $p^n \cdot z_j(p^n) = 0$ for every n. Then,

$$\lim_{n\to\infty} p^{n_k} \cdot z_{lj}(p^{n_k}) = 0 \implies \bar{p} \cdot z^*_{lj} = 0 \implies \bar{p} \cdot x^*_j = \bar{p} \cdot \omega_j \implies x^*_j \in \mathbf{B}(\bar{p}, \bar{p} \cdot \omega_j).$$

To conclude the prove, we will show $x^*_j \succsim_j x_j$ for every $x_j \in \mathbf{B}(\bar{p}, \bar{p} \cdot \omega_j)$ to obtain the desired contradiction. Let $x_j \in \mathbf{B}(\bar{p}, \bar{p} \cdot \omega_j)$. Let $\lambda^n = \frac{p^n \cdot \omega_j}{\bar{p} \cdot \omega_j}$ for every n, and note $\lambda^n \to 1$ and $\lambda^n \geq 0$ for every n. Since $\bar{p} \cdot x_j \leq \bar{p} \cdot \omega_j$, multiplying both sides by λ^n yields $\bar{p} \cdot \lambda^n x_i \leq p^n \cdot \omega_j$ for every n, i.e., $\lambda^n x_j \in \mathbf{B}(p^n, p^n \cdot \omega_j)$. By definition, $z_j(p^n) = x^*_j(p^n, p^n \cdot \omega_j) - \omega_j$. Hence, $z_j(p^n) + \omega_j \succsim_j \lambda^n x_j$. By continuity of $\succsim_j$, we obtain:

$$\lim_{n\to\infty} z_j(p^n) + \omega_j = z^*_j + \omega_j = x^*_j \succsim_j x_j = \lim_{n\to\infty} \lambda^n x_j.$$

□

Observation 17. Note that properties (i)-(iii) do not rely on $\sum_i \omega_i \gg 0$, nor on strict monotonicity. Local non-satiation suffices for Walras' Law.

6.3. **Aggregate excess demand.** So we derived the aggregate excess demand from an economy, and showed that it satisfies the five basic properties. Now we shall analyze excess demand function as a primitive in its own right, "forgetting" the economy that it came from.

This approach allows us to obtain results for many different models, as long as their equilibria are characterized by the zeroes of an excess demand. For example we can incorporate production, and show that a private ownership economy, under some assumptions, has an excess demand function that satisfies the five properties. And there are other models that can also be captured by the demand function as a reduced form.

We use property (ii) to restrict the domain in z in various ways. Some times we "normalize" prices, and express all prices in terms of one good. For example in terms of good one: p_l/p_1, $l = 1, \dots, L$. Another normalization is $p_l / \sum_{k=1}^L p_k$, so prices are in Δ^o:

[7] It is tempting to claim $z^*_{lj} = z_{lj}(\bar{p})$, but this may not be true since $z_{lj}(p)$ may no be continuous at $p = \bar{p}$.

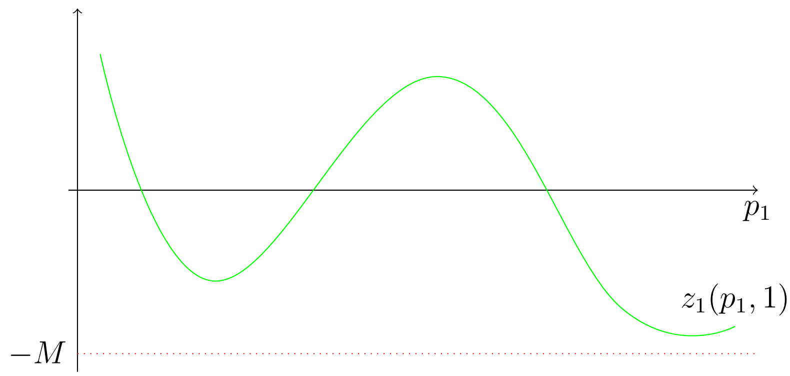

FIGURE 5. Aggregate excess demand for good 1 with $L = 2$/

Consider a function $z : \Delta^o \to \mathbf{R}$ that satisfies the properties:

(1) z is continuous;
(3) z satisfies ***Walras' Law***, meaning that $p \cdot z(p) = 0$.
(4) z is ***bounded below***, meaning that here is $M > 0$ such that $z_l(p) \geq -M$ for all $l = 1, \ldots, L$ and p.
(5) z satisfies the following boundary condition: If $\{p^n\}$ is a sequence in Δ^O and $\bar{p} = \lim_{n\to\infty} p^n$, with $p \in \partial\Delta$, then there is l such that the sequence $\{z_l(p^n)\}$ is unbounded.

We refer to aggregate excess demand functions satisfying the 5 properties with the understanding that we use homogeneity to go back and forth between the two domains. Either the domain is $\mathbf{R}^L_{++}$ and we impose all 5 properties, or the domain is Δ^o, and we can obtain a function on $\mathbf{R}^L_{++}$ by imposing homogeneity. Specifically, if we define an excess demand function $z : \Delta^o \to \mathbf{R}$, then the domain can be extended to all of $\mathbf{R}^L_{++}$ by $z(p) = z(\frac{1}{\sum_l p_l} p)$.

7. Existence of competitive equilibria

Consider the case when $L = 2$. Then Walras Law implies that $p_1 z_1(p) + p_2 z_2(p) = 0$. We can normalize the price of good 2 to be 1. This allows us to graph excess demand as a function of p_1 alone. Moreover, $p_1 z_1(p_1, 1) + p_2 z_2(p_1, 1) = 0$ implies that we can focus on the market for good 1. Whenever $z_1(p_1, 1) = 0$ we know that $z_2(p_1, 1) = 0$.

We present an existence result for excess demand functions satisfying the five properties. The proof is taken from Geanakoplos (2003).

Theorem 18. *Let $z : \Delta^o \to \mathbf{R}$ satisfy properties (1)-(5). Then there is $p^* \in \Delta^O$ with $0 = z(p^*)$.*

We use the following result, which subsumes the role of the boundary condition.

Lemma 19. *Let $z : \Delta^o \to \mathbf{R}$ satisfy properties (1)-(5). Then there is $\varepsilon > 0$ and $\bar{p} \in \Delta^\varepsilon$ s.t.*

$$p \in \partial \Delta^\varepsilon \Rightarrow \bar{p} \cdot z(p) > 0.$$

The proof of Lemma 19 is part of your homework. Now we turn to the proof of the theorem. The proof relies on the following famous result, Brouwer's fixed-point theorem:

Theorem 20. *Let $X \subseteq \mathbf{R}^n$ be (nonempty) compact and convex. If $f : X \to X$ is continuous, then there is $x^* \in X$ with $x^* = f(x^*)$.*

Proof. Let ε and $\bar{p}$ be as in the statement of the lemma.

Define the functions m and ϕ as follows:

$$\begin{aligned} m(\tilde{p}, p) &= \tilde{p} \cdot z(p) - \|\tilde{p} - p\|^2 \\ \phi(p) &= \operatorname{argmax}_{\tilde{p} \in \Delta^\varepsilon} m(\tilde{p}, p), \end{aligned}$$

for $\tilde{p}, p \in \Delta^\varepsilon$.

You should prove that ϕ is a function (it takes singleton values), and that is satisfies the hypotheses of Brower's fixed point theorem (use the Maximum Theorem). So $\phi : \Delta^\varepsilon \to \Delta^\varepsilon$ is continuous. Brower's fixed point theorem implies that there is $p^* \in \Delta^\varepsilon$ with $p^* = \phi(p^*)$. We shall prove that $0 = z(p^*)$.

Notice first that if p^* is in the interior of Δ^ε then the first order condition for the problem in the definition of ϕ means that

$$0 = \frac{\partial m(\tilde{p}, p^*)}{\partial \tilde{p}}|_{\tilde{p}=p^*} = \left(z(p^*) - 2(\tilde{p} - p^*)\right)|_{\tilde{p}=p^*} = z(p^*)$$

and we are done. To prove the theorem, we need then to show that p^* *must* be interior.

Suppose then, towards a contradiction, that p^* is not interior. Then $\bar{p} \cdot z(p^*) > 0$ by the lemma. Let $p(\lambda) = \lambda \bar{p} + (1 - \lambda) p^*$. Note that

$m(p^*, p^*) = p^* \cdot z(p^*) = 0$, by Walras' Law. Now:

$$\begin{aligned} m(p(\lambda), p^*) - m(p^*, p^*) &= p(\lambda) \cdot z(p^*) - \|p(\lambda) - p^*\|^2 \\ &= \lambda \bar{p} \cdot z(p^*) + (1-\lambda) p^* \cdot z(p^*) - \lambda^2 \|\bar{p} - p^*\|^2 \\ &= \lambda \left(\bar{p} \cdot z(p^*) - \lambda \|\bar{p} - p^*\|^2 \right), \end{aligned}$$

as Walras' Law implies

$$(1-\lambda) p^* \cdot z(p^*) = 0,$$

and $\|p(\lambda) - p^*\|^2 = \lambda^2 \|\bar{p} - p^*\|^2$.

Since $\bar{p} \cdot z(p^*) > 0$ there is $\lambda > 0$ small enough that

$$\bar{p} \cdot z(p^*) - \lambda \|\bar{p} - p^*\|^2 > 0.$$

Then $m(p(\lambda), p^*) > m(p^*, p^*)$, which is absurd. □

Missing: examples of non-existence. Discontinuous offer curves because of lack of convexity. One example with lexicographic preferences.

7.1. **The Negishi approach.** We give a proof of existence that is based on adjusting welfare weights instead of adjusting prices: this is called the Negishi approach (after Negishi (1960)); you can think of it as a way of using the second welfare theorem to prove existence of Walrasian equilibria. After all, it is easy to see that there must exist Pareto optimal allocations, and the second welfare theorem says that they can be obtained as Walrasian equilibria with transfers. The challenge is to show that they can be obtained as Walrasian equilibria without transfers, meaning transfers that equal zero.

Let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy.

Theorem 21. *If $\succeq_i$ is continuous, strictly monotonic and convex, for all $i = 1, \ldots, I$, and $\sum_{i=1}^I \omega_i \gg 0$, then $\mathcal{E}$ has a Walrasian equilibrium.*

Our proof of Theorem 21 relies on Kautani's fixed point theorem:

Theorem 22. *Let $X \subseteq \mathbf{R}^n$ be compact and convex. If $\Pi : X \to 2^X$ is a correspondence such that, for all $x \in X$, $\Pi(x)$ is a nonempty and convex set, and such that the graph*

$$\{(x, y) \in X \times X : y \in \Pi(x)\}$$

is closed (we say that Π is upper hemicontinuous), then there is $x^* \in X$ with $x^* \in \Pi(x^*)$.[8]

[8]The notion of upper hemicontinuous correspondence is more general, but here it reduces to the correspondence having a closed graph.

A triple (x, p, T), where $x \in \mathbf{R}_+^{LI}$, $p \in \mathbf{R}_+^L$ and $T \in \mathbf{R}^I$ is a ***Walrasian equilibrium with transfers (WET)*** if

(1) $p \cdot x_i = p \cdot \omega_i + T_i$ and $x'_i \succ x_i$ implies that $p \cdot x'_i > p \cdot \omega_i + T_i$,
(2) x is an allocation$_=$($\sum_i x_i = \bar{\omega}$, or supply = demand), and
(3) $\sum_i T_i = 0$.

For the rest of this proof, we refer to allocation$_=$as allocations. We shall also need the following version of the second welfare theorem, stated here without proof.

Lemma 23 (Second welfare theorem)**.** *If x is a Pareto optimal allocation, then there is $p \in \Delta$ and $T \in \mathbf{R}^I$ such that (x, p, T) is a WET.*

Let u_i be a utility function representing $\succeq_i$. Suppose wlog that $u(0) = 0$. Observe then that strict monotonicity implies that if $u(x) = 0$ then $x = 0$.

Let

$$\mathcal{U} = \{v \in \mathbf{R}^I : v_i = u_i(x_i), i = 1, \ldots, n \text{ for some Pareto optimal allocation } x\}$$

be the ***utility possibility frontier*** in $\mathcal{E}$.

Lemma 24. *Let $v \in \mathcal{U}$, and let x and x' be Pareto optimal allocations with*

$$v_i = u_i(x_i) = u_i(x'_i) i = 1, \ldots, I$$

. If (x, p, T) is a Walrasian equilibrium with transfers, then so is (x', p, T).

Proof. It is obvious that (x', p, T) satisfies (2) and (3) in the definition of Walrasian equilibrium with transfers.

To prove that (x', p, T) satisfies (1), note first that if $z_i \succ_i x'_i$ then $z_i \succ_i x_i$ as $u_i(x_i) = u_i(x'_i)$. Therefore, $p \cdot z_i > p \cdot \omega_i + T_i$.

To finish the proof, we need to show that $p \cdot x'_i = p \cdot \omega_i + T_i$. But $x'_i \succeq x_i$ (as $u_i(x_i) = u_i(x'_i)$) implies that $p \cdot x'_i \geq p \cdot \omega_i + T_i$ because $p \cdot x'_i < p \cdot \omega_i + T_i$ would imply (by strict monotonicity) the existence of a $z_i \succ_i x_i$ with $p \cdot z_i < p \cdot \omega_i + T_i$. Now, $p \cdot x'_i \geq p \cdot \omega_i + T_i$ for all $I = 1, \ldots, I$, $\sum_i x'_i = \bar{\omega}$, and $\sum_i T_i = 0$ implies that $p \cdot x'_i = p \cdot \omega_i + T_i$ for all i. □

Lemma 25. *There exists $M > 0$ such that if (x, p, T) is a Walrasian equilibrium with transfers, then $-M < T_i < M$ for all $i = 1, \ldots, I$.*

Proof. Observe that for all WET (x,p,T), and all $i=1,\ldots,I$,

$$p\cdot(x_i-\omega_i)=T_i.$$

The set of allocations, say A, and Δ is compact, so there exists $M>0$ such that

$$-M<\min\{p\cdot(x_i-\omega_i):p\in\Delta,x\in A,i=1,\ldots,I\}\leq\max\{p\cdot(x_i-\omega_i):p\in\Delta,x\in A,i=1,\ldots,I$$

□

For $v\in\mathcal{U}$, let $\Pi(v)\subseteq[-M,M]$ be the set of T for which there exists a Pareto optimal x allocation with $v=u(x)$ and (x,p,T) is a WET.

Lemma 26. *The correspondence $v\mapsto\Pi(v)$ is convex valued and has a closed graph.*

Proof. First we show that $\Pi(v)$ is a convex set, for $v\in\mathcal{U}$. So let $T,T'\in\Pi(v)$. By Lemma 24 there is a Pareto optimal x and $p,p\in\Delta$ such that $v=u(x)$ and (x,p,T) and (x,p',T') are both WET.

Let $\mu\in(0,1)$, $\hat{p}=\mu p+(1-\mu)p'$ and $\hat{T}=\mu T+(1-\mu)T'$. We shall prove that $(x,\hat{p},\hat{T})$ is a WET. Properties (2) and (3) of WET are immediately satisfies, as (x,p,T) and (x,p',T') are both WET. To prove Properties (1) note that $p\cdot x_i=p\cdot\omega_i+T_i$ and $p'\cdot x_i=p'\cdot\omega_i+T_i'$ imply $\hat{p}\cdot x_i=\hat{p}\cdot\omega_i+\hat{T}_i$. Moreover, $z_i\succ x_i$ implies that $p\cdot z_i>p\cdot\omega_i+T_i$, and $p'\cdot z_i>p'\cdot\omega_i+T_i'$. Thus $\hat{p}\cdot z_i>\hat{p}\cdot\omega_i+\hat{T}_i$. This shows that $\Pi(v)$ is a convex set.

Now we prove that $v\mapsto\Pi(v)$ has a closed graph. Let $\{v^n\}$ be a convergent sequence in $\mathcal{U}$ and $\{T^n\}$ be a convergent sequence in $[-M,M]$, with $T^n\in\Pi(v^n)$ for all n. Let x^n be a Pareto optimal allocation with $v^n=u(x^n)$ for all n. Choose $p^n\in\Delta$ such that (x^n,p^n,T^n) is a WET. By the compactness of the set of allocations, and the compactness of Δ, after considering a subsequence, we can suppose that (x^n,p^n,T^n) convergence to a triple (x,p,T).

To finish the proof, we have to prove that (x,p,T) is a WET. Again, properties (2) and (3) are immediate. To prove (1) note first that $p^n\cdot x_i^n=p^n\cdot\omega_i+T_i^n$ implies that $p\cdot x_i=p\cdot\omega_i+T_i$. Next, suppose that $z_i\succ x_i$. Then the continuity of $\succeq_i$ implies that for n large enough $z_i\succ x_i^n$. Then for n large enough, $p^n\cdot z_i>p^n\cdot\omega_i+T_i^n$. Thus $p\cdot z_i\geq p\cdot\omega_i+T_i$.

Before we continue, we show that $p\gg 0$. We know that x is an allocation, so $\sum_i x_i=\bar{\omega}$, and $\bar{\omega}\gg 0$. So, $p\in\Delta$ means that $p\cdot\sum_i x_i>$

0. Therefore, there is some consumer i for which $p \cdot x_i > 0$. We show that (1) holds for this consumer. For consumer i, we can rule out that $z_i \succ_i x_i$ and $p \cdot z_i = p \cdot \omega_i + T_i = p \cdot x_i$ because there would then exist, by continuity of $\succeq_i$, $\delta \in (0,1)$ with $\delta z_i \succ_i x_i$. But then (and this argument should be familiar by now),

$$p \cdot (\delta z_i) = \delta p \cdot x_i < p \cdot x_i = p \cdot \omega_i + T_i,$$

a contradiction of the property we have already established that $z_i \succ_i x_i \Rightarrow p \cdot z_i \geq p \cdot \omega_i + T_i$. For any l, then $x_i + e_l \succ_i x_i$ (by strict monotonicity) and therefore $p_l > 0$. Thus $p \gg 0$.

Now consider an arbitrary consumer i and suppose that $z_i \succ_i x_i$. Consider first the possibility that $p \cdot x_i = 0$. If $x_i = 0$ then $z_i \succ_i x_i$ would imply that $p \cdot z_i > 0 = p \cdot x_i$ as $z_i > 0$ when $z_i \succ_i 0$ and $p \gg 0$. If $x_i \neq 0$ then $p \cdot x_i > 0$, again because $p \gg 0$. So suppose that $p \cdot x_i > 0$. Then we can just repeat the argument we made for the special consumer above, for which we knew that $p \cdot x_i > 0$. This implies that (1) holds for all consumers. □

Let $\Delta_I = \{\lambda \in \mathbf{R}^I_+ : \sum_{i=1}^I \lambda_i = 1\}$ be the simplex in $\mathbf{R}^I$. Now observe that for each $v \in \mathcal{U}$ there is a unique value of $\alpha > 0$ such that $\alpha v \in \Delta_I$. In words, the ray defined by $v \geq 0$ in $\mathbf{R}^I$ intersects Δ_I once and only once. The function that maps $v \in \mathcal{U}$ into Δ_I is one-to-one. It is also possible to show that it is onto, and that it and its inverse is continuous. Let h be its inverse. The function h is called the ***radial projection*** of Δ_I onto $\mathcal{U}$.

Lemma 27. *The function $h : \Delta_I \to \mathcal{U}$ is a continuous bijection.*

Lemma 28. *The correspondence $\lambda \mapsto \Pi(h(\lambda))$ has convex values and a closed graph.*

Proof. This follows from Lemma 26. If λ^n is a sequence, $T^n \in \Phi(\lambda^n)$ for each n, and $(\lambda, T) = \lim_{n\to\infty}(\lambda^n, T^n)$ □

Define

$$\eta(T) = \text{argmin}\{\lambda \cdot T : \lambda \in \Delta_I\}$$

for $T \in [-M, M]$. The correspondence $T \mapsto \eta(T)$ has convex values and a closed graph (an application of the maximum theorem).

Consider the correspondence

$$F : \Delta_I \times [-M, M] \to \Delta_I \times [-M, M]$$

defined by $F(\lambda, T) = \eta(T) \times \Pi(h(\lambda)$. Then F is convex valued and has a closed graph. Kakutani's fixed point theorem implies that there exists $(\lambda^*, T^*) \in \Delta_I \times [-M, M]$ with $(\lambda^*, T^*) \in F((\lambda^*, T^*))$.

Let (x^*, p^*, T^*) be a WET with $h(\lambda^*) = u(x^*)$. We shall show that $T^* = 0$, and thereby prove that (x^*, p^*) is a Walrasian equilibrium.

Suppose then, towards a contradiction, that $T_h^* > 0$ for some i. Then there is j with $T_j^* < 0$ as $\sum_i T_i^* = 0$. But then the definition of η, and $\lambda^* \in \eta(T^*)$ implies that $\lambda_h = 0$. By definition of h, then, $v_h = 0$. This means that $u_h(x_h) = 0$ which is only possible if $x_h = 0$ (as $u_i(0) = 0$ and u_i is strictly monotone). Then

$$0 = p^* \cdot x^h = p^* \cdot \omega^h + T^h.$$

Hence $T^h \leq 0$, contradicting that $T^h > 0$.

8. Uniqueness

Definition 29 (Strong Weak Axiom for Excess Demand functions)**.** Let $z : \Delta^o : \mathbf{R}^L_+ \to \mathbf{R}_+$ be an excess demand function. We say that z satisfies the ***Strong Weak Axiom*** if:

$$[z(p^*) = 0,\ p \neq p^*] \implies p^* \cdot z(p) > 0.$$

The strong weak axiom is motivated by a one-consumer exchange economy. Let $x^*(p, p \cdot \omega)$ be the demand function of this consumer. Then $p^* \in \Delta^o$ is an equilibrium price iff $x^*(p^*, p^* \cdot \omega) = \omega$ (supply equals demand when there is a single consumer).

Let $p \neq p^*$, and suppose that $x^*(p, p \cdot \omega) \neq x^*(p^*, p^* \cdot \omega)$. By WARP, it must be the case that $p^* \cdot x^*(p, p \cdot \omega) > p^* \cdot x^*(p^*, p^* \cdot \omega) = p^* \cdot \omega$, i.e., the new optimal bundle was unaffordable under the equilibrium price vector. The reason is that the bundle consisting of the endowment is affordable under any price vector, and $x^*(p^*, p^* \cdot \omega) = \omega$. Therefore,

$$p^* \cdot (x^*(p, p \cdot \omega) - \omega) = p^* \cdot z(p) > 0.$$

Theorem 30. *If $z : \Delta^o \to \mathbf{R}^L$ satisfies conditions (i) to (v) and the Strong Weak Axiom, then there is a unique competitive equilibrium.*

Proof. By the equilibrium existence theorem, a competitive equilibrium must exist. To see why it must be unique, suppose that there are two different competitive equiilbriums, and use the Strong Weak Axiom to reach a contradiction. □

We motivated the Strong Weak Axiom in a one consumer economy using the Weak Axiom of Revealed Preference. With more than one consumer, we would need that aggregate demand satisfies WARP. When there are multiple heterogeneous consumers, however, it is very hard to guarantee that an aggregate demand will satisfy WARP. Our next condition may seem to have a lot of economic content, but it turns out to be stronger than SWA.

Definition 31 (Gross Substitutes)**.** Let $z : \mathbf{R}^L_{++} \to \mathbf{R}^L$ be an excess demand function. We say that z satisfies the ***gross substitutes property*** if, $\forall\ p, p' \in \mathbf{R}^L_{++}$, $p_k < p'_k$ for some k, and $p_l = p'_l\ \forall\ l \neq k$ imply $z_l(p) < z_l(p')\ \forall\ l \neq k$.

Like the SWA, the meaning of GS is simple in the case of a single consumer. Consider an economy with one consumer and two goods: coffee and tea. The two goods are substitutes, meaning that if the price of coffee increases, then the demand for tea must increase. Since there is a single consumer, aggregate demand will have the same property. Now, if there are many heterogeneous consumers, then the price increase will have consequences the incomes of the consumers who own coffe, or for whose who own shares in the firms that produce coffee. It is hard to know what the final impact of these consequences will be. It could result in lower demand for tea. In all, with many agents and many goods, the GS property is not very plausible.

Proposition 32. *If z satisfies the gross substitutes property, then it satisfies the Strong Weak Axiom.*

A proof of Proposition 32 can be found in Arrow and Hahn (1971).

By Theorem 30, Proposition 32 implies that the gross substitutes property is a sufficient condition for uniqueness of competitive equilibrium. We show this fact directly in the next result:

Theorem 33. *If the excess demand function $z : \mathbf{R}^L_{++} \to \mathbf{R}^L$ satisfies conditions (i) to (v) and the gross substitutes property, then there is a unique competitive equilibrium in Δ^o.*

Proof. By Theorem18, properties (i) -(v) from Proposition 16 assure the existence of $p^* \in \Delta^o$ such that $z(p^*) = 0$. Let $p \in \Delta^o$ be such that $p \neq p^*$. Choose $\lambda > 0$ such that $\lambda p \geq p^*$ and $\exists\ h$ for which $\lambda p_h = p^*_h$. (This is achieved by letting $h = \text{argmax}_l\{p^*_l/p_l\}$ and setting $\lambda = p^*_h/p_h > 0$.) Therefore, $\lambda p \geq p^*$ implies that for some k, $\lambda p_k > p^*_k$, i.e., the set $L^+ = \{k \in L : \lambda p_k > p^*_k\} \neq \emptyset$. Increase the price of each good in L^+ one by one to get from price vector p^* to λp. By applying gross substitutes, it must be that the excess demand function of good $h \notin L^+$ increases with each price increase, so that $z_h(\lambda p) > z_h(p^*)$. As $z_h(p^*) = 0$, and $z_h(\lambda p) = \lambda z_h(p)$ by homogeneity of z, then $z_h(p) > 0$. Therefore, p is not an equilibrium. □

By homogeneity of degree zero, there is of course always multiple equilibria in $\mathbf{R}^L_{++}$. That is why Theorem 33 qualifies the statment to talk about uniqueness in Δ^o.

9. Representative Consumer

Consider a collection of preferences $(\succsim_i)_{i=1}^I$ over $\mathbf{R}^L_+$ such that each $\succeq_i$ gives rise to a continuous demand function x_i^*. We want to know when there is a preference relation $\succeq$, giving rise to a demand function x^* with the property that: $x^R(p, p\cdot\bar{\omega}) = \sum_{i=1}^I x_i^*(p, p\cdot\omega_i)$, where $\bar{\omega} = \sum_i \omega_i$.

Let
$$\vec{\omega} = (\omega_1, \omega_2, \dots, \omega_I) \in \mathbf{R}^{IL}_+.$$
Note that $\vec{\omega}$ represents the distribution of the aggregate endowment in the economy. Under this notation, $\bar{\omega} = \iota\cdot\vec{\omega}$, where ι is a vector of ones. We write the aggregate demand function in terms of the distribution of income as:
$$z(p, \vec{\omega}) = \sum_{i=1}^I x_i^*(p, p\cdot\omega_i) - \iota\cdot\vec{\omega}.$$

Definition 34 (Representative Consumer)**.** A collection of preferences $(\succsim_i)_{i=1}^I$ ***admits a representative consumer*** if there is a preference relation $\succsim^R$ on $\mathbf{R}^L_+$ and an associated continuous demand function $x^R : \mathbf{R}^L_+ \times \mathbf{R}_+ \to \mathbf{R}^L_+$ such that
$$x^R(p, p\cdot\bar{\omega}) = \sum_{i=1}^I x_i^*(p, p\cdot\omega_i) \quad \forall\, \vec{\omega} \in \mathbf{R}^{IL}_+,\ p \in \mathbf{R}^L_+.$$
If a collection of preferences admits a representative consumer, note that its associated demand function satisfies
$$z^R(p, \bar{\omega}) = x^R(p, p\cdot\bar{\omega}) - \bar{\omega} = z(p, \vec{\omega}) \quad \forall\, \vec{\omega} \in \mathbf{R}^{IL}_+.$$
In this case, we say that z admits a representative consumer.

Observation 35. Let $(\succsim_i, \omega_i)_{i=1}^I$ be an exchange economy. If the collection of preferences $(\succsim_i)_{i=1}^I$ admits a representative consumer, then
$$z(p, \vec{\omega}') = z(p, \vec{\omega}) \quad \forall, \vec{\omega}, \vec{\omega}' \text{ s.th. } \bar{\omega} = \iota\cdot\vec{\omega} = \iota\cdot\vec{\omega}'.$$

The observation suggests that a representative consumer will only exist under very stringent conditions.

We shall see that *homothetic* preferences is central to the question of when an economy accepts a representative consumer. Recall that a consumer has a homothetic preference relation if $x \succsim_i y$ implies $\alpha x \succsim_i \alpha y$ for all $x, y \in \mathbf{R}^L_+$ and $\alpha \geq 0$.

Theorem 36 (Antonelli's Theorem)**.** *The aggregate excess demand function z admits a representative consumer if and only if there is a homothetic preference relation $\succsim$ on $\mathbf{R}^L_+$ such that each individual demand x^*_i is generated by $\succsim$.*

You may have seen a result on aggregate demand and Gorman forms, and you may be wondering what the relation is to Antonelli's theorem. The Gorman form characterizes demand with linear expansion paths (Engel curves), but the usual result on the Gorman form is local. If you consider demand when income becomes very small, close to 0, then the income expansion paths have to pass through zero. This means that income expansion curves must behave like they do for homothetic preferences.

9.0.1. *Digression: The Cauchy equation.* Consider the following equation $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all $x_1, x_2 \in \mathbf{R}$. This is an equation in which the unknown is the (real) function f: a so-called ***functional equation***. This particular equation is called ***Cauchy's equation***. We want to know if Cauchy's equation has a solution, and what the solution is. It obviously has solutions, for example $f(x) = x$. The following result characterizes all continuous solutions.

Proposition 37. *Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy that $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all $x_1, x_2 \in \mathbf{R}$. Then there is $c \in \mathbf{R}$ (a constant) such that $f(x) = cx$. Note that $c = f(1)$.*

Proof. First, let $n > 1$ be a positive integer and $r \in \mathbf{R}$. Note that

$$f(nr) = f(r+(n-1)r) = f(r)+f((n-1)r) = f(r)+f(r)+f((n-2)r) = \ldots = nf(r).$$

In second place, let $q = n/m \in \mathbf{Q}$ be a rational number, with $n, m \in \mathbf{Z}$ being positive integers. Then $f(n/m) = nf(1/m)$, which means that

$$mf(n/m) = mnf(1/m) = nf(m/m) = nf(1).$$

Hence $f(q) = qf(1)$. The same holds true when $q \leq 0$.

Now, since $f(q) = qf(1)$ for all rational numbers q, and f is continuous, $f(x) = xf(1)$ for all real numbers x. □

9.0.2. *Proof of Antonelli's theorem.*

Proof. ($\Leftarrow$) Let $\succsim$ be a preference on $\mathbf{R}^L_+$ such that it generates every demand function x^i and it is homothetic. Let x^R be the demand generated by this preference relation. Note that $x^R(p,m) = mx^R(p,1)$ for all m as x^R is linear homogeneous in income. Then,

$$\begin{aligned}\sum_{i=1}^{I} x^*_i(p, p\cdot\omega_i) &= \sum_{i=1}^{I} x^R(p, p\cdot\omega_i)\\ &= x^R(p,1)\sum_{i=1}^{I} p\cdot\omega_i\\ &= x^R(p,1)(p\cdot\bar{\omega})\\ &= x^R(p, p\cdot\bar{\omega})\end{aligned}$$

($\Rightarrow$) Let $p \in \mathbf{R}^L_{++}$ and $\omega \in \mathbf{R}^L_+$. Let $\vec{\omega}_i = (0,\ldots,0,\omega,0,\ldots,0)$, with ω in the i-th position. Then

$$\begin{aligned}z(p,\vec{\omega}_i) &= x^R(p, p\cdot\omega) - \omega\\ &= \sum_{j\neq i} x^*_j(p, p\cdot 0) + x^*_i(p, p\cdot\omega) - \omega\\ &= x^*_i(p, p\cdot\omega) - \omega\\ \implies\ x^R(p, p\cdot\omega) &= x^*_i(p, p\cdot\omega)\quad \forall\, i\in I.\end{aligned}$$

As the consumer i, price vector p and endowment ω are arbitrary, $x^R = x^*_i\ \forall i$. Let $\succsim^R$ generate x^R, so that $\succsim^R$ also generates x^*_i. We shall prove that x^R is linear homogeneous in income, i.e., $x^R(p,\lambda\cdot m) = \lambda x^R(p,m)$ $\forall\ \lambda > 0$, $(p,m) \in \mathbf{R}^L_{++}\times\mathbf{R}^L_+$. This is a neccesary and sufficient for homotheticity.

First, let $m = m_1 + m_2$, and choose for $i = 1,2$, $p\cdot\omega_i = m_i$ and $\omega_j = 0$ $\forall j \neq i$. Since we have a representative consumer,

$$\begin{aligned}x^R(p,m) &= x^*_1(p,m_1) + x^*_2(p,m_2) + \sum_{j\geq 3} x^*_j(p, p\cdot 0)\\ &= x^R(p,m_1) + x^R(p,m_2).\end{aligned}$$

In particular, for a positive integer n, $x^R(p, nm) = x^R(p,(n-1)m) + x^R(p, m)$. This means that $x^R(p, nm) = nx^R(p, m)$. Let $q = \frac{k}{l} \in \mathbf{Q}_+$, with $k, l \in \mathbf{N}$. Then

$$\begin{aligned}
&x^R(p, qm) = kx^R\left(p, \frac{m}{l}\right) \\
\Longrightarrow\ & lx^R(p, qm) = kx^R\left(p, \frac{lm}{l}\right) \\
\Longrightarrow\ & x^R(p, qm) = \left(\frac{k}{l}\right) x^R(p, m) \\
\Longrightarrow\ & x^R(p, qm) = qx^R(p, m)
\end{aligned}$$

Finally, since x^R is continuous by hypothesis:

$$x^R(p, \lambda m) = \lambda x^R(p, m) \quad \forall\, \lambda > 0.$$

□

Observation 38. The exercise in the proof is known as the solution to the *Cauchy Equation*. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy that $f(x_1 + x_2) = f(x_1) + f(x_2)$ for any $(x_1, x_2) \in \mathbf{R}^2$. Then $\forall\ x \in \mathbf{R}$, $f(x) = cx$ for some constant c. In our case, let $f(m) = x^R(p, m)$, for a fixed p let $c = x^R(p, 1)$. Therefore, $x^R(p, m) = mx^R(p, 1)$.

9.1. **Samuelsonian Aggregation.** Let $(\succsim)_{i=1}^I$ be a collection of preferences where each $\succsim_i$ is represented by a utility function $u_i : \mathbf{R}_+^L \to \mathbf{R}$.

Let $W : \mathbf{R}^I \to \mathbf{R}$ be a strictly monotone increasing function. We refer to W as a ***social welfare function***.

Consider the following problem:

$$\text{(P1)} \qquad \max_{(x_1,\ldots,x_I)\in\mathbb{R}_+^{IL}} W\left(u_1(x_1), \ldots, u_I(x_I)\right) \quad \text{subject to} \quad p \cdot \sum_{i=1}^I x_i \le m.$$

The problem above implies choosing an aggregate consumption bundle $z = \sum_{i=1}^I x_i \in \mathbf{R}^{IL}$ for the economy. The objective is to maximize the social welfare in the economy, as given by W. The constraint reflects an aggregate budget constraint.

Consider the alternative problem:

(P2)

$$\max_{z\in\mathbb{R}_+^{IL}} \left\{ \max_{(x_1,\ldots,x_I)\in\mathbb{R}_+^{IL}} W\left(u_1(x_1),\ldots,u_I(x_I)\right) \quad \text{s.t.} \quad \sum_{i=1}^{I} x_i = z \right\}$$
$$\text{subject to} \quad p\cdot z \leq m.$$

Therefore, define the value function of (P1) as

$$U(z) = \sup\{W\left(u_1(x_1),\ldots,u_I(x_I)\right) : (x_1,\ldots,x_I) \in \mathbb{R}_+^{IL} \text{ and } \sum_{i=1}^{I} x_i = z\},$$

and rewrite (P2):

(P2')
$$\max_{z\in\mathbb{R}_+^{IL}} U(z) \quad \text{subject to} \quad p\cdot z \leq m.$$

Let $x_i^*(p, m_i)$ be the demand function generated by $\succsim_i$, where m_i is the income of consumer i. Consider the following problem:

(P4)
$$\max_{\{m_i\}_{i=1}^{I}} W\left(u_1(x_1^*(p,m_1)),\ldots,u_I(x_I^*(p,m_I))\right)$$
$$\text{subject to} \quad \sum_{i=1}^{I} m_i \leq m.$$

Interpret these problems. Problem 1 maximizes social welfare function by choosing an individual bundle for each consumer subject to the aggregate budget constraint. This induces an optimal aggregate bundle $z \in \mathbf{R}^L$. The second problem is a maximization in two steps. First, given an arbitrary aggregate bundle of consumption $z \in \mathbf{R}^L$, the social planner maximizes the social welfare function, i.e., decides on the optimal allocation of the aggregate bundle among the consumers. Then, given the income restriction of the economy, the social planner decides the optimal aggregate consumption bundle. Problem P2' makes explicit that we can write the second problem as that of a representative consumer whose utility function is the value function resulting from the first of the two nested problems above. Therefore, the utility function of our representative consumer represents the optimal division of an aggregate consumption bundle among the consumers. Another way of addressing this problem is in terms of income.

9.2. **Eisenberg's Theorem.** Let $(\succsim_i)_{i=1}^I$ be a collection of preferences where each $\succsim_i$ is represented by $u_i : \mathbf{R}_+^L \to \mathbf{R}$. Identify an *economy* with a vector of endowments $\vec{\omega} \in \mathbf{R}_+^{IL}$ (so that for each $\vec{\omega} \in \mathbf{R}_+^{IL}$, $(\succsim_i, \omega_i)_{i=1}^I$. An economy has a ***fixed structure of endowments*** if there is $\alpha = (\alpha_i)_{i=1}^I \in \mathbf{R}_+^I$ with $\sum_i \alpha_i = 1$ and $\omega_i = \alpha_i \bar{\omega}$.

Theorem 39 (Eisenberg's Theorem)**.** *Consider an economy with a fixed structure of endowments, given by α. Let each $\succsim_i$ be represented by a continuous and homogenous degree one utility fuction $u_i i$. Then the aggregate demand of the economy is generated by a representative consumer, whose utility function $U : \mathbf{R}_+^L \to \mathbf{R}$ is given by:*

$$(9) \qquad U(x) = \max_{(x_1,\dots,x_I)\in\mathbb{R}_+^{IL}} \left\{ \prod_{i=1}^{I} \Big(u_i(x_i)\Big)^{\alpha_i} \quad s.t. \quad x = \sum_{i=1}^{I} x_i \right\}.$$

10. Determinacy

Consider a collection of preferences $(\succeq_i)_{i=1}^I$ over $\mathbf{R}_+^L$. For each given vector of endowments:

$$\vec{\omega} = (\omega_1, \dots, \omega_I) \in \mathbf{R}_+^{IL},$$

we have an exchange economy $(\succeq_i, \omega_i)_{i=1}^I$. So for fixed preferences $(\succeq_i)_{i=1}^I$ we can identify a set of possible (exchange) economies with a set of vectors of initial endowments.

Let E be an open subset of $\mathbf{R}_+^{IL}$. Assume that all vectors of endowments $\vec{\omega} \in E$ satisfy $\sum_i \omega_i \gg 0$, and the consumers' preferences $\succeq_i$ are strictly convex, strictly monotone, and continuous. Then each $\succeq_i$ gives rise to a demand function x_i^* and aggregate excess demand is

$$z^*(p; \vec{\omega}) = \sum_{i=1}^{I} x_i^*(p, p \cdot \omega_i) - \sum_{i=1}^{I} \omega_i.$$

The function $p \mapsto z^*(p; \vec{\omega})$ satisfies properties (1)-(5). Note that we now make the dependence of z^* on $\vec{\omega}$ explicit. As a function of p (meaning, for fixed $\vec{\omega}$, z^* satisfies properties (i)-(v).

As an example, consider an economy with two goods. Normalize $p_2 = 1$, and focus on p_1 and z_1. In this case, we would expect the economy to have an excess demand function such as the one depicted below.

10.1. **Digression: Implicit Function Theorem.** REMIND OF THE INVERSE Fn THM using a picture.

Here's a quick calculation to remind you of what the IFT says.

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a $\mathcal{C}^1$ function. Let v be a variable of interest and q a parameter of the model. Suppose that the solution to our model is characterized *implicitly* by $f(v, q) = 0$.

We want to find a function h defined on a neighborhood of a q_0 such that $f(v_0, q_0) = 0$, $v_0 = h(q_0)$ and $f(h(q), q) = 0$ for all q in the neighborhood. Taking derivatives, we find that

$$\frac{\partial f}{\partial v} \cdot \frac{\partial h}{\partial q} + \frac{\partial f}{\partial q} = 0 \implies h'(q) = -\frac{\frac{\partial f}{\partial q}}{\frac{\partial f}{\partial v}}.$$

Note that in order to perform the previous excercise we need to assume $\frac{\partial f}{\partial v} \neq 0$. This is intuitive: as $f(v, q) = 0$ describes our solution, if f does not vary in v, even though f may change as we vary q, this does not tell us anything the behavior of v.

A function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$, for which A is open, is said to be of class r if it has $0 \leq r \leq \infty$ continuous derivatives. We write $\mathcal{C}^r$ to denote the property of being of class r.

Let $g : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^m$ be a $\mathcal{C}^1$ function. The *Jacobian matrix* of g is

$$\begin{aligned} Dg(x, y) = \begin{bmatrix} D_x g(x, y) & D_y g(x, y) \end{bmatrix} &= \begin{bmatrix} \frac{\mathrm{d}g}{\partial x_1} & \cdots & \frac{\mathrm{d}g}{\partial x_n} & \frac{\mathrm{d}g}{\partial y_1} & \cdots & \frac{\mathrm{d}g}{\partial y_k} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} & \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} & \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial y_k} \end{bmatrix}_{m \times (n+k)} \end{aligned}$$

Theorem 40 (Implicit Function Theorem)**.** *Let $A \subseteq \mathbf{R}^n$, and $B \subseteq \mathbf{R}^m$ be open sets. Let $f : A \times B \to \mathbf{R}^n$ be $\mathcal{C}^r$ with $1 \leq r \leq \infty$. Let $(\bar{v}, \bar{q}) \in A \times B$ be such that $f(\bar{v}, \bar{q}) = 0$. Then, if*

$$D_v f(v, q)\Big|_{(v,q)=(\bar{v},\bar{q})}$$

is a non-singular matrix, there are open sets $A' \subseteq A$, $B' \subseteq B$, and a $\mathcal{C}^r$ function $h : B' \to A'$ such that $h(\bar{q}) = \bar{v}$, and $f(v, q) = 0$ for $(v, q) \in A' \times B'$ if and only if $v = h(q)$. Moreover,

$$D_q h(\bar{q}) = -\left[D_v f(v, q)\Big|_{(v,q)=(\bar{v},\bar{q})} \right]^{-1} \cdot D_q f(\bar{v}, \bar{q}).$$

This version of the IFT can be found in Mas-Colell (1989).

10.2. **Regular and Critical Economies.** We develop methods for analyzing parametrized models. To this end, we focus on exchange economies and interpret endowments as parameters. It is, however, possible to use the same ideas for different parametrizations. For example, we could use preferences as the parameters of the model, or firms' technologies in a POE.

Fix a collection of preferences $\succeq_i$ on $\mathbf{R}^L_+$, $1 \leq i \leq I$. For each collection of endowments $\vec{\omega} = (\omega_1, \ldots, \omega_I) \in \mathbf{R}^{IL}_+$ we have an exchange economy $(\succeq_i, \omega_i)_{i=1}^I$. With fixed preferences, we identify an economy with the vector of endowments $\vec{\omega}$, and let $E \subseteq \mathbf{R}^{IL}_{++}$ be a set of economies. Assume that E is open.

Given an economy $\vec{\omega} \in E$, the resulting demand for agent i is $x_i^*(p, p \cdot \omega_i)$. We assume that demand functions, as functions of prices and income $(p, m) \mapsto x_i^*(p, m)$, and with domain $\mathbf{R}^{L+1}_{++}$ are C^1. This assumption relies on a notion of smooth preferences that we are not going to get into.

To make the dependence of the model on its parameters (endowments) explicit, we write the aggregate excess demand as

$$z(p, \vec{\omega}) = \sum_{i=1}^{I} x_i^*(p, p \cdot \omega_i) - \sum_{i=1}^{I} \omega_i.$$

Before we used homogeneity to normalize prices so that they are in Δ^o, but now we will use a different normalization. In particular, homogeneity of degree zero implies that

$$z(p_1, \ldots, p_L) = z\left(\frac{p_1}{p_L}, \ldots, \frac{p_{L-1}}{p_L}, \frac{p_L}{p_L}\right),$$

which means restricting to prices in $\mathbf{R}^{L-1}_{++}$ with the price of the Lth good being fixed at 1. Moreover, by Walras' law we can restrict attention to all market except for one. To this end let $\hat{z} : \mathbf{R}^{L-1}_{++} \times E \to \mathbf{R}^{L-1}$ be defined by:

$$\hat{z}_l(p_1, p_2, \ldots, p_{L-1}; \vec{\omega}) = z_l(p_1, p_2, \ldots, p_{L-1}, 1; \vec{\omega}) \tag{10}$$

for $l = 1, \ldots, L-1$. In particular,

$$\hat{z}(p_1, \ldots, p_{L-1}) = 0 \text{ iff } z(p_1, \ldots, p_{L-1}, 1) = 0.$$

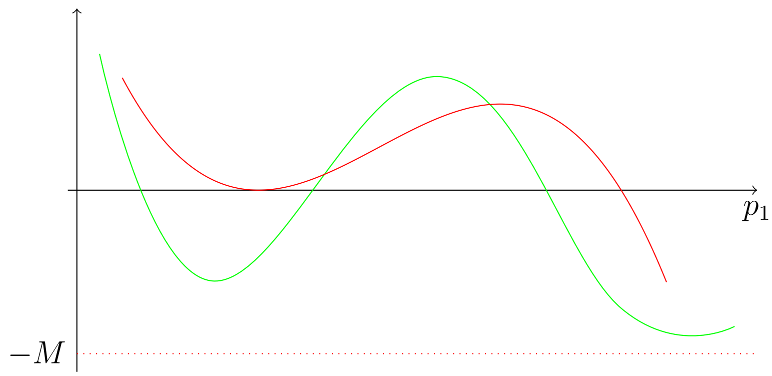

FIGURE 6. Aggregate excess demand $\hat{z}_1(p_1, 1, \vec{\omega})$.

Given an economy $\vec{\omega}$, let $\mathcal{P}(\vec{\omega})$ denote its competitive equilibria. So

$$\mathcal{P}(\vec{\omega}) = \left\{ p \in \mathbf{R}^{L-1}_{++} : \hat{z}(p, \vec{\omega}) = 0 \right\}.$$

Definition 41 (Regular Equilibrium)**.** An equilibrium price $p^* \in \mathcal{P}(\vec{\omega})$ is a ***regular equilibrium*** of $\vec{\omega}$ if the matrix

$$D_p \hat{z}(p; \vec{\omega}) \Big|_{p=p^*}$$

is non-singular.

An economy $\vec{\omega}$ is said to be a ***regular economy*** if all the equilibrium prices in $\mathcal{P}(\vec{\omega})$ are regular. An economy that is not regular is ***critical***.

Recall our initial example of an economy with two goods. In that case $\hat{z} : \mathbf{R}_{++} \times \mathbf{R}^{2I} \to \mathbf{R}$. Therefore, the condition of regularity is equivalent to the derivative of the excess demand function being different from zero at every equilibrium price. Consider the aggregate excess demands in Figure 6. The economy with the green excess demand is regular, while that with the red excess demand is critical.

If there are more than two goods, consider an infinitesimal change in prices $dp = (dp_1, dp_2, \ldots, dp_{L-1})$. The directional derivative of the function $\hat{z}$ at the price vector $p \in \mathbf{R}^L_{++}$ in the direction of dp is given by $D\hat{z}(p) \cdot dp$.[9] If $D\hat{z}(p)$ has full rank at p, then $D\hat{z}(p) \cdot dp \neq 0$ for every $dp \neq 0$. This implies that the normalized excess demand function

[9]Let $f : \mathbf{R}^n \to \mathbf{R}$ be a differentiable function with gradient vector $\nabla f(x)$. The *directional derivative* of f in the direction of vector $u \in \mathbf{R}^n$ is given by $Df(x; u) = \nabla f(x) \cdot u$. If $f : \mathbf{R}^n \to \mathbf{R}^m$, then $Df(x; u) = [\nabla f^1(x), \ldots, \nabla f^m(x)]'u = [\nabla f^1(x) \cdot u, \ldots, \nabla f^m(x) \cdot u]' = [Df^1(x; u), \ldots, Df^m(x; u)]$.

$\hat{z}$ changes for infinitesimal changes in prices, i.e., we stop being at equilibrium. This is the intuition for our next concept and result.

Definition 42 (Local Uniqueness)**.** An equilibrium $p^* \in \mathcal{P}(\vec{\omega})$ is ***locally unique*** if there exists $\varepsilon > 0$ such that, $\forall\ p \in \mathbf{R}^{L-1}_{++}$,

$$\|p - p^*\| < \varepsilon \implies p \notin \mathcal{P}(\vec{\omega}).$$

Proposition 43. *Let $\hat{z}$ be C^1 and $p^* \in \mathcal{P}(\vec{\omega})$ be a regular equilibrium. Then, p^* is locally unique. Furthermore, there are neighborhoods B_1 of $\vec{\omega}$ in E, and B_2 of p^* in $\mathbf{R}^{L-1}_{++}$, and a function $h : B_1 \to B_2$ such that*

$$\hat{z}(h(\vec{\omega}), \vec{\omega}) = 0 \quad \forall\, \vec{\omega} \in B_1,$$

and

$$D_{\vec{\omega}} h(\vec{\omega}) = -\left[D_p \hat{z}(p, \vec{\omega})\Big|_{p=p^*} \right]^{-1} \cdot D_{\vec{\omega}} \hat{z}(p^*, \vec{\omega}).$$

Proof. Inmediate from the Implicit Function Thm □

Proposition 44. *A regular economy has a finite number of equilibria.*

Proof. Consider prices as subsets of the simplex, Δ, a re-normalization. Local uniqueness is preserved under the re-normalization.

Note that $\mathcal{P}(\vec{\omega}) = z^{-1}(0; \vec{\omega})$. Since z is a continuous function, $\mathcal{P}(\vec{\omega})$ is a closed set. Moreover $\mathcal{P}(\vec{\omega})$ is compact because it is a closed subset of a compact set. For each price $p \in \mathcal{P}(\vec{\omega})$, let N_p be an open ball with center p and for which p is the only equilibrium price in N_p. Such balls exist because equilibrium prices are locally unique. Now

$$\mathcal{P}(\vec{\omega}) \subseteq \cup_{p \in \mathcal{P}(\vec{\omega})} N_p,$$

an open cover. So $\mathcal{P}(\vec{\omega})$ has a finite subcover, and therefore is unique.

Another proof is as follows: Suppose towards a contradiction that $\mathcal{P}(\vec{\omega})$ is infinite. Let p_n be a sequence of distinct prices in $\mathcal{P}(\vec{\omega})$. p_n is in Δ so it has a convergent subsequence that we, in an abuse of notation, also denote by p_n. If $p^* = \lim_{n \to \infty} p_n$ then $p^* \in \mathcal{P}(\vec{\omega})$ by continuity of aggregate excess demand. But then p^* is not locally isolated; a contradiction. □

Definition 45 (Index)**.** The ***index*** of $p \in \mathcal{P}(\vec{\omega})$ is defined as:

$$\text{index}(p) = (-1)^{L-1} \cdot \text{sign}\left(\left| D_p \hat{z}(p, \vec{\omega}) \right| \right),$$

where $\left| D_p \hat{z}(p, \vec{\omega}) \right|$ is the determinant of the matrix $D_p \hat{z}(p, \vec{\omega})$.

Note that for every regular economy $\vec{\omega}$, index$(p) \in \{-1, 1\}$ $\forall$ $p \in \mathcal{P}(\vec{\omega})$. We state without proof the following theorem.

Theorem 46 (Index Theorem)**.** *If $\vec{\omega}$ is a regular economy, then*

$$\sum_{p \in \mathcal{P}(\vec{\omega})} \text{index}(p) = 1.$$

The index theorem can be used to establish uniqueness: if you can show that any competitive equilibrium in an economy has index one, then there can only be one equilibrium. Finally, the index theorem implies the following curiosity.

Corollary 47. *A regular economy has an odd number of equilibria.*

Consider our initial example of an economy with two goods. See in the graph below the index of each equilibrium in the economy, and verify that they sum up to one.

10.3. **Digression: Measure Zero Sets and Transversality.**

Definition 48 (Rectangle and Volume)**.** A ***rectangle*** in $\mathbf{R}^n$ is a set of the form

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = \{x \in \mathbf{R}^n : a_i \leq x_i \leq b_i, i = 1, \dots, n\},$$

where $[a_i, b_i] = \{x \in \mathbf{R} : a_i \leq x \leq b_i\}$ is an interval in $\mathbf{R}$. The ***volume*** of a rectangle $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is

$$\text{vol}(R) = \prod_{i=1}^{n} (b_i - a_i).$$

Definition 49 (Measure Zero)**.** A set $A \subseteq \mathbf{R}^n$ is measure zero if $\forall$ $\varepsilon > 0$ there exist a collection of rectangles $R_1, R_2, \dots$ such that

$$A \subseteq \bigcup_{i=1}^{\infty} R_i, \quad \text{and} \quad \sum_{i=1}^{\infty} \text{vol}(R_i) < \varepsilon.$$

Consider the following two examples of measure zero sets in $\mathbf{R}^2$: a finite set and a straight line.

Under the same reasoning, note that a line segment is not measure zero in $\mathbf{R}$ (because it is an interval), but it is in $\mathbf{R}^n$ $\forall$ $n > 2$. Similarly, a plane is not measure zero in $\mathbf{R}^2$, but it is in $\mathbf{R}^m$ $\forall$ $m > 2$.

Observation 50. No open set has measure zero.

Consider a parametrized familiy of equations $f : \mathbf{R}^{N+K} \to \mathbf{R}^N$. Define a *system of equations* as $f(v, q) = 0$, where $v \in \mathbf{R}^N$ is a vector of unknowns, and $q \in \mathbf{R}^K$ is a vector of parameters.

Theorem 51 (Transversality Theorem)**.** *Let $f : \mathbf{R}^{N+K} \to \mathbf{R}^N$ be $\mathcal{C}^1$. If the $N \times (N+K)$ Jacobian matrix $Df(v, q)$ has full rank (rank N) at each (v, q) with $f(v, q) = 0$, then there is a measure zero set $C \subseteq \mathbf{R}^K$ such that, $\forall\ q \in \mathbf{R}^K \setminus C$, the $N \times N$ matrix $D_v f(v, q)$ is non-singular whenever $f(v, q) = 0$.*

The rough idea behind the theorem is that when $Df(v, q)$ is full rank, the function f can change locally in any direction. This means that when we are at a critical solution, a change in parameters can "perturb away" the critical point.

10.4. **Genericity of regular economies.** Return to the assumptions we made earlier. Suppose that $\hat{z} : \mathbf{R}^{L-1}_{++} \times E \to \mathbf{R}^{L-1}$ is C^1 and satisfies properties (i)-(v).

Theorem 52. *There is a measure zero set $C \subseteq \mathbf{R}^{IL}_{+}$ such that, if $\vec{\omega} \in E \setminus C$, then $\vec{\omega}$ is a regular economy.*

Proof. The result follows from the Transversality Theorem.

The matrix $D\hat{z}(p; \vec{\omega})$ is $(L-1) \times (L-1+IL)$. Write this matrix as

$$D\hat{z}(p; \vec{\omega}) = \begin{bmatrix} D_p \hat{z}(p; \vec{\omega}) & D_{\omega_1} \hat{z}(p; \vec{\omega}) & \cdots & D_{\omega_I} \hat{z}(p; \vec{\omega}) \end{bmatrix},$$

where $D_{\omega_i} \hat{z}(p; \vec{\omega})$ is the $(L-1) \times L$ Jacobian matrix with respect to the endowment vector of consumer i. We shall prove that any one of the matrices $D_{\omega_i} \hat{z}(p; \vec{\omega})$ has full rank. If we show that any $D_{\omega_i} \hat{z}(p; \vec{\omega})$ has full rank, then we have that $D\hat{z}(p; \vec{\omega})$ has $L-1$ linearly independent columns and thus it is also full rank.

So lets calculate $D_{\omega_1} \hat{z}(p; \vec{\omega})$, and show that it has full rank. The matrix $D_{\omega_1} \hat{z}(p; \vec{\omega})$ may be written as

$$\begin{aligned} D_{\omega_1} \hat{z}(p; \vec{\omega}) &= \begin{bmatrix} D_{\omega_{11}} \hat{z}(p; \vec{\omega}) & D_{\omega_{21}} \hat{z}(p; \vec{\omega}) & \cdots & D_{\omega_{L1}} \hat{z}(p; \vec{\omega}) \end{bmatrix} \\ &= \begin{bmatrix} \frac{d\hat{z}(p; \vec{\omega})}{\partial \omega_{11}} & \frac{d\hat{z}(p; \vec{\omega})}{\partial \omega_{21}} & \cdots & \frac{d\hat{z}(p; \vec{\omega})}{\partial \omega_{L1}} \end{bmatrix} \end{aligned}$$

Recall that

$$\hat{z}_l(p; \vec{\omega}) = \sum_{i=1}^{I} x_i^*(p, p \cdot \omega_i) - \bar{\omega}_l \quad \forall\, l = 1, \ldots, L_1.$$

Then, compute for every $l = 1, \dots, L-1$,

$$\frac{\partial \hat{z}_l(p;\vec{\omega})}{\partial \omega_{h1}} = \begin{cases} \frac{\partial x^*_{h1}(p, p\cdot\omega_1)}{\partial m_1} p_h - 1 & \text{if } h = l \\ \\ \frac{\partial x^*_{h1}(p, p\cdot\omega_1)}{\partial m_1} p_h & \text{if } h \neq l \end{cases}$$

For notational purposes, we let m_1 denote the income of consumer 1. Therefore we may write the complete matrix $D_{\omega_1}\hat{z}(p;\vec{\omega})$ as

$$D_{\omega_1}\hat{z}(p;\vec{\omega}) = \begin{bmatrix} \frac{\partial x^*_{11}}{\partial m_1} p_1 - 1 & \frac{\partial x^*_{11}}{\partial m_1} p_2 & \cdots & \frac{\partial x^*_{11}}{\partial m_1} p_{L-1} & \frac{\partial x^*_{11}}{\partial m_1} \\ \frac{\partial x^*_{21}}{\partial m_1} p_1 & \frac{\partial x^*_{21}}{\partial m_1} p_2 - 1 & \cdots & \frac{\partial x^*_{21}}{\partial m_1} p_{L-1} & \frac{\partial x^*_{21}}{\partial m_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial x^*_{L-1,1}}{\partial m_1} p_1 & \frac{\partial x^*_{L-1,1}}{\partial m_1} p_2 & \cdots & \frac{\partial x^*_{L-1,1}}{\partial m_1} p_{L-1} - 1 & \frac{\partial x^*_{L-1,1}}{\partial m_1} \end{bmatrix}_{(L-1)\times L}$$

See that the L-th column of the previous matrix takes into account the fact that we have normalized $p_L = 1$. Take the L-th column and multiply it by p_l to subtract it from the l-th column. This operation does not change the rank of the matrix. Repeating the operation, we obtain:

$$\begin{bmatrix} -1 & 0 & \cdots & 0 & \frac{\partial x^*_{11}}{\partial m_1} \\ 0 & -1 & \cdots & 0 & \frac{\partial x^*_{21}}{\partial m_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \frac{\partial x^*_{L-1,1}}{\partial m_1} \end{bmatrix}_{(L-1)\times L}$$

The first $L-1$ columns of the previous matrix are linearly independent, implying that $D_{\omega_1}\hat{z}(p;\vec{\omega})$ has full rank. Therefore, the Jacobian matrix $D\hat{z}(p;\vec{\omega})$ has also full rank. Transversality implies that there exists a measure zero set $C \in \mathbf{R}^{IL}$ such that, if $\vec{\omega} \in \mathbf{R}^{IL} \setminus C$, then $D_p\hat{z}(p;\vec{\omega})$ has full rank for every $p \in E(\vec{\omega})$. In other words, every economy $\vec{\omega} \in \mathbf{R}^{IL} \setminus C$ is regular, where C is a measure zero set. □

11. Observable Consequences of Competitive Equilibrium

11.1. **Digression on Afriat's Theorem.** A ***data set*** as a collection $(x^k, p^k)_{k=1}^K$ where $x^k \in \mathbf{R}^L_+$ and $p^k \in \mathbf{R}^L_{++}$ for every $k = 1, \dots, K$.

Definition 53. A data set $(x^k, p^k)_{k=1}^K$ is rationalizable by $u : \mathbf{R}^L_+ \to \mathbf{R}$ if, $\forall\, y \in \mathbf{R}^L_{++}$,

$$p^k \cdot y \leq p^k \cdot x^k \implies u(y) \leq u(x^k) \quad \forall\, k = 1, \dots, K.$$

Definition 54 (Revealed Preference)**.** Given a data set $(x^k, p^k)_{k=1}^K$, define a binary relation $\succsim^R$ on $\mathbf{R}_+^L$ as:

$$x \succsim^R y \iff \exists\, k \text{ s.th. } x = x^k \text{ and } p^k \cdot x^k \geq p^k \cdot y,$$
$$\textit{and } x \succ^R y \iff \exists\, k \text{ s.th. } x = x^k \text{ and } p^k \cdot x^k > p^k \cdot y.$$

We refer to the binary relation $\succsim^R$ as the revealed preference relation.

Definition 55 (Weak Axiom of Revealed Preference)**.** A data set satisifes the Weak Axiom of Revealed Preference (WARP) if there is no k and l such that $x^k \succsim^R x^l$ and $x^l \succ^R x^k$.

Definition 56. A data set satisfies the Generalized Axiom of Revealed Preference (GARP) if there is no sequence $x^{k_1}, x^{k_2}, ..., x^{k_n}$ such that

$$x^{k_1} \succsim^R x^{k_2} \succsim^R \cdots \succsim^R x^{k_n}, \textit{ and } x^{k_n} \succ^R x^{k_1}.$$

Observation 57. If $L \leq 2$, WARP and GARP are equivalent. If $L \geq 3$, then GARP $\implies$ WARP, but not conversely.

Theorem 58 (Afriat's Theorem)**.** *Consider a data set* $(x^k, p^k)_{k=1}^K$. *The following statements are equivalent:*

(1) *the data set is rationalizable by a locally nosatitated utility function;*
(2) *the data set satisfies GARP;*
(3) *there are numbers* $U^k, \lambda^k > 0$ *for* $k = 1, \ldots, K$, *such that*

$$U^k \leq U^l + \lambda^l p^l \cdot (x^k - x^l);$$

(4) *the data set is rationalizable by a strictly monotonic and concave utility function.*

See Chambers and Echenique (2016) for a proof and detailed discussion of Afriat's theorem.

11.2. **Sonnenschein-Mantel-Debreu Theorem: *Anything goes*.** In this section we consider a set of important results regarding what can be obtained as an equilibrium for a well-behaved economy. The answer is that, in a sense, anything can happen. The Sonnenschein-Mantel-Debreu Theorem says that, off the boundaries of the simplex, any function that satisfies the basic properties of an excess demand function can be obtained as the excess demand of a well-behaved exchange economy. Note that properties (iv) and (v) are about the boundary behavior. Homogeneity will be subsumed in the use of Δ^o as the domain of f.

Theorem 59 (Sonnenschein-Mantel-Debreu Theorem)**.** *Let $f : \Delta^o \to \mathbf{R}^L$ be a continuous function that satisfies Walras' Law. For any $\varepsilon > 0$ there exists an exchange economy $(\succsim_i, \omega_i)_{i=1}^I$ with $I = L$ and each $\succsim_i$ strictly monotonic, continuous and strictly convex such that the aggregate excess demand function of this economy coincides with f on Δ^ε.*

The proof of the Sonnenschein-Mantel-Debreu Theorem is complicated, but here are some of the basic ideas.[10] The proof "decomposes" f into individual aggregate excess demand functions that satisfy WARP. For each individual consumer i, we let $\omega^i = e_i$, the ith unit vector (recall that $I = L$). So agent i is endowed with one unit of good i. Then let $g_i(p)$ be the projection of e_i onto the orthogonal complement of p. You can think of $g_i(p)$ as the residuals of a least squares regression of $\omega_i = e_i$ on the line spanned by p. Then $g_i(p)$ will satisfy Walras law, as it was chosen to live in the orthogonal complement to p. It is also continuous, and satisfies WARP because it is the outcome of an optimization problem. Moreover, if $h_i(p) > 0$ is a scalar, then the function $p \cdot h_i(p) g_i(p)$ is also continuous and satisfies WARP and Walras Law.

Theorem 60 (Mantel)**.** *Let $f : \Delta^o \to \mathbf{R}^L$ be $\mathcal{C}^2$ and satisfy Walras' Law. Then, $\forall\ \varepsilon > 0$, there exists an exchange economy $(\succsim_i, \omega_i)_{i=1}^I$ with each $\succsim_i$ homothetic, monotonic and strictly convex such that its aggregate excess demand function coincides with f on Δ^ε.*

Corollary 61. *For any compact set $K \subseteq \Delta^o$, there exists an exchange economy with $I = L$ and each $\succsim_i$ strictly monotonic, continuous, and strictly convex such that K is the set of equilibrium prices of the economy.*

11.3. **Brown and Matzkin: Testable Restrictions On Competitve Equilibrium.** If we imagine that we can observe more than just prices, so that we can see how equilibrium varies with endowments (or, observe outcomes "on the equilibrium manifold"), then we can avoid the negative conclusion in the SMT theorem. These ideas are due to Brown and Matzkin (1996).

Recall that the equilibrium manifold is defined by $\{(p, \vec{\omega}) : z(p; \vec{\omega}) = 0\}$ for an economy with exchange demand function z. In words, the equilibrium manifold for a fixes set of preferences $(\succeq_i)_{i=1}^I$ is the set

[10] See Shafer and Sonnenschein (1982) or Chambers and Echenique (2016) for a sketch of the main ideas behind the proof.

of price and endowment combinations for which markets clear in the resulting exchange economy.

Suppose we have the following two observations for an economy with two agents and two goods: (p^1, ω^1) and (p^2, ω^2), i.e., we have two observations of the equilibrium manifold.

GRAPH OF EQUILIBRIUM MANIFOLD

Each endowment vector ω^1 and ω^2 induces an Edgeworth Box economy. For illustrative purposes, suppose that each Edgewroth Box is given by:

GRAPH OF TWO SEPARATE EDGEWROTH BOXES

Superimpose both Edgeworth Boxes by aligning the origin of consumer 1's consumption set, and draw the equilibrium price vectors as follows:

GRAPH OF TWO EDGEWROTH BOXES SUPERIMPOSED

As the economy is in equilibrium under the two observations, then it must be that both individuals are maximizing their respective utilities. However, note that this two choices violate WARP for consumer 1. There are no preferences $\succsim_1$ and $\succsim_2$ for which there exist allocations (x_1, x_2) and (x_1', x_2') such that $((x_1, x_2), p^1)$ is a Walrasian Equilibrium of the economy $((\succsim_1, \omega_1^1), (\succsim_2, \omega_2^1))$ and $((x_1', x_2'), p^2)$ is a Walrasian Equilibrium of the economy $((\succsim_1, \omega_1^2), (\succsim_2, \omega_2^2))$. Therefore, this pair of observations is not consistent with the notion of competitive equilibrium.

12. The Core

Let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy.

- An *allocation*$_{\leq}$ in $\mathcal{E}$ is a vector $x = (x_i)_{i=1}^I \in \mathbf{R}_+^{IL}$, such that $\sum_{i=1}^I x_i \leq \sum_{i=1}^I \omega_i = \bar{\omega}$.
- An *allocation*$_{=}$ in $\mathcal{E}$ is a vector $x = (x_i)_{i=1}^I \in \mathbf{R}_+^{IL}$, such that $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i = \bar{\omega}$.
- A nonempty subset $S \subseteq \{1, \ldots, I\}$ of agents is called a *coalition*.
- Let S be a coalition. A vector $(y_i)_{i\in S}$ is an S-*allocation*$_{\leq}$ if $\sum_{i\in S} y_i \leq \sum_{i\in S} \omega_i$.
- Let S be a coalition. A vector $(y_i)_{i\in S}$ is an S-*allocation*$_{=}$ if $\sum_{i\in S} y_i = \sum_{i\in S} \omega_i$.

Definition 62. A coalition S ***blocks*** the allocation$_\leq x$ in $\mathcal{E}$ if there exists an S-allocation$_\leq (y_i)_{i \in S}$ such that $y_i \succ_i x_i \; \forall \; i \in S$.

- An allocation$_\leq$is ***weakly Pareto optimal*** if it not blocked by the coalition I of all consumers,
- ***individually rational*** if no coalition consisting of a single consumer blocks it,
- and a ***core allocation*** if there is no coalition that blocks it.

Let $C(\mathcal{E})$ be the set of core allocation$_\leq$of $\mathcal{E}$. We refer to $C(\mathcal{E})$ as the ***core*** of the economy $\mathcal{E}$. Let $P(\mathcal{E})$ be the set of *Pareto Optimal* allocation$_\leq$of the economy $\mathcal{E}$, and let $W(\mathcal{E})$ be the set of ***Walrasian Equilibrium allocation***$_\leq$.

Note that $C(\mathcal{E})$, $W(\mathcal{E})$, and $P(\mathcal{E})$ are subsets of $\mathbf{R}_+^{IL}$.

Definition 63. A coalition S ***weakly blocks*** the allocation x if there exists an S-allocation$_\leq (y_i)_{i \in S}$ such that $y_i \succeq_i x_i \; \forall \; i \in S$, and $y_j \succ_j x_j$ for some $j \in S$.

Observation 64. If each preference relation is continuous and strictly monotonic, then a coalition blocks an allocation if and only if it weakly blocks it.

Proof. ($\Rightarrow$) If a coalition blocks an allocation, then it weakly blocks it by the definition of strict preference.

($\Leftarrow$) Let x be an allocation and S a coalition that weakly blocks it. Then, there exists an S-allocation $(y_i)_{i \in S}$ such that $y_i \succsim_i x_i \; \forall \; i \in S$, and $y_j \succ_j x_j$ for some $j \in S$. By continuity of $\succsim_j$, there exists $\delta \in (0,1)$ such that $(1-\delta)y_j \succ_j x_j$. By strict monotonicity of each $\succsim_i$, $\frac{\delta y_j}{|S|-1} + y_i \succ_i y_i$ $\forall \; i \in S \setminus \{j\}$. Let $z_j = (1-\delta)y_j$ and $z_i = \frac{\delta y_j}{|S|-1} + y_i \; \forall \; i \in S \setminus \{j\}$. $z = (z_i)_{i \in S}$ is an S-allocation since

$$\sum_{i \in S} z_i = (1-\delta)y_j + \sum_{i \in S \setminus \{j\}} \frac{\delta y_j}{|S|-1} + y_i = \sum_{i \in S} y_i = \sum_{i \in S} \omega_i.$$

By transitivity, $z_i \succ_i x_i$ for every $i \in S$, so S blocks the allocation x. $\square$

For the remainder of our treatment of the core, we will work with preferences that are continuous and strictly monotonic. So there will be no difference between weak blocks and blocks. We will also refer to allocations, and not distinguish between allocation$_{\leq}$and allocation$_{=}$. (The only relevant allocations will be allocation$_{=}$.)

12.1. **Pareto Optimality, The Core and Walrasian Equiilbria.**

Observation 65. If each preference relation is continuous and strictly monotonic, then all core allocations are Pareto Optimal, i.e., $C(\mathcal{E}) \subseteq P(\mathcal{E})$.

- An allocation x of $\mathcal{E}$ is *individually rational* if $x_i \succsim_i \omega_i \ \forall \ i = 1, \ldots, I$.

Observation 66. If $x \in C(\mathcal{E})$, then x is individually rational.

Example 67. *Let $I = 2$. Given continuous and strictly increasing preferences, the core is the intersection of the set of Pareto optimal allocations and the set of individually rational allocations. We have already shown that every core allocation is both Pareto Optimal and individually rational.*

The following is a sort of strong version of the first welfare theorem, except that the notion of blocking we're using now precludes the need for local non-satiation.

Theorem 68. *Every Walrasian Equilibrium allocation is a core allocation, i.e., $W(\mathcal{E}) \subseteq C(\mathcal{E})$.*

Proof. Let $(x^*, p^*) \in \mathbf{R}^{IL}_{+} \times \mathbf{R}^{L}_{++}$ be a Walrasian Equilibrium. Towards contradiction, suppose $x^* \notin C(\mathcal{E})$. Then, there exists a coalition S and an S-allocation $(y_s)_{s\in S}$ such that $\sum_{s\in S} y_s = \sum_{s\in S} \omega_s$, and, $\forall s \in S$, $y_s \succ_s x^*_s$. Since x^* is an equilibrium allocation, the latter implies $y_s \notin \mathbf{B}(p^*, p^* \cdot \omega_s)$ for every $s \in S$, i.e., $p^* \cdot y_s > p^* \cdot \omega_s$. Summing over s and pricing both resulting bundles, we obtain a contradiction:

$$p^* \cdot \left(\sum_{s\in S} y_s \right) > p^* \cdot \left(\sum_{s\in S} \omega_s \right).$$

Thus, we conclude $W(\mathcal{E}) \subseteq C(\mathcal{E})$. □

12.2. **Debreu-Scarf Core Convergence Theorem.** We next discuss the validity of the competitive hypothesis: the idea that agents act as price-takers. We start from bargaining outcomes, in which agents will typically posses market power. The bargaining outcome is modeled through the game theoretic notion of the core, which says that no gains from multilateral exchange are left unexploited. Note that we do not model the detailed institutional framework for trade (the extensive-form game, if we were to adopt a non-cooperative paradigm). Instead we are agnostic about how trade takes place, and impose that whatever the outcome is, it has to exhaust all the gains from trade. So it has to be in the core.

The second, crucial, idea is to ensure that no agent is special. We replicate an economy, so that there will be many copies of the same agent. Each agent will have a large number of identical "twins." This will reduce the bargaining power of each individual agent. In the limit all the core outcomes must be Walrasian allocations.

Definition 69 (Replica Economy)**.** Let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy. and $N \geq 1$ an integer. The ***N*-th replica** of $\mathcal{E}$ is the exchange economy

$$\mathcal{E}^N = (\succsim_{i,n}, \omega_{i,n})_{i=1,\ldots,I,n=1,\ldots,N}$$

in which agents are indexed by (i,n) with $i = 1,\ldots,I$, $n = 1,\ldots,N$, and satisfy that:

$$\forall\, n = 1,\ldots,N, \quad \succsim_i = \succsim_{i,n} \quad \text{and} \quad \omega_i = \omega_{i,n}.$$

Note that the replica $\mathcal{E}^N$ has IN agents.

Definition 70 (Equal Treatment Property)**.** An allocation

$$x = (x_{i,n})_{i=1,\ldots,I,n=1,\ldots,N}$$

of $\mathcal{E}^N$ has the equal treatment property if $x_{i,n} = x_{i,m}$ for every $n, m = 1,\ldots,N$ and $i = 1,\ldots,I$.

Lemma 71. *Let $\succsim_i$ be strictly monotonic, continuous and strictly convex for every $i = 1,\ldots,I$. Every allocation in $C(\mathcal{E}^N)$ has the equal treatment property.*

Proof. The result is trivial for $N = 1$. Let $N \geq 2$, and consider the contrapositive statement. Let x be an allocation of $\mathcal{E}^N$ that does not have the equal treatment property. For each $i = 1,\ldots,I$, let $n(i)$ be the agent $(i, n(i))$ such that

$$x_{i,n} \succsim_i x_{i,n(i)} \quad \forall\, n = 1,\ldots,N.$$

Hence, $n(i)$ is the (weakly) worst off agent among all the replicas of i. Let $S = \{n(1), n(2), \ldots, n(I)\}$ be a coalition of all such agents. We shall prove that coalition S weakly blocks the allocation x. This is sufficient for $x \notin C(\mathcal{E}^N)$ by continuity and strict monotonicity.

Define the bundle $y_{i,n(i)}$ by:

$$y_{i,n(i)} = \frac{1}{N}\sum_{n=1}^{N} x_{i,n} \quad \forall\, i.$$

As $x_{i,n} \succsim_i x_{i,n(i)}$ $\forall$ $n = 1,...,N$, then $y_{i,n(i)} \succsim_i x_{i,n(i)}$ $\forall$ i by (strict) convexity. As x does not satisfy the equal treatment property, there is at least one $i \in \{1,\dots,I\}$ and one $m \in \{1,\dots,N\}$ such that $x_{i,m} \neq x_{i,n(i)}$. Therefore, for such i, by strict convexity we have

$$y_{i,n(i)} \succ_i x_{i,n(i)}.$$ [11]

Finally, let us prove that $(y_{i,n(i)})_{i=1}^{I}$ is an S-allocation.

$$\begin{aligned}
\sum_{i=1}^{I} y_{i,n(i)} &= \sum_{i=1}^{I} \frac{1}{N}\sum_{n=1}^{N} x_{i,n} \\
&= \frac{1}{N}\sum_{i=1}^{I}\sum_{n=1}^{N} x_{i,n} \\
&= \frac{1}{N}\sum_{i=1}^{I}\sum_{n=1}^{N} \omega_{i,n} \quad \text{since } x \text{ is an allocation of } \mathcal{E}^N \\
&= \frac{1}{N}\sum_{i=1}^{I} N\omega_i \quad \text{since } \omega_{i,n} = \omega_i \text{ for every } n \text{ and } i \\
&= \sum_{i=1}^{I} \omega_i = \sum_{i=1}^{I} \omega_{i,n(i)}.
\end{aligned}$$

Therefore, we have proved that $(y_{i,n(i)})_{i=1}^{I}$ is an S-allocation that (weakly) blocks x, so $x \notin C(\mathcal{E}^N)$. □

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[11] If $N = 2$ the result is straightforward. Let $N \geq 3$, and note that $x_{i,m} \succsim_i x_{i,n(i)}$ and $x_{i,m} \neq x_{i,n(i)}$) imply $\frac{1}{2}x_{i,m} + \frac{1}{2}x_{i,n(i)} \succ_i x_{i,n(i)}$ by strict convexity. Furthermore, as $x_{i,n} \succsim_i x_{i,n(i)}$ for every $n = 1,...,N$, then by convexity:

$$\frac{1}{N-2}\sum_{n\notin\{m,n(i)\}} \succsim_i x_{i,n(i)}.$$

Therefore, see that $2/N \in (0,1)$ and note that by strict convexity we obtain:

$$\frac{2}{N}\left(\frac{1}{2}x_{i,m} + \frac{1}{2}x_{i,n(i)}\right) + \left(1 - \frac{2}{N}\right)\frac{1}{N-2}\sum_{n\notin\{m,n(i)\}} = y_{i,n(i)} \succ_i x_{i,n(i)}.$$

By the previous lemma, we can represent the core allocations of $\mathcal{E}^N$ as vectors in $\mathbf{R}^{IL}_+$. So we shall write $C(\mathcal{E}^N)$ as a subset of $\mathbf{R}^{IL}_+$. The same is true of Walrasian allocations.

Observation 72. Let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy in which preferences $\succeq_i$ are continuous, strictly monotonic and strictly convex.

(i) The core of a replica economy decreases with the number of replicas:

$$\forall\, N, \quad C(\mathcal{E}^N) \supseteq C(\mathcal{E}^{N+1}) \supseteq C(\mathcal{E}^{N+2}) \supseteq \cdots$$

(ii) The equilibrium allocations of $\mathcal{E}^N$ may be represented as allocations in $\mathcal{E}$, i.e., the elements in $W(\mathcal{E}^N)$ can be represented in $\mathbf{R}^{IL}_+$.

(iii) $W(\mathcal{E}^N) = W(\mathcal{E})$ for every N.

(iv) An equilibrium allocation of $\mathcal{E}$ is in the core of every replica economy $\mathcal{E}^N$:

$$W(\mathcal{E}) \subseteq \bigcap_{N=1}^{\infty} C(\mathcal{E}^N).$$

Statement *(iii)* needs some explanation. We know that $W(\mathcal{E}^N)$ are in the core, and therefore satisfy the equal treatment property. So we can represent $W(\mathcal{E}^N)$ as a subset of $\mathbf{R}^{IL}$ for all N. Now, in a Walrasian equilibrium (x, p) of $\mathcal{E}^N$, any two agents of the same type face the same budget, and have the same preferences. So a consumption bundle is optimal for each one of them iff it is optimal for the original prototype in $\mathcal{E}$. By the equal treatment property, the supply = demand property holds for $\mathcal{E}^N$ iff it holds in $\mathcal{E}$.

The following result is due to Debreu and Scarf (1963).[12]

Theorem 73 (Debreu-Scarf Core Convergence Theorem)**.** *Let $\mathcal{E} = (\succsim_i, \omega_i)_{i=1}^I$ be an exchange economy. Suppose that, for all $i = 1, \ldots, I$, $\succeq_i$ is continuous, strictly monotonic and strictly convex, and that $\omega_i > 0$. Suppose also that $\bar{\omega} \gg 0$.*

Then $x^* \in C(\mathcal{E}^N)\ \forall\ N \geq 1$ implies that $x^* \in W(\mathcal{E})$. In other words,

$$W(\mathcal{E}) = \bigcap_{N=1}^{\infty} C(\mathcal{E}^N).$$

[12] The proof here follows Kim Border's notes on Debreu-Scarf.

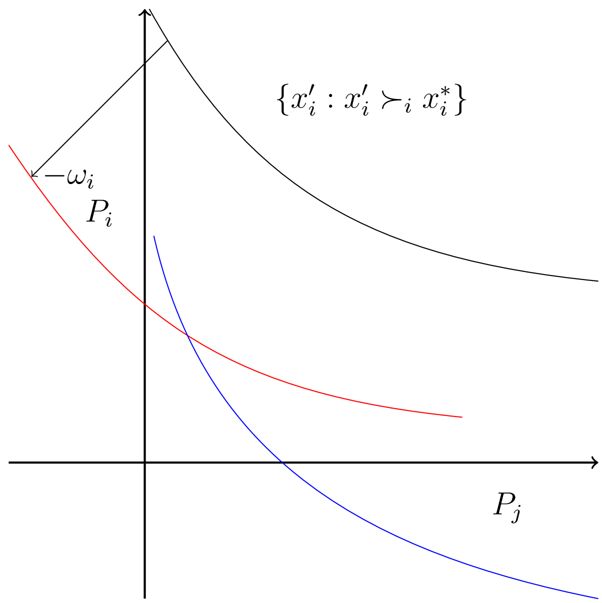

FIGURE 7. Proof of Debreu-Scarf Theorem

Proof. Let $x^* \in \cap_{N=1}^{\infty} C(\mathcal{E}^N)$.

Define

$$P_i = \{z_i \in \mathbf{R}^L : z_i + \omega_i \succ_i x_i^*\},$$

and let P be the convex hull of $\cup_{i=1}^{I} P_i$.[13]

Note that individual rationality of x^* rules out that $0 \in \cup P_i$. By imposing the core property we get much more than individual rationality, and will be able to rule out that $0 \in P$. For this it is important that we work with arbitrary replicas of $\mathcal{E}$ as we need to represent convex combinations with rational coefficients as coalitions in some (arbitrarily large) economy.

We shall prove that $0 \notin P$. Suppose then towards a contradiction that $0 \in P$. P is an open set, so there is a neighborhood V of 0 with $V \subseteq P$. Then there is a neighborhood V' in V contained in $\mathbf{R}^L_{--} = \{-z : z \in \mathbf{R}^L_{++}\}$. See Figure 8. Now, since $V' \subseteq P$ there is a collection

$$z_1, \ldots, z_I$$

[13] Note the difference with the approach in the second welfare theorem. We work with $\cup_{i=1}^{I} P_i$ instead of $\sum_{i=1}^{I} P_i$, and $\cup_{i=1}^{I} P_i$ may not be a convex set.

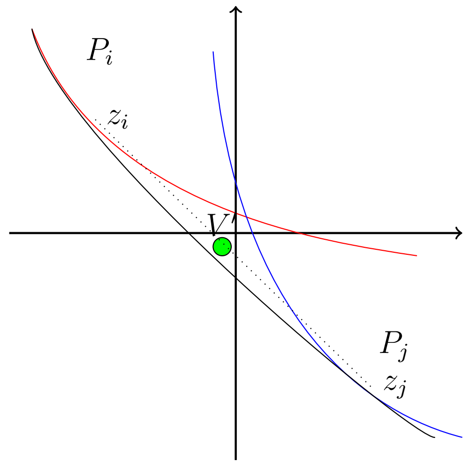

FIGURE 8. Proof of Debreu-Scarf Theorem

with $z_i \in P_i$ for all i, and

$$\sum_{i=1}^{I} \alpha_i z_i \in V'.$$

The function

$$(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_I) \mapsto \sum_{i=1}^{I} \tilde{\alpha}_i z_i$$

is continuous, so there is $(\alpha'_1, \ldots, \alpha'_I)$ with $\sum_{i=1}^{I} \alpha'_i z_i \in V'$ and for which $\alpha'_i \in \mathbf{Q}_+$ for all $i = 1, \ldots, I$.

Let $\beta_i \in \mathbf{N}$ and N be such that $\alpha'_i = \beta_i / N$. Consider a coalition S in $\mathcal{E}^N$ with β_i copies of agent type i. Define the vector $(y_h)_{h \in S}$ with $y_h = z_i + \omega_i$ for the type of agent i that h is a member of. This means that $y_h \succ_h x^*_h$, as $z_i \in P_i$. In addition:

$$\begin{aligned} \sum_{h \in S} y_h &= \sum_{i=1}^{I} \beta_i (z_i + \omega_i) \\ &= N \sum_{i=1}^{I} \frac{\beta_i}{N} z_i + \sum_{i=1}^{I} \beta_i \omega_i \\ &\ll \sum_{h \in S} \omega_h, \end{aligned}$$

as

$$\sum_{i=1}^{I} \frac{\beta_i}{N} z_i \ll 0.$$

Agents' preferences are monotonic, so the coalition S blocks x^* in $\mathcal{E}^N$, contradicting that $x^* \in C(\mathcal{E}^N)$.[14]

The rest of the proof is similar to the proof of the second welfare theorem, but simpler since we do not need to go from individual bundles to aggregate bundles.

The set P is convex and nonempty, and $0 \notin P$. By the separating hyperplane theorem, there is $p^* \in \mathbf{R}^L$, non-zero, such that $p^* \cdot z \geq 0$ for all $z \in P$. Then $x_i \succ_i x_i^*$ implies that $x_i - \omega_i \in P_i$, which means that $p^* \cdot x_i \geq p^* \cdot \omega_i$.

Let $x_i^k = x_i^* + (1/k, \ldots, 1/k)$ for $k \geq 1$. Then $x_i^k \succ_i x_i^*$, which means that $p^* \cdot x_i^k \geq p^* \cdot \omega_i$ for all k. Then $p^* \cdot x_i^* \geq p^* \cdot \omega_i$ as $x_i^* = \lim_{k \to \infty} x_i^k$. Then $p^* \cdot x_i^* \geq p^* \cdot \omega_i$ for all i, and $\sum_{i=1}^I (p^* \cdot x_i^* - p^* \cdot \omega_i) = p^* \cdot \sum_{i=1}^I (x_i^* - \omega_i) = 0$ (as x^* is an allocation), imply that $p^* \cdot x_i^* = p^* \cdot \omega_i$ for all i.

So we have established that if $x_i \succ_i x_i^*$ then $p^* \cdot x_i \geq p^* \cdot x_i^* = p^* \cdot \omega_i$

Now we can show that $p^* > 0$. Strict monotonicity implies that $x_i^* + e_l \succ_i x^*$. Then, $p^* \cdot (x_i^* + e_l) \geq p^* \cdot x_i^*$ gives $p_l^* \geq 0$. Since $p^* \neq 0$ we have $p^* > 0$.

We have $\bar{\omega} \gg 0$. So $p^* > 0$ implies that $p^* \cdot \bar{\omega} > 0$. Then there is j with $p^* \cdot \omega_j > 0$. For such j, if we had some $x_j \succ_j x_j^*$ and $p^* \cdot x_j = p^* \cdot x_j^*$ we would obtain a contradiction by the same reasoning as in the proof of the second welfare theorem: namely that there would be $\delta > 0$ with $(1 - \delta)x_j \succ_j x_j^*$ but $p^* \cdot (1 - \delta)x_j = p^* \cdot (1 - \delta)x_j^* < p^* \cdot x_j^*$.

Now, using what we know about this particular consumer j, we can show that $p^* \gg 0$. We have that $x_j^* + e_l \succ_j x^*$ implies $p^* \cdot (x_j^* + e_l) > p^* \cdot x_j^*$. So $p_l > 0$.

Finally, let i be any consumer. Then $p^* \gg 0$ and $\omega_i > 0$ implies that $p^* \cdot \omega_i = p^* \cdot x_i^* > 0$. This implies that we can repeat the argument we made above for consumer j, and show that $x_i \succ_i x_i^*$ then $p^* \cdot x_i > p^* \cdot x_i^*$. □

[14]We obtain a vector $(y_h)_{h \in S}$ that sums up to less than $\sum_{h \in S} \omega_h$ but this implies that S blocs since we can increase y_h a little bit so as to satisfy the equality with the sum of endowments.

13. Partial equilibrium

Economists often want to focus on the market for one good in isolation. The ability to do so depends on making certain assumptions, which we proceed to spell out.

Consider an exchange economy in which each preference is quasilinear. Specifically, let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy in which each $\succeq_i$ is represented by a utility function

$$u_i(x_1, \dots, x_L) = v_i(x_1) + \sum_{l=2}^{L} x_l.$$

Al consumers regard goods $l = 2, \dots, L$ as perfect substitutes. As a consequence, in equilibrium, these goods should have the same price (meaning that consumers should exchange one unit of good l for one unit of good l', for all $l, l' \neq 1$). We can normalize the price of all those goods to be 1, and regard good $\sum_{l=2}^L x_l$ as a ***composite good***. Denote this composite good by m, and x_1 by x. Effectively, then $\mathcal{E}$ becomes a two-good economy in which consumers have utility $u_i(x,m) = v_i(x) + m$.

Let p be the price of good x.

Consider the maximization problem for consumer i:

$$\begin{array}{l} \max_{(x,m)} v_i(x) + m \\ \text{s.t. } px + m \leq W. \end{array}$$

Allow negative consumption of m (a common, but not innocent assumption). Then the problem becomes to maximize

$$v_i(x) + (W - px).$$

Note that the solution $x^*(p)$ is the solution to maximizing $v_i(x) - px$ and does not depend on W. In contrast the demand for "money" is $m^*(p,W) = W - px^*(p)$. These facts about x^* and m^* are very important. All "wealth effects," meaning all changes in income W, translate one for one into changes in the demand for m. The demand for x does not depend on W.

Assume: v_i is strictly increasing, C^2 and concave.

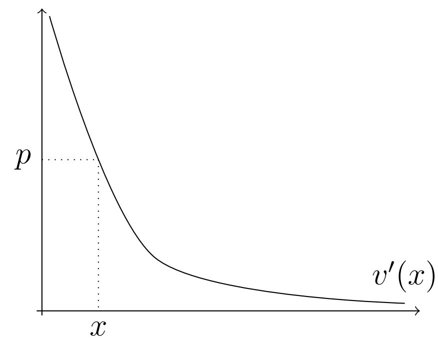

FIGURE 9. Quasi-linear demand

The FOC for the consumer's problem is then

$$v'(x) - p \begin{cases} \leq 0 & \text{if} x = 0 \\ = 0 & \text{if} x > 0 \end{cases}$$

We should emphasize that it is this simple because we are allowing for $m < 0$.

Now we have that $v_i'(x_i^*(p)) = p$ when $x_i^*(p) > 0$, so that x_i^* is the inverse of the marginal utility v_i'.

The indirect utility of consumer i is

$$\psi_i(p, W) = v_i(x_i^*(p)) + m^*(p, W) = (v_i(x_i^*(p)) - px_i^*(p)) + W = \int_p^\infty x_i^*(s)ds + W,$$

when the (improper) integral $\int_p^\infty x_i^*(s)ds$ is well defined. See Figure 10

The term

$$CS_i(p) = v_i(x_i^*(p)) - px_i^*(p) = \int_p^\infty x_i^*(s)ds$$

is usually called ***consumer surplus***. It is a measure of the well being of a consumer, in the obvious sense that it's (up to the value of W) her indirect utility. It has a simple meaning, for magnitude $x < x_i^*(p)$, the consumer was willing to pay an additional $v'(x)$ in moiney for a bit more of the good, but only paid p. So the cosumer gained $v'(x) - p$, If we "add" up all these gains, stemming from the excess willingness to pay over what was actually paid, we obtain $\int_0^{x_i^*(p)} [v_i'(x) - p]dx = v_i(x_i^*(p)) - px_i^*(p)$. Moreover,

$$v_i'(x) = s \text{ iff } x_i^*(s) = x, \text{ hence } \int_0^{x_i^*(p)} [v_i'(x) - p]dx = \int_p^{+\infty} x_i^*(s)ds$$

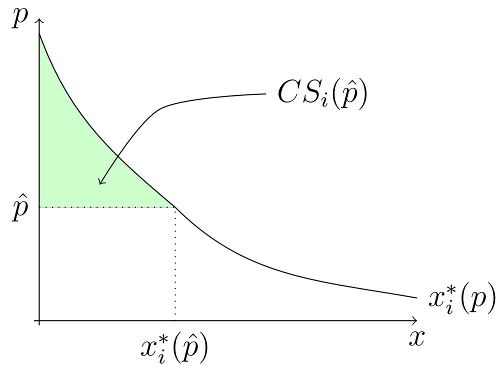

FIGURE 10. Consumer surplus

Note that the formula expressing consumer surplus as a function of demand allows us to empirically measure consumer surplus, if we can empirically estimate consumer demand (and we often can).

13.1. **Aggregate demand and welfare.** Let $X^*(p) = \sum_{i=1}^I x_i^*(p)$, which defines a utility v (by integrating its inverse). So there exists obviously a representative consumer.

Note also that the sum $\sum_{i=1}^I \psi_i(p, W_i)$ is, up to the value of $\sum_{i=1}^I W_i$, equal to

$$\sum_{i=1}^I (v_i(x_i^*(p)) - p x_i^*(p)) = \sum_{i=1}^I \int_p^\infty x_i^*(s) ds = \int_p^\infty \sum_{i=1}^I x_i^*(s) ds = \int_p^\infty X^*(s) ds.$$

We can often estimate aggregate demand, which gives us a measure of collective welfare.

In fact, consider the problem of finding a Pareto optimal allocation

$$\begin{aligned} &\max v_1(x_1) + m_1 \\ &s.t. \begin{cases} v_i(x_i) + m_i & \geq \bar{u}_i \forall i = 2, \ldots, I \\ \sum_i x_i & \leq \sum_i \omega_{i,x} \\ \sum_i m_i & \leq \sum_i \omega_{i,m} \end{cases} \end{aligned}$$

This problem is equivalent to maximizing $\sum_i v_i(x_i)$ subject to $\sum_i x_i \leq \sum_i \omega_{i,x}$.

The Lagrangian is

$$L((x_i);\lambda) = \sum_i v_i(x_i) + \lambda\left(\sum_i \omega_{i,x} - \sum_i x_i\right) = \sum_i [v_i(x_i) - \lambda x_i] + \lambda \sum_i \omega_{i,x}$$

The first-order conditions are the same as for individual maximization when $\lambda = p$. This is a version of the welfare theorems. So we can set the solution to be $x_i^*(\lambda)$. Then the value of the lagrangian at the maximum is equal to $\sum_{i=1}^{I} CS_i(\lambda)$. So in a sense what we're tying to do is to maximize the sum of consumer surpluses.

13.2. **Production.** Suppose that instead x is produced from "money" using a cost function $c : \mathbf{R}_+ \to \mathbf{R}_+$. This means that there is a production possibility set

$$Y = \{(x, -c(x)) : x \in \mathbf{R}_+\}.$$

Pareto efficient allocation can be obtained as before by solving:

$$\max v_1(x_1) + m_1$$
$$s.t. \begin{cases} v_i(x_i) + m_i \geq \bar{u}_i & \forall i = 2, \dots, I \\ \sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} \omega_{i,x} + x^f \\ \sum_{i=1}^{I} m_i \leq \sum_{i=1}^{I} \omega_{i,m} - m^f \\ (x^f, -m^f) \in Y \end{cases}$$

Suppose that $\sum_{i=1}^{I} \omega_{i,x} = 0$. Then at a solution we must have that $\sum_i x_i = x^f$. Then $(x^f, -m^f) \in Y$ means that $m^f = c(\sum_i x_i)$. Finally, $\sum_i m_i = \sum_i \omega_{i,m} - m^f$ implies that efficiency is characterized by

$$\max_{(x_1,\dots,x_I)\geq 0} \sum_{i=1}^{I} v_i(x_i) - c(\sum_{i=1}^{I} x_i)$$

Assume: c is C^1, with $c' > 0$, and c is strictly convex.

The FOCs for an interior solution demands that $v_i'(x_i) = c'(x)$, where $x = \sum_i x_i$. This solution can be decentralized by setting the price of x to be $p = c'(x)$. Then $v_i'(x_i) = p = c'(x)$, $x_i = x_i^*(p)$, and a firm maximizes profits by choosing x at prices p.

13.3. **Public goods.** A public good is a good that all consumers must consume in the same quantity.

If the public good consumed by i is x, then all consumers consume the quantity x of the public good. Each consumer j obtains then utility $v_j(x)$.

Efficiency demands that

$$\max_{x \geq 0} \sum_{i=1}^{n} v_i(x) - c(x).$$

The FOCs for an interior solution now mean that

$$\sum_{i=1}^{n} v_i'(x) = c'(x)$$

The condition $\sum_{i=1}^{n} v_i'(x) = c'(x)$, equating marginal cost to the sum of marginal utilities is often called *Samuelson's condition* for efficient provision of public goods.[15] See Figure 11. Let x^* be the solution to this maximization problem, so x^* is the (unique) efficient level of the public good.

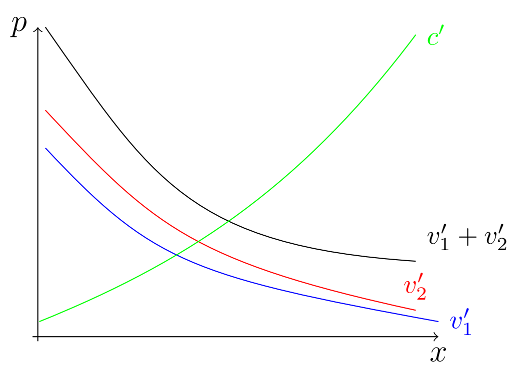

FIGURE 11. Samuelson's condition for public good provision

Compare to private provision of public good. Suppose that the price of the public good is p. Then each agent i is choosing a quantity x_i to buy, at price p. Then i solves the problem

$$\max_{x_i \geq 0} v_i(x_i + \sum_{j \neq i} x_j) - px_i.$$

[15]Paul Samuelson introduced this way of analyzing public goods in Samuelson (1954).

The FOC is then

$$v_i'(\sum_j x_j) - p = 0 \text{ if } x_i > 0$$
$$v_i'(\sum_j x_j) - p \leq 0 \text{ if } x_i = 0.$$

Profit maximization requires that $p = c'(\sum_j x_j)$.

Suppose, to simplify the exposition, that $v_1' < v_2' < \cdots < v_I'$.

Then since p does not depend on i we must have $v_I'(\hat{x}) = p$ and $v_i'(\hat{x}) - p < 0$ for $i = 1, \ldots, I-1$. Hence $x_i = 0$ for $i = 1, \ldots, I-1$, and $x_I = \hat{x}$.

Note that since $v_i' > 0$ we have that $\sum_i v_i'(\hat{x}) > v_I'(\hat{x}) = c'(\hat{x})$. Since c' is monotone increasing and $\sum_i v_i'$ is strictly decreasing,

$$\hat{x} < x^*.$$

In words, there is *underprovision* of the public good.

13.4. **Lindahl equilibrium.** Imagine that we could fool consumers into believing that the quantity consumed of the public good only depends on their individual purchases. Sort of like selling the Hollywood sign to a very gullible tourist. They would then pay an (individualized!) price p_i for each unit of the public good, and solve the maximization problem

$$\max_{x_i \geq 0} v_i(x_i) - p_i x_i$$

The FOC for an interior solution is $v_i'(x_i) = p_i$. Now let x^* be the efficient level of the public good and set $p_i = v_i'(x^*)$. It is then optimal for each consumer i to purchase the same level x^* of the public good.

The firm solves the problem of

$$\max_{q \geq 0} (\sum_i p_i) q - c(q).$$

Since $\sum_i p_i = \sum_i v_i'(x^*)$ and x^* satisfies the Samuelson condition of optimality, we have that $\sum_i p_i = c'(x^*)$. The firm optimizes by choosing the level x^* of public good when it sells it to each consumer i at price p_i.

This outcome is called a Lindahl equilibrium. The idea is to illustrate the role of the inability to exclude agents from consuming the public good as the source of inefficiency in the private provision of public goods.

14. Two-sided matching model with transferable utility

We turn to a model of a two-sided market due to Shapley and Shubik (1971): B is a set of ***buyers*** and S is a set of ***sellers***. The sets B and S are finite, nonempty, and disjoint. Each buyer seeks to buy one and only one unit of an indivisible good. Each seller has one unit to sell, but sellers are different from each other, and offer potentially different goods.

If $i \in B$ buys from $j \in S$, they generate a surplus $\alpha_{i,j}$. Here $\alpha_{i,j}$ is given; part of the primitive of the model. You can think of buyer i getting some value from consuming j's good, and of j having some cost of providing that good to i. Then $\alpha_{i,j}$ would be the difference between value and cost.

When i buys from j, she obtains some utility u_i and j gets some profit v_j, so that $u_i + v_i = \alpha_{i,j}$. For example, imagine that the cost for j is zero. Then i's utility is $u_i = \alpha_{i,j} - v_j$, where v_j is the price (and therefore the profit) that j gets from i. In this case, i buys from j if $\alpha_{i,j} - v_j \geq \alpha_{i,h} - v_h$ for all h. Put differently,

$$u_i + v_h \geq \alpha_{i,h},$$

for all i and h.

A ***matching*** is a matrix $(x_{i,j})_{i\in B, j\in S}$ such that $x_{i,j} \geq 0$, and for all $(i,j) \in B \times V$,

$$\sum_{h\in V} x_{i,h} \leq 1$$
$$\sum_{h\in B} x_{h,j} \leq 1.$$

If $x_{i,j} \in \{0,1\}$ we can interpret $x_{i,j} = 1$ as i buying from j. The above inequalities mean that each buyer buys from at most one seller, and each seller sell to at most one buyer.

An ***assignment*** is a pair of vectors $((u_i)_{i\in B}, (v_j)_{j\in S})$ such that $u_i \geq 0$, $v_j \geq 0$ and such that there exists a matching $x_{i,j}$ such that $\sum_{i\in B} u_i + \sum_{j\in S} v_j = \sum_{i\in B, j\in S} \alpha_{i,j} x_{i,j}$. An assignment (u,v) is in the ***core*** if

$$u_i + v_j \geq \alpha_{i,j}.$$

The core terminology arises from a notion of blocks. A pair (i, j) can ***block*** an assignment if $u_i + v_j < \alpha_{i,j}$ by trading among themselves and sharing the surplus $\alpha_{i,j}$. In this model, only pairs can "generate value." So the only relevant coalitions are pairs, and the core is the set of assignments that no pair can block.

At the same time, the core requirement ensures that any agent is optimizing in the sense that

$$u_i = \max\{\alpha_{i,h} - v_h : h \in S\} \qquad v_j = \max\{\alpha_{h,j} - u_h : h \in B\}$$

We can now analyze the model by means of linear programming duality. There will be a duality between the problem of efficiently (meaning surplus-maximizing), matching buyers and sellers, and the problem of finding a core assignment.

Consider the problem of efficiently matching buyers and sellers. This is the problem of choosing a matching between buyers and sellers to maximize total surplus.

$$\begin{aligned} &\max\nolimits_{(x_{i,j})} \textstyle\sum_{i\in B, j\in S} x_{i,j}\alpha_{i,j} \\ &\text{s.t.} \begin{cases} x_{i,j} \geq 0 \\ (\forall i \in B) \sum_{j\in S} x_{i,j} \leq 1 \\ (\forall j \in S) \sum_{i\in B} x_{i,j} \leq 1. \end{cases} \end{aligned}$$

This is a linear program. We set up the corresponding Lagrange multiplier, and use the minmax theorem. Let u_i be the Lagrange multiplier associated to the constraint that $\sum_{j\in S} x_{i,j} \leq 1$. Let v_j be the Lagrange multiplier associated to $\sum_{i\in B} x_{i,j} \leq 1$.

$$L(x; (u, v)) = \sum_{i\in B, j\in S} x_{i,j}\alpha_{i,j} + \sum_{i\in B} u_i(1 - \sum_{j\in S} x_{i,j}) + \sum_{j\in S} v_j(1 - \sum_{i\in B} x_{i,j}).$$

Then we have that

$$\max_{(x_{i,j})} \min_{((u_i),(v_j))} L(x; (u, v)) = \min_{((u_i),(v_j))} \max_{(x_{i,j})} L(x; (u, v)).$$

Note that

$$\begin{aligned} L(x; (u, v)) &= \sum_{i\in B, j\in S} x_{i,j}\alpha_{i,j} + \sum_{i\in B} u_i(1 - \sum_{j\in S} x_{i,j}) + \sum_{j\in S} v_j(1 - \sum_{i\in B} x_{i,j}) \\ &= \sum_{i\in B, j\in S} x_{i,j}(\alpha_{i,j} - u_i - v_j) + \sum_{i\in B} u_i + \sum_{j\in S} v_j. \end{aligned}$$

Hence,

$$\max_{(x_{i,j})} \min_{((u_i),(v_j))} L(x;(u,v)) = \min_{((u_i),(v_j))} \max_{(x_{i,j})} \sum_{i\in B} u_i + \sum_{j\in S} v_j + \sum_{i\in B, j\in S} x_{i,j}(\alpha_{i,j} - u_i - v_j).$$

But then $x_{i,j}$ are Lagrange multipliers in the problem of minimizing $\sum_{i\in B} u_i + \sum_{j\in S} v_j$ subject to $(\alpha_{i,j} - u_i - v_j) \leq 0$. This gives us the ***dual*** to the problem we started from:

$$\min \textstyle\sum_{i\in B} u_i + \sum_{j\in S} v_j$$
$$\text{s.t.} \begin{cases} u_i + v_j \geq \alpha_{i,j} \\ u_i \geq 0 \\ v_j \geq 0 \end{cases}$$

Observation 74. If $x_{i,j}$ solves the surplus maximization problem, and (u, v) solves the dual, then

$$\sum_{i,j} \alpha_{i,j} x_{i,j} = \sum_i u_i + \sum_j v_j,$$

and (u, v) is a core assignment. Note that this holds for any pair of solutions to the two problems. We can pair up any optimal matching with any core assignment.

Observation 75. There is a solution to the surplus maximization problem in which $x_{i,j} \in \{0, 1\}$ and where, if $x_{i,j} = 1$, then $u_i + v_j = \alpha_{i,j}$.

Proof. The extreme points of the set of matchings consist of matrices with $0-1$ values. The primal problem is a linear programming problem, so a solution always exists that is a extreme point. As for the second statement, complementary slackness in the dual problem says that

$$0 = \sum_{i\in B, j\in S} x_{i,j} \underbrace{(\alpha_{i,j} - u_i - v_j)}_{\leq 0}.$$

So $x_{i,j} > 0$ implies that $u_i + v_j = \alpha_{i,j}$. □

Suppose that $\alpha_{i,j} > 0$ for all i, j, and that $|B| = |S|$. Then the matching constraints will hold with equality at a solution, and all agents will be matched to someone.

Proposition 76. *Let (u, v) and (u', v') be assignments in the core. Let $\bar{u}_i = \max\{u_i, u'_i\}$ and $\underline{v}_j = \min\{v_j, v'_j\}$. Then $(\bar{u}, \underline{v})$ is an assignment in the core.*

Proof. Note that for any $i \in B$ and $j \in S$, $\bar{u}_i + \underline{v}_j \geq \alpha_{i,j}$ (for say that $\underline{v}_j = v_j$ then $\bar{u}_i + \underline{v}_j \geq u_i + v_j \geq \alpha_{i,j}$), and $\bar{u}_i, \underline{v}_j \geq 0$.

Choose one optimal matching $x_{i,j}$ that is an extreme point. Then $x_{i,j} = 1$ means that $u_i + v_j = u'_i + v'_j$. Then $\bar{u}_i = u_i$ iff $\underline{v}_j = v_j$. So $\bar{u}_i + \underline{v}_j = u_i + v_j$. Thus,

$$\sum_{i\in B} u_i + \sum_{j \in S} v_j = \sum_{i \in B, j \in S} x_{i,j}(u_i + v_j) = \sum_{i \in B, j\in S} x_{i,j}(\bar{u}_i + \underline{v}_j) = \sum_{i \in B} \bar{u}_i + \sum_{j \in S} \underline{v}_j.$$

The last equality uses the fact that if $x_{i,j} = 0$ for all $j \in S$ then $u_i = u'_i = 0$, and analogously for v_j, v'_j.

Thus $(\bar{u}, \underline{v})$ satisfies the constraints of the dual program, and has the same value for the objective function. □

An analogous result is true if we take the maximum of v_j and v'_j and the minimum of u_i and u'_i. This is called the lattice structure of the core assignments. It means that buyers share some interests with other buyers, and sellers with other sellers. There are common interests for agents on the same side of the market, and opposing interest for agents on opposite side of the market.

As a consequence we have

Corollary 77. *There exists core assignments* (u^*, v_*) *and* (u_*, v^*) *such that for any core assignment* (u, v)*,*

$$\begin{aligned} u_i^* &\geq u_i \geq u_{*i} \\ v_j^* &\geq v_j \geq v_{*i} \end{aligned}$$

Think of (u^*, v_*) and (u_*, v^*) as core assignments with minimal, respectively maximal, prices.

14.1. **Pseudomarktes.**

14.1.1. *Notation:* The simplex $\{x \in \mathbf{R}^n_+ : \sum_{j=1}^n x_j = 1\}$ in $\mathbf{R}^n$ is denoted by $\Delta^n \subseteq \mathbf{R}^n$, while the set $\{x \in \mathbf{R}^n_+ : \sum_{j=1}^n x_j \leq 1\}$ is denoted by $\Delta^n_- \subseteq \mathbf{R}^n$. When n is understood, we simply use the notation Δ and Δ_-.

A function $u : \Delta^n_- \to \mathbf{R}$ is

- ***concave*** if, for any $x, z \in \Delta_-$, and $\lambda \in (0,1)$, $\lambda u(z) + (1-\lambda)u(x) \leq u(\lambda z + (1-\lambda)x)$;
- ***quasi-concave*** if, for each $x \in \Delta_-$, the set $\{z \in \Delta_- : u(z) \geq u(x)\}$ is convex.
- ***semi-strictly quasi-concave*** if, for any $x, z \in \Delta_-$, $u(z) < u(x)$ and $\lambda \in (0,1)$ imply that $u(z) < u(\lambda z + (1-\lambda)x)$.
- ***expected utility*** if it is linear. In this case we identify u with a vector $u \in \mathbf{R}^n$ and denote $u(x)$ as $u \cdot x$.
- C^1 if it can be extended to a continuously differentiable function defined on an open set that contains Δ_-.

A ***discrete allocation problem*** is a tuple $(N, O, (u^i)_{i\in I}$, where $N = \{1, \ldots, n\}$ is a finite set of agents, O is a set of ***objects***, and $u^i : \Delta_-^{|O|} \to \mathbf{R}$ is a utility function for agent i.

There is an implicit normalization in u^i. We suppose that an agent gets the outside option $\emptyset$ with probability $1 - \sum_o x_o$, and u takes this into account.

Let $L = |O|$.

Allocations and Pareto optimality. An ***allocation*** in Γ is a vector $x \in \mathbf{R}_+^{LN}$, which we write as $x = (x^i)_{i=1}^N$, with $x^i \in \Delta_-^L$, such that

$$\sum_{i\in I} x_s^i = q_s$$

for all $i \in I$ and all $s \in S$. When $x_s^i \in \{0,1\}$ for all i and all s, x is a deterministic allocation.

The notion of efficiency comes in three flavors: An allocation x is ***weak Pareto optimal*** (wPO) if there is no allocation y such that $u^i(y^i) > u^i(x^i)$ for all i; ε***-weak Pareto optimal*** (ε-PO), for $\varepsilon > 0$, if there is no allocation y such that $u^i(y^i) > u^i(x^i) + \varepsilon$ for all i; and ***Pareto optimal*** (PO) if there is no allocation y such that $u^i(y^i) \geq u^i(x^i)$ for all i and $u^j(y^j) > u^j(x^j)$ for some j.

Equilibrium. A ***Hylland-Zeckhauser equilibrium*** is a pair (x, p) such that $x \in \Delta_-^N$, and $p = (p_o)_{o\in O} \in \mathbf{R}_+^L$ is a price vector such that

(1) $\sum_{i=1}^N x^i = (1, \ldots, 1)$; and
(2) x^i maximizes i's utility within his budget:

$$x^i \in \text{argmax}\{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq \alpha + (1-\alpha)p \cdot \omega^i\};$$

The following important result is due to Hylland and Zeckhauser (1979).

Theorem 78. *Suppose that each u^i is continuous, monotonic and quasi-concave. Then there exists a Hylland-Zeckhauser equilibrium (x, p) in which x is Pareto efficient and envy-free.*

Note: in this model the first welfare theorem fails. It is possible to construct examples of HZ equilibria that are Pareto dominated. The theorem says, however, that there exists one that is Pareto efficient.

Let

$$\begin{aligned} B^i(p) &= \{x \in \Delta_- : p \cdot x \leq 1\} \\ d^i(p) &= \operatorname{argmax}\{u^i(x) : x \in B^i(p)\} \\ \underline{d}^i(p) &= \operatorname{argmin}\{p \cdot x : x \in d^i(p)\} \\ z^i(p) &= \underline{d}^i(p) - \{(1, \dots, 1)\} \text{ and } z(p) = \sum_{i=1}^N z^i(p). \end{aligned}$$

Let $\bar{p} > N$.

Lemma 79. *$\underline{d}^i$ is upper hemi-continuous on $[0, \bar{p}]^L$*

We omit the proof of Lemma 79.

Consider the correspondence $\phi : [0, \bar{p}]^L \to [0, \bar{p}]^L$ defined by

$$\phi_l(p) = \{\min\{\max\{0, \zeta_l + p_l\}, \bar{p}\} : \zeta \in z(p)\}.$$

Lemma 80. *ϕ is upper hemi-continuous, convex- and compact- valued.*

The proof of Lemma 80 essentially follows from Lemma 79.

By Kakutani's fixed point theorem there is $p^* \in [0, \bar{p}]^L$ with $p^* \in \phi(p^*)$. We shall prove that p^* is an equilibrium price. Note that there exists $\zeta \in z(p^*)$ such that

$$p_l^* = \min\{\max\{0, \zeta_l + p_l^*\}, \bar{p}\}. \tag{11}$$

Lemma 81. *$p^* \cdot \zeta \geq 0$.*

Proof. If $p^* \cdot \zeta < 0$ then there is some good l with $p_l^* > 0$ and $\zeta_l < 0$. By Equation 11, then, we cannot have $p_l^* = \bar{p}$ because $\zeta_l < 0$ and then $\zeta_l + p_l^* < \bar{p}$. So $p_l^* = \max\{0, p_l^* + \zeta_l\}$, which is not possible as $\zeta_l < 0$ and $p_l^* > 0$. □

Lemma 82. *$p_l^* < \bar{p}$ for all $l \in [L]$*

Proof. Suppose towards a contradiction that there is l for which $p^*_l = \bar{p}$. Then $p^*_l > 0$, so Equation 11 means that $\bar{p} \leq \zeta_l + p^*_l = \zeta_l + \bar{p}$. Let $\zeta = \sum_i x^i - \mathbf{1}$, with $x^i \in d^i(p^*)$.

Note that $p^* \cdot x^i \leq 1$ for each i, so, adding over i, we obtain that $p^* \cdot \sum_i x^*_i \leq N$. Subtract $p^* \cdot \mathbf{1}$ and we obtain that $p^* \cdot \zeta \leq N - p^* \cdot \mathbf{1}$.

Now, $p^* \cdot \mathbf{1} \geq p^*_l = \bar{p}$. But we assumed that $\bar{p} > N$. So $p^* \cdot \zeta < 0$, contradicting Lemma 81. □

Lemma 83. $\zeta = 0$

Proof. By Lemma 82 and Equation (11),

$$p^*_l = \max\{0, \zeta_l + p^*_l\} \tag{12}$$

for all $l \in [L]$.

Equation 12 implies two things. First, that $\zeta_l > 0$ is not possible for any l. Hence $\zeta \leq 0$. Second, that if $\zeta_l < 0$ then $p^*_l = 0$.

Suppose then, towards a contradiction, that that $\zeta_l < 0$ for some good l, and correspondingly that $p^*_l = 0$. Now, $\zeta_l < 0$ and $\zeta \leq 0$ means that

$$0 > \sum_l \zeta_l = \sum_l (\sum_i x^i_l - 1) = \sum_i (\sum_l x^i_l - 1).$$

So there is some agent i for which $\sum_l x^i_l < 1$. Agent i can then increase his consumption of good l without violating the constraint that consumption lie in Δ_-. Given that $p^*_l = 0$, the increase in consumption of good l would also not violate the budget constraint. So there exist a bundle in $B^i(p)$ with strictly more of good l, and the same amount of every other good, than x^i. This contradicts the strict monotonicity of u^i, and the fact that $x^i \in d^i(p^*)$. □

I learned this proof from Antonio Miralles (HZ's proof seems to be in the original unpublished working paper, but I do not have a copy).

15. General equilibrium under uncertainty

Uncertainty is modeled through a set S (finite) of ***states of the world***. If the uncertainty is over the weather in the future, S could be $S = \{\text{rain}, \text{shine}\}$. Agents trade and consume ***contingent goods***; defined as functions from S into $\mathbf{R}^n$. The interpretation is that there are n physical goods, and that a contingent good x is a contract that delivers $x(s) \in \mathbf{R}^n$ upon the realization of $s \in S$.

Note that each contingent good is a vector in $\mathbf{R}_+^{|S|n}$. A vector $x \in \mathbf{R}_+^{n|S|}$ specifies a level of consumption of each physical good conditional on a state of the world. For example if $S = \{s_1, s_2, s_3\}$ and $n = 2$ then the vector

$$(2, 3, 1, 2, 2, 2)$$

implies consumption of $(2, 3)$ if the first state of the world, s_1, occurs, but $(1, 2)$ if the second state of the world, s_2, occurs.

Let $L = n|S|$. An exchange economy now has the meaning that trade takes place in contingent goods.

15.1. **Two-state, two-agent economy.** Consider the following example: $I = 2$, $n = 1$ and $S = \{s_1, s_2\}$. Suppose that each agent i has preferences represented by a utility

$$U_i(x_1, x_2) = \pi u_i(x_1) + (1 - \pi) u_i(x_2).$$

These are expected utility preferences. Suppose that u_i is strictly increasing, C^1, and strictly concave. Agents' endowments are (ω_1, ω_2) with $\bar{\omega}_1 = \bar{\omega}_2$. This assumption means that there is no ***aggregate risk*** (or no systemic risk). All risk is ***idiosyncratic***.

We can represent the Pareto optimal allocations in the Edgeworth box.

Pareto optimality is characterized by

$$\frac{\pi u_1'(x_1)}{(1-\pi) u_1'(x_2)} = \frac{\pi u_2'(\bar{\omega}_1 - x_1)}{(1-\pi) u_2'(\bar{\omega}_2 - x_2)},$$

for interior allocations. This means that $x_1 = x_2$ for, suppose that that were not the case. Suppose wlog that $x_1 > x_2$; so we have as a consequence of $\bar{\omega}_1 = \bar{\omega}_2$ that $\bar{\omega}_1 - x_1 < \bar{\omega}_2 - x_2$. Then the strict concavity of u_1 and u_2 would mean that

$$\frac{\pi u_1'(x_1)}{(1-\pi) u_1'(x_2)} < \frac{\pi}{(1-\pi)} < \frac{\pi u_2'(\bar{\omega}_1 - x_1)}{(1-\pi) u_2'(\bar{\omega}_2 - x_2)}.$$

That $x_{i,1} = x_{i,2}$ means that agents will get ***full insurance***. They are both risk averse and have the same beliefs about the states of the world. Agents have incentives to trade with each other until they achieve full insurance. There is no aggregate risk, so full insurance is feasible.

At a competitive equilibrium, prices will satisfy

$$\frac{p_1}{p_2} = \frac{\pi}{(1-\pi)},$$

so prices will reflect the agents' shared beliefs about the state of the world.

Note that, while these properties are very appealing, they rest of very particular assumptions on the economy: no aggregate risk, risk aversion, and expected utility preferences with a common belief over the state of the world.

15.2. **Pari-mutuel betting.** Agents' probabilistic assessments are aggregated into a single "odds ratio." We can think of the pari-mutuel market as a system for aggregating information in the form of different priors.

Suppose that there are I agents betting on L horses. Each agent i has beliefs over which horse will win the race in the form of a probability distribution $\pi_i \in \Delta_L$. So $\pi_{i,l}$ is the probability that i assigns to the lth horse winning the race. Suppose that for each l there is i with $\pi_{i,l} > 0$, a monotonicity condition.

Agents are going to bet on horses with the objective of maximizing their expected payoff. Let $\beta_{i,l}$ be how much i bets on l (expressed in monetary units), for $i = 1, \ldots, I$ and $l = 1, \ldots, L$.

In the pari-mutuel system, if horse l wins, then the total amount of bets in the race get allocated to the agents who bid on l, and it is allocated proportional to their bets. So agent i gets a payoff that equals

$$\frac{\beta_{i,l}}{\sum_{h=1}^{I} \beta_{h,l}} \left(\sum_{h=1}^{I} \sum_{k=1}^{L} \beta_{h,k} \right).$$

The total amount of bets are $\sum_{h=1}^{I} \sum_{k=1}^{L} \beta_{h,l}$, and these are distributed proportionally among the agents who bet on horse l (if no-one bets on l this magnitude is not defined).

The ***parimutuel odds*** on horse l are:

$$\left[\frac{1}{\sum_{h=1}^{I} \beta_{h,l}} \sum_{h=1}^{I} \sum_{k=1}^{L} \beta_{h,k} - 1 \right] \text{ to } 1.$$

So, for example, if the odds on horse 3 are 5-to-1, it means that a bet of \$1 dollar on horse 3 pays \$6 (meaning a net gain of \$5 on the \$1).[16]

[16]Of course, in an actual horse race, the track's profit is first subtracted from the total proceeds $\sum_h \sum_k \beta_{h,k}$.

We assume that agents have a budget $b_i > 0$ for their bets, so they are constrained by $\sum_l \beta_{i,l} = b_i$ (wlog we assume equality). Moreover, assume (as a normalization) that $\sum_i b_i = 1$. This means that if horse l wins, then i gets a payoff of $\beta_{i,l}/\sum_{h=1}^I \beta_{h,l}$. Or that the parimutuel odds on horse l are $[(\sum_{h=1}^I \beta_{h,l})^{-1} - 1]$ to 1.

We can translate the problem into a more familiar model if we express the agents' decisions as the purchase of an asset, instead of as a monetary bet. Each agent i chooses a number $x_{i,l} \in \mathbf{R}_+$ of ***tickets*** to bet on horse l. So that if horse l wins then they obtain a monetary payoff in the amount of $x_{i,l}$. The expected payoff from a vector of tickets $x \in \mathbf{R}^L_+$ is $\sum_l \pi_{i,l} x_{i,l}$.[17]

Now, $\beta_{i,l}$ is the amount of money that i bets on l; so if p_l is the price for betting on l, meaning the price of one unit of $x_{i,l}$, then $\beta_{i,l} = p_l x_{i,l}$. So, now if l wins then i gets a payoff equal to

$$\frac{\beta_{i,l}}{p_l} = x_{i,l}.$$

This payoff must equal $\beta_{i,l}/\sum_{h=1}^I \beta_{h,l}$, so we have

$$p_l = \sum_{h=1}^{I} \beta_{h,l}$$

So we have two representations of the problem. In the first, each agent i chooses $\beta_i \in \mathbf{R}^L_+$ to maximize $\sum_l \pi_{i,l}\beta_{i,l}/p_l$ subject to $\sum_l \beta_{i,l} = b_i$. And we know that $\sum_l p_l = 1$. In the second representation of the problem we have each agent i choosing $x_i \in \mathbf{R}^L_+$ to maximize $\sum_l \pi_{i,l} x_{i,l}$ subject to $\sum_l p_l x_{i,l} = b_i$. In this case we require that "in equilibrium" $\sum_i x_{i,l} = 1$. These are market clearing constraints for each l. They make sense because of horse l wins then there is $1 = \sum_i b_i$ monetary units to be distributed among the agents.

A ***pari-mutuel equilibrium*** is a pair (β, p), where $\beta = (\beta_{i,l})$ is such that β_i maximizes agent i's expected payoff ($\sum_l \pi_{i,l}\beta'_{i,l}/p_l$; taking p as given) subject to her budget constraint ($\sum_l \beta'_{i,l} = b_i$), and $p_l = \sum_{h=1}^I \beta_{h,l}$.

Given that $p_l = \sum_i \beta_{i,l}$, we have $\sum_l p_l = \sum_l \sum_i \beta_{i,l} = \sum_i b_i = 1$, so we can think of p as a vector of "market probabilities." Moreover, recall

[17]The vector of tickets should not be confused with a portfolio of L Arrow-Debreu securities because the payoff upon horse l winning depends on how much other agents bet on horse l.

that the track odds are $1/\sum_i \beta_{i,l} - 1 = \frac{1-p_l}{p_l}$ to 1. Therefore, prices p correspond one-to-one to track odds.

In the second representation of the problem: Agent i's problem is to maximize $\sum_l \pi_{i,l} x_{i,l}$ subject to $\sum_l p_l x_{i,l} = b_i$. The solution is given by

$$x_{i,l} > 0 \Rightarrow \theta_i = \frac{\pi_{i,l}}{p_l}, \text{ where } \theta_i = \max\{\frac{\pi_{i,l}}{p_l} : 1 \le l \le L\}.$$

Let $x = (x_{i,l})$ denote a vector of ticket choices.

Consider the function

$$\phi(x) = \sum_{i=1}^{I} b_i \log \left(\sum_{l=1}^{L} \pi_{i,l} x_{i,l} \right)$$

Suppose that x^* solves the problem of maximizing ϕ subject to $x_{i,l} \ge 0$ and $\sum_i x_{i,l} \le 1$. Note that x^* exists, and that $\sum_{l=1}^{L} \pi_{i,l} x^*_{i,l} > 0$ for all i; hence

$$\left.\frac{\partial\phi(x))}{\partial x_{i,l}}\right|_{x=x^*} = \frac{b_i \pi_{i,l}}{\sum_{k=1}^{L} \pi_{i,k} x^*_{i,k}}$$

is well defined.

The following result is due to Eisenberg and Gale (1959).

Theorem 84. *A pari-mutuel equilibrium* (β, p) *exists in which*

$$p_l = \max\{ \left.\frac{\partial\phi(x))}{\partial x_{i,l}}\right|_{x=x^*} : 1 \le i \le I\} \tag{13}$$

$$\beta_{i,l} = p_l x^*_{i,l}. \tag{14}$$

Moreover, equilibrium price p is unique.[18]

The idea in the proof is to consider the problem of allocating the vector $\mathbf{1} = (1, \ldots, 1) \in \mathbf{R}^L$ among the agents $i \in \{1, \ldots, I\}$ so as to maximize the social welfare function $W(U_1, \ldots, U_I) = \Pi_{i=1}^{I} U_i^{b_i}$, where $U_i(x_i) = \sum_l \pi_{i,l} x_{i,l}$. The prices p support the resulting solution (x_i) (and are therefore obtained as dual variables from the problem of allocating the vector $\mathbf{1}$).

Proof. Consider the choice of $x = (x_i) \ge 0$ in the program to maximize ϕ subject to the constraint that $\sum_i x_{i,l} \le 1$. The relevant Lagrangian

[18] The equilibrium βs are not generally unique.

is

$$L(x;\lambda) = \phi(x_1,\dots,x_I) + \sum_{l=1}^{L} \lambda_l (1 - \sum_{i=1}^{I} x_{i,l}) \tag{15}$$

$$= \sum_i [b_i \log(\sum_l \pi_{i,l} x_{i,l}) - \sum_l \lambda_l x_{i,l}] + \sum_l \lambda_l \tag{16}$$

Note that x^* satisfies the (KKT) first-order condition:

$$\left.\frac{\partial \phi(x)}{\partial x_{i,l}}\right|_{x=x^*} = \frac{b_i \pi_{i,l}}{\sum_l \pi_{i,l} x^*_{i,l}} \begin{cases} = \lambda_l \text{ if } x^*_{i,l} > 0 \\ \leq \lambda_l \text{ if } x^*_{i,l} = 0 \end{cases}$$

So we have $p_l = \lambda_l$ when p is defined as in the statement of the theorem.

We shall prove that x^*_i is utility maximizing, which will imply that $(\beta_{i,l})$, with $\beta_{i,l} = p_l x^*_{i,l}$, is utility maximizing. Observe that

$$\sum_l p_l x^*_{i,l} = \sum_{l:x^*_{i,l}>0} p_l x^*_{i,l} = \sum_{l:x^*_{i,l}>0} (\frac{x^*_{i,l} b_i \pi_{i,l}}{\sum_k \pi_{i,k} x^*_{i,k}}) = b_i,$$

as x^* satisfies the KKT conditions.

Note that x^* maximizes $L(x,(\lambda_l))$, given λ_l. That's just the standard saddle-point property of Lagrangians. Given the expression on Equation 16, this means that x^*_i maximizes $b_i \log(\sum_l \pi_{i,l} x_{i,l}) - \sum_i \lambda_l x_{i,l}$.

Recall that $\lambda_l = p_l$. So if y_i is in $i's$ budget set, we have $\sum_{l=1}^L \lambda_l y_{i,l} \leq \sum_{l=1}^L \lambda_l x^*_{i,l}$. So we must have that $\sum_l \pi_{i,l} y_{i,l} \leq \sum_l \pi_{i,l} x^*_{i,l}$.

Now, $\beta_{i,l} = \lambda_l x^*_{i,l}$ and $1 = \sum_i x^*_{i,l}$ implies that

$$\lambda_l = \sum_i \lambda_l x^*_{i,l} = \sum_i \beta_{i,l}.$$

We now turn to uniqueness. Suppose that (β, p) and (β', p') are two pari-mutuel equilibria. We shall prove that $p = p'$.

For each i, let $\mu_i = \max\{\pi_{i,l}/p_l : 1 \leq l \leq L\}$ and $\mu'_i = \max\{\pi_{i,l}/p'_l : 1 \leq l \leq L\}$. Observe that $\beta_{i,l}\mu_i p_l = \beta_{i,l}\pi_{i,l}$ because p_l canceles out when $\beta_{i,l} > 0$ (and both sides of the equation are zero otherwise). Since, $\pi_{i,l} \leq p'_l \mu'_l$, we obtain that $\beta_{i,l}\mu_i p_l \leq \beta_{i,l}\mu'_i p'_l$. Similarly, $\beta'_{i,k}\mu'_i p'_k \leq \beta'_{i,k}\mu_i p_k$.

Given that the elements of p and p' are strictly positive, these inequalities imply that

$$\beta_{i,l}\beta'_{i,k}\mu'_i\frac{p'_k}{p_k} \leq \beta_{i,l}\beta'_{i,k}\mu_i \leq \beta_{i,l}\beta'_{i,k}\mu'_i\frac{p'_l}{p_l}.$$

Cancel out μ'_i, which is strictly positive, and summing up:

$$b_i\sum_k \beta'_{i,k}\frac{p'_k}{p_k} = \sum_{l,k}\beta_{i,l}\beta'_{i,k}\frac{p'_k}{p_k} \leq \sum_{l,k}\beta_{i,l}\beta'_{i,k}\frac{p'_l}{p_l} = b_i\sum_l \beta_{i,l}\frac{p'_l}{p_l}$$

Again, $b_i > 0$ so $\sum_k \beta'_{i,k}\frac{p'_k}{p_k} \leq \sum_l \beta_{i,l}\frac{p'_l}{p_l}$. Adding over i gives

$$\sum_k p'_k\frac{p'_k}{p_k} \leq \sum_l p'_l = 1.$$

But then, using the Cauchy-Schwartz inequality we have that:

$$1 = (\sum_k p'_k)^2 = (\sum_k \sqrt{p_k}\frac{p'_k}{\sqrt{p_k}})^2 \leq \sum_k p_k \sum_k \left(\frac{p'_k}{\sqrt{p_k}}\right)^2 = \sum_k \frac{(p'_k)^2}{p_k} \leq 1.$$

This means that

$$\frac{(\sum_k \sqrt{p_k}\frac{p'_k}{\sqrt{p_k}})^2}{\sum_k p_k \sum_k \left(\frac{p'_k}{\sqrt{p_k}}\right)^2} = 1,$$

so the correlation between the vectors $\sqrt{p_k}$ and $p'_k/\sqrt{p_k}$ is $= 1$. Thus there is a scalar θ such that $\sqrt{p_k} = \theta p'_k/\sqrt{p_k}$. Which implies that p and p' are collinear. Since they are probabilities, we must have $p = p'$. □

In a pari-mutuel market, then, equilibrium "market probabilities" are given by

$$p_l = \max\{\frac{b_i\pi_{i,l}}{\sum_k \pi_{i,k}x^*_{i,k}} : 1 \leq i \leq I\}.$$

So that the market probability that horse l wins the race is, in a sense, the largest weighted subjective probability that some agent assigns to l winning.

Example 85. *Suppose that all agents agree on the probability that each horse wins, so that $\pi_{i,l} = \pi_l$ for all i. Then in any parimutuel equilibrium (β, p) an agent i will bet only on horse for which π_l/p_l is maximal. This means that $\pi_l/p_l = \pi_k/p_k$ for all l, k (otherwise we would have $\beta_{i,l} = 0$ for all i for some l, which by $p_l = \sum_i \beta_{i,l}$ would give $p_l = 0$, and this is not possible). Hence $\pi_l = p_l$, as both the (π_l) and (p_l) vectors add up to one.*

Example 86. *Suppose that* $I = L = 2$ *and* $\pi_1 = (1/2, 1/2) = (b_1, b_2)$. *We claim that* $p = (1/2, 1/2)$ *regardless of* π_2. *First, if* $\pi_2 = \pi_1$ *we obtain the desired conclusion by the argument in Example 85. So suppose that this is not the case and wlog that* $\pi_{2,1} < \pi_{1,1}$. *If* $p_1 > p_2$ *then both agents will only bet on horse 2 so we must have* $p_1 < p_2$. *This means that agent 1 bets on 1 and agent 2 bets on 2. Now,* $p_l = \sum_i \beta_{i,l}$ *so* $p_1 = b_1$ *and* $p_2 = b_2$. *Hence* $p_1 = p_2$, *a contradiction.*

15.3. **Arrow-Debreu and Radner equilibria.** Let $(\succeq_i, \omega_i)$ be an exchange economy, with $L = n|S|$ goods. We assume that each $\succeq_i$ is represented by a utility function U_i. We refer to the Walrasian equilibria of such an economy as ***Arrow-Debreu*** equilibria (and to contingent commodities as Arrow-Debreu commodities, but more about this later).

Now, in general we have that events unfold over time. In an Arrow-Debreu model one would trade contingent commodities, contingent on all possible events and dates at which the events might occur. It turns out that a simple insight shows that such complexity is not really needed.

Suppose that there are two time periods: $t = 0$ and $t = 1$. We can re-interpret the exchange economy model as trade in contingent goods taking place at time $t = 0$, and consumption taking place in $t = 1$.

Suppose instead that at time $t = 0$ agents buy contingent quantities of one good only: say good 1. Then agents solve the following problem P_i:

$$\max_{z_i \in \mathbf{R}^S, x_i \in \mathbf{R}^L} U_i(x_i)$$
$$\text{s.t.} \begin{cases} q \cdot z_i & \leq 0 \\ p_s \cdot x_s & \leq p_s \cdot \omega_{i,s} + p_{1,s} z_{i,s} \quad \forall s \in S \end{cases}$$

Each z_i is a ***portfolio*** of state-contingent contracts for delivery of good 1. The vector $q \in \mathbf{R}^S_+$ is a vector of prices at time $t = 0$ for the state-contingent quantities of good 1. The prices p_s are called ***spot prices***, and are valid for each of the spot markets that may open at time $t = 1$.

Definition 87. An allocation $(x_i)_{i=1}^I$ with portfolio choices $(z_i)_{i=1}^I$, spot prices $(p_s)_{s \in S}$, and contingent-good prices q constitute a ***Radner equilibrium*** if

(1) For each $i = 1, \ldots, I$, (x_i, z_i) solve consumer i's problem P_i, given (p_s) and q.

(2) For each $s \in S$:

$$\sum_{i=1}^{I} x_{i,s} = \sum_{i=1}^{I} \omega_{i,s},$$

and

(3) $\sum_{i=1}^{I} z_i = 0$.

Proposition 88. *Suppose that each U_i is strictly monotonic.*

(1) *Let (x^*, p^*) be an Arrow-Debreu equilibrium. Then there is $(z_i^*)_{i=1}^{I}$ and q such that (x^*, p^*, z^*, q^*) is a Radner equilibrium.*
(2) *Let (x^*, p^*, z^*, q^*) be a Radner equilibrium, then there is $\mu_s^* > 0$, $s \in S$, such that $(x^*, (\mu_s^* p_s^*)_{s \in S})$ is an Arrow-Debreu equilibrium.*

16. Asset Pricing

We now turn to a different two-period model. We assume consumption at both times $t = 0$ and $t = 1$. We also assume that there is a single physical good.

So time is indexed by $t = 0, 1$. There is uncertainty at time $t = 0$, captured through a state space, and uncertainty is resolved at time $t = 1$.

- Let $S = \{s_1, s_2, \ldots, s_m\}$ be a set of ***states***.
- A column vector $c \in \mathbf{R}^{1+m}$ is called a ***cash flow***.
- A column vector $a = (a_1, a_2, \ldots, a_m)' \in \mathbf{R}^m$ is called an ***asset***, where a_k is the payment of asset a (in terms of *the* good) in period 1 under state s_k for $k = 1, \ldots, m$.
- Let $\{a^1, a^2, \ldots, a^J\}$ be a collection of J assets. Collect them all in a matrix A with j-th column equal to a^j. That is:

$$A = \begin{bmatrix} a^1 & a^2 & \cdots & a^J \end{bmatrix}_{m \times J}$$

Examples of assets include: a ***risk-free*** asset $a^{rf} = (1, 1, \ldots, 1) \in \mathbf{R}^m$, or the ***Arrow-Debreu security*** $a_k^{AD} = e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ which delivers a unit of the good if and only if the realized state is s_k.

Another example is an ***option***.[19] Suppose that $s_j = j$, for example the state could be the value of a stock market index, and consider an

[19] We're describing an option to buy, a so-called call option. An option to sell is termed a put option.

option to buy the index at a fixed "strike" price p. We can write this as an asset

$$a = (0, \dots, 0, j - p, (j+1) - p, \dots m - p),$$

where $j - p \geq 0$ and $(j-1) - p) < 0$. An option to buy at price p will only be exercised when the price exceeds p, and will give a payoff $s - p$.[20]

Let $q_j \in \mathbf{R}_+$ be the price of asset a^j. Purchasing one unit of asset a^j generates the following cash flow:

$$(-q_j, a_1^j, a_2^j, \dots, a_m^j)' \in \mathbf{R}^{1+m}.$$

The cash flow generated by selling one unit of asset a^j at price q_j is:

$$(q_j, -a_1^j, -a_2^j, \dots, -a_m^j)' \in \mathbf{R}^{1+m}.$$

- Let $q = (q_1, \dots, q_J) \in \mathbf{R}_+^J$ be the vector of asset prices, where q_j is the price of asset a^j.
- Define W as the following matrix:

  $$W = \begin{bmatrix} -q \\ A \end{bmatrix}_{(1+m)\times J}$$

  Note that the j-th column of W, W_j, is the cash flow generated by purchasing one unit of asset a^j. That is:

  $$W_j = \begin{pmatrix} -q_j \\ a_1^j \\ \vdots \\ a_m^j \end{pmatrix}_{(1+m)\times 1}$$

- A column vector $z \in \mathbf{R}^J$ is called a *portfolio*. A portfolio z generates a cash flow Wz. In period zero, the porfolio z pays $-\sum_{j=1}^J z_j q_j$, and in period 1 if state $k \in \{1, \dots, m\}$ is realized, it pays $\sum_{j=1}^J z_j a_k^j$. That is,

(17)

$$Wz = \begin{bmatrix} -q \\ A \end{bmatrix} z = \begin{bmatrix} -q_1 & \cdots & -q_j & \cdots & -q_J \\ a_1^1 & \cdots & a_1^j & \cdots & a_1^J \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_m^1 & \cdots & a_m^j & \cdots & a_m^J \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_J \end{pmatrix} = \begin{pmatrix} -\sum_{j=1}^J z_j q_j \\ \sum_{j=1}^J z_j a_1^j \\ \vdots \\ \sum_{j=1}^J z_j a_m^j \end{pmatrix}_{(1+m)\times 1}$$

[20]More jargon: when $s > p$ we say that the option is "in the money;" otherwise it is "out of the money."

We denote by $\langle W\rangle$ the set of cash flows that can be achieved through a portfolio of the assets $a^1, \dots, a^J$ at prices q.

$$\langle W\rangle = \{Wz : z \in \mathbf{R}^J\}$$

Observation 89. $\langle W\rangle$ is a linear subspace of $\mathbf{R}^{1+m}$. It is the linear subspace generated by the columns of W.

Definition 90. A ***market*** is a pair (A, q).

Definition 91. An ***arbitrage opportunity*** is a cash flow $c \in \mathbf{R}^{1+m}$ such that $c > 0$. We say that a market (A, q) is ***free of arbitrage opportunities*** if there is no arbitrage opportunity in $\langle W\rangle$.

So a market is free of arbitrage opportunities iff there is no portfolio $z \in \mathbf{R}^J$ such that $Wz > 0$. This means that there is no portfolio that generates a strictly positive payoff in some state or time, without also generating some negative payoff.

- Let $\omega \in \mathbf{R}^{1+m}_+$ be an ***endowment*** vector.
- The *budget set* associated to market (A, q) and endowment ω is given by:

  $$\mathbf{B}(\omega, A, q) = \{x \in \mathbf{R}^{1+m} : \exists\, z \in \mathbf{R}^J \text{ s.th. } x \leq \omega + Wz\}.$$

Theorem 92 (Fundamental Theorem of arbitrage pricing)**.** *Let (A, q) be a market. The following statements are equivalent:*

(i) *for any continuous and strictly monotonic utility function $u : \mathbf{R}^{1+m}_+ \to \mathbf{R}$ the following problem has a solution*

$$\max_x \quad u(x) \quad \text{subject to} \quad x \in \mathbf{B}(\omega, A, q); \tag{18}$$

(ii) *the market (A, q) is free of arbitrage opportunities;*
(iii) *$\exists\, \pi \in \mathbf{R}^{1+m}_{++}$ such that $\pi W = 0$;*
(iv) *$\mathbf{B}(\omega, A, q)$ is compact and there is $\pi \in \mathbf{R}^{1+m}_{++}$ such that*

$$B(\omega, A, q) \subseteq \{x \in \mathbf{R}^{1+m}_+ : \pi \cdot x \leq \pi \cdot \omega\}.$$

Proof. $(i) \Rightarrow (ii)$. If $c \in \mathbf{R}^{1+m}$ is an arbitrage opportunity in market (A, q), then there exists a portfolio z^* s.th. $c = Wz^* > 0$. For any $x = \omega + W\hat{z} \in B(\omega, A, q)$,

$$x < x' = x + c = \omega + W(\hat{z} + z^*) \in B(\omega, A, q)$$

So $u(x) < u(x')$ as u is strictly monotonic. Then (18) has no solution.

$(ii) \Rightarrow (iii)$. Suppose that (A, q) admits no arbitrage opportunity. Then

$$\langle W \rangle \cap \mathbf{R}^{1+m}_{+} = \{0\}.$$

Let $\Delta \subseteq \mathbf{R}^{1+m}_{+}$ be the simplex in $\mathbf{R}^{1+m}_{+}$, so

$$\Delta = \left\{ p \in \mathbf{R}^{1+m}_{+} : \sum_i p_i = 1 \right\}.$$

Then, $\langle W \rangle \cap \Delta = \emptyset$. $\langle W \rangle$ is a closed and convex set, and Δ is compact and convex.

Then, $\exists\, \pi \in \mathbf{R}^{1+m}$ such that

$$\pi \cdot c < \pi \cdot p \quad \forall\, c \in \langle W \rangle \text{ and } p \in \Delta \tag{19}$$

by the Strict Separating-hyperplane Theorem.[21] Note that $0 \in \langle W \rangle$ and e_l (the l-th unit vector) in $\mathbf{R}^{1+m}$ is in Δ, so $\pi \cdot 0 < \pi \cdot e_l = \pi_l\ \forall\, l$. Hence, $\pi \gg 0$.

To show that $\pi W = 0$, suppose towards a contradiction that $\pi W \neq 0$. Then we can choose $z \in \mathbf{R}^J$ such that $\pi W z > 0$. Then for all α we have

$$\alpha \pi W z = \pi W(\alpha z) \leq \sum \{\pi \cdot p : p \in \Delta\}.$$

This is a contradiction because we can choose $\alpha > 0$ to make $\alpha \pi W z$ arbitrarily large while $p \mapsto \pi \cdot p$ is a bounded function over $p \in \Delta$.

$(iii) \Rightarrow (iv)$. Let $x \in \mathbf{B}(\omega, A, q)$. Then

$$x \leq \omega + Wz \Rightarrow \pi x \leq \pi \omega,$$

as $\pi W z = 0$. Therefore,

$$\mathbf{B}(\omega, A, q) \subseteq \{x \in \mathbf{R}^{1+m} : \pi x \leq \pi \omega\}.$$

The right-hand side set of the above equation is a standard "Arrow-Debreu budget set," so it is compact. $\mathbf{B}(\omega, A, q)$ is a closed subset of a compact set, so it is compact.

$(iv) \Rightarrow (i)$. If the budget set $\mathbf{B}(\omega, A, q)$ is compact, then problem (18) has a solution because u is continuous. □

[21] Compare this version of the Separating-hyperplane Theorem with the one given in Lemma 15. Note that it is necessary that at least one of the sets is compact. Consider $X = \{(x, y) \in \mathbf{R}^2 : x \geq 0, y \geq 1/x\}$ and $Z = \{(x, y) \in \mathbf{R}^2 : y \leq 0\}$.

16.1. **Digression: Farkas Lemma.** In the proof of Theorem 92 we basically proved the following useful result.

Lemma 93. *Let W be a $(n \times m)$ matrix. Then one and only one of the following statements is true.*

(1) *There is $z \in \mathbf{R}^m$ such that $Wz > 0$.*
(2) *There is $\pi \in \mathbf{R}^n_{++}$ such that $\pi W = 0$.*

Proof. In the proof of the theorem, we proved that (1) being false implies that (2) must be true. All that is left is to observe that (1) and (2) cannot hold at the same time because it would lead to $0 = \pi W z > 0$, which is absurd. □

16.2. **State prices.** The vector π in the theorem may be intriguing. It has the interpretation as a price vector: in fact as a vector of Arrow-Debreu prices.

$$\pi W = \pi \begin{bmatrix} -q \\ A \end{bmatrix} = \pi \begin{bmatrix} -q_1 & \cdots & -q_j & \cdots & -q_J \\ a_1^1 & \cdots & a_1^j & \cdots & a_1^J \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_m^1 & \cdots & a_m^j & \cdots & a_m^J \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}_{(1+m)\times J}$$

Therefore, for every $j = 1, \ldots, J$,

$$(\pi_0, \pi_1, \ldots, \pi_m) \begin{bmatrix} -q_j \\ a_1^j \\ \vdots \\ a_m^J \end{bmatrix} = \pi_0(-q_j) + \sum_{i=1}^m \pi_i a_i^j = 0.$$

We may write

$$q_j = \sum_{i=1}^m \left(\frac{\pi_i}{\pi_0}\right) a_i^j \quad \forall\, j = 1, \ldots, J. \tag{20}$$

Hence, we may write the price of asset j, q_j, at time $t = 0$ as a weighted sum of it future payments under the different states $a_1^j, \ldots, a_m^j$. The weight on the payoff a_i^j in state s_i, is $\frac{\pi_i}{\pi_0}$.

Normalize the vector π by defining

$$\bar{\pi} = \left(1, \frac{\pi_1}{\pi_0}, \frac{\pi_2}{\pi_0}, \ldots, \frac{\pi_m}{\pi_0}\right).$$

$\pi W = 0$ implies that $\frac{\pi}{\pi_0} W = \bar{\pi} W = 0$.

Therefore, rewrite (20) as:

$$q_j = \sum_{i=1}^{m} \bar{\pi}_i a_i^j \quad \forall\, j = 1, \ldots, J. \tag{21}$$

From the expression above it becomes clear that $\bar{\pi} \in \Delta \subseteq \mathbf{R}^m_{++}$ is a price vector. The price of a unit of good (of money, say) at time $t = 0$ is 1. The price of a unit of good at time $t = 1$ conditional on state s_i occurring is $\bar{\pi}_i$. We refer to $\bar{\pi}$ as a *normalized state price vector*, and call the expression for q_j given in (21) the *present value price* of asset a^j.

The state price vector $\bar{\pi}$ allows us to price assets. For example, let $y_i \in \mathbf{R}$ for every state $i = 1, \ldots, m$, and define the column vector $\tilde{y} = (y_1, y_2, \ldots, y_m)' \in \mathbf{R}^m$, so that $\tilde{y}$ is an asset. We can use $\bar{\pi}$ to calculate the price of asset $\tilde{y}$ as: $\sum_{i=1}^m \bar{\pi}_i y_i$.

A ***risk-free*** asset is $a^{rf} = (1, \ldots, 1) \in \mathbf{R}^m$. In a market with no arbitrage, then, the price of the risk free asset is given by:

$$q^{rf} = \sum_{i=1}^{m} \bar{\pi}_i. \tag{22}$$

Define the *rate of return* of the risk free asset as

$$R^{rf} = \frac{1}{q^{rf}} = \frac{1}{\sum_{i=1}^m \bar{\pi}_i}. \tag{23}$$

The rate R^{rf} is known as the *risk-free rate*.

16.2.1. *A risk-neutral probability measure.* For any asset j, the ***expected rate of return on*** j is the expected value of the random variable $s \mapsto a^j_s/q^j$, which we denote by $(R^j_s)_{s\in S}$, or by $\tilde{R}^j$.[22]

The expectation of $\tilde{R}^j$ depends of course on the probability measure used. It turns out that a particular probability measure is useful. Let

$$p_i = \frac{\bar{\pi}_i}{\sum_{h=1}^m \bar{\pi}_h}, \text{ so that } p = (p_1, \ldots, p_m) \in \Delta$$

is a probability distribution over S. The probability distribution p is termed the ***risk free probability measure***. The name comes from

[22] A random variable $s \mapsto X_s$ is denoted by $\tilde{X}$.

the following calculation:

$$\frac{q^j}{q^{rf}} = \sum_{h=1}^m p_h a_h^j = \mathbf{E}_p \tilde{a}^j.$$

Thus,

$$R^{rf} = \frac{1}{q^{rf}} = \sum_{i=1}^m p_i \frac{a_s^j}{q^j} = E_p \tilde{R}_s^j.$$

So the expected return of asset j, *when the expectation is calculated using risk-neutral probabilities,* equals the risk-free rate.

16.3. **Application: Put-call parity.** An *option* to buy an asset a^j at a price K is called a *call option.* The price K is called the *strike price.* Since the option is exercised only when profitable, i.e., if the market price is higer than K, the call option on a^j has payoff $(a_i^j - K)^+$ in state $i = 1, \dots, m$.[23] Hence, the price of the call option is given by:

$$q^c = \sum_{i=1}^m \bar{\pi}_i (a_i^j - K)^+.$$

An option to sell an asset a^j at strike price K is called a *put option.* Since the option is exercised only when profitable, i.e., if the market price is lower than K, the put option on a^j has payoff $(K - a_i^j)^+$ in state $i = 1, \dots, m$. Hence, the price of the put option is given by:

$$q^p = \sum_{i=1}^m \bar{\pi}_i (K - a_i^j)^+.$$

Note that

$$\begin{aligned} q^c - q^p &= \sum_{i=1}^m \bar{\pi}_i (a_i^j - K)^+ - \sum_{i=1}^m \bar{\pi}_i (K - a_i^j)^+ \\ &= \sum_{i=1}^m \bar{\pi}_i (a_i^j - K) \\ &= q^j - \sum_{i=1}^m \bar{\pi}_i K \\ &= q^j - K q^{rf}. \end{aligned}$$

[23]For any $x \in \mathbf{R}$, $x^+ = \max\{x, 0\}$.

Then, we obtain:

$$q^j - Kq^{rf} = q^c - q^p. \tag{24}$$

Equation (24) is called ***put-call parity***. The left hand side of (24) is the payoff from selling one unit of asset j and borrowing K dollars at the risk-free rate. The right-hand-side is the expense that results from purchasing a call option and selling a put option for the same strike price of K. No-arbitrage demands that these two quantities must be the same because they result in the same state-contingent payoffs.

16.4. **Market incompleteness.** Let (q, A) be a financial market, and define the matrix W as before. Suppose that the market is free of arbitrage.

Definition 94. The market (q, A) is ***complete*** if $\dim(\langle W \rangle) = |S| = m$. Otherwise say that the market (q, A) is ***incomplete***.

When a market is complete, agents can use the assets to carry out transfers of the good (of "money," or "wealth") across states.

Proposition 95. *The market* (q, A) *is complete iff* $\dim(\langle A \rangle) = m$.

Proof. When (q, A) is free of arbitrage, $q = \bar{\pi} \cdot A$. So q is a linear combination of the rows of A. Therefore, A and W have the same row rank. □

The importance of this proposition is that market completeness is a matter of A, which is exogenous. Not prices q. In a more general model with more than one good, market completeness is however dependent on prices.

When the market is free of arbitrage, then $\pi W = 0$ implies that $\pi \in \langle W \rangle^\perp$, so that $\langle W \rangle^\perp \neq \emptyset$. Since $\pi \neq 0$, $\dim(\langle W \rangle^\perp) \geq 1$.

Observation 96. A market is complete iff there is a unique (up to a scalar multiple) solution to the system of equations $\pi W = 0$. The uniqueness is unique up to a scalar multiple: For example, normalizing $\pi_0 = 1$, when the market is complete there is a unique $\bar{\pi} \in \mathbf{R}^M_{++}$ with $q = \pi A$.

To see why Observation 96 is true: Let $\langle W \rangle^\perp$ denote the set of vectors orthogonal to the vectors in $\langle W \rangle$, called the ***orthogonal complement*** of $\langle W \rangle$:

$$\langle W \rangle^\perp = \{z \in \mathbf{R}^{1+m} : z \cdot y = 0 \text{ for all } y \in \langle W \rangle\}.$$

Note that

$$\mathbf{R}^{1+m} = \langle W \rangle + \langle W \rangle^{\perp}.$$

Given lack of arbitrage, $\dim(\langle W \rangle^{\perp}) \geq 1$. The subspace $\langle W \rangle^{\perp}$ has dimension 1 iff the dimension of $\langle W \rangle$ is m. And π is unique (up to a scalar multiple) iff $\langle W \rangle^{\perp}$ has dimension 1.

Uniqueness of the state prices that result from the "fundamental theorem" is a matter of market completeness. For this reason, Observation 96 is some times called the "second fundamental theorem of asset pricing."

Theorem 97. *In the one-good, two-period model with C^{∞} utilities. If the market is incomplete then there is a set of vectors of endowments of measure zero such that for all endowments outside of this set, every equilibrium allocation is Pareto inefficient.*

The theorem is stated informally. Since market completeness in this model is a matter of A, not q, we can make sense of incompleteness independently of prices (which are endogenous, of course, and determined in equilibrium).

A natural question to ask is about Pareto efficiency constrained by the transfers across states that are possible given the asset structure A. It turns out that, in the one-good model, equilibrium allocations are constrained Pareto efficient, but this is no longer true when we allow for multiple goods. In fact, generically, equilibrium allocations are constrained Pareto inefficient.

16.5. **The Capital Asset Pricing Model (CAPM).** The traditional CAPM is :

$$\mathbf{E}\tilde{R}^j = R^{rf} + \beta(\mathbf{E}\tilde{R}^m - R^{rf}),$$

where R^m is the "market" rate of return. In practice, R^m is taken to be the return of some market index such as the SP500. The CAPM equation is a linear regression:

$$\beta = \frac{\mathbf{Cov}(\tilde{R}^j, \tilde{R}^m)}{\mathbf{V}(\tilde{R}^m)}.$$

The CAPM means that the expected return of an asset j is given by the risk-free rate + a *risk premium* $\beta(\mathbf{E}R^m - R^{rf})$ that depends on the "beta" of the asset j. The beta of an asset depends on how closely it is related to the market returns. An asset that varies closely with the market returns (has high beta) has high systemic risk and therefore commands a larger risk premium.

The idea behind the CAPM is that optimally choosing a portfolio will allow an agent to fully diversify all risk that is idiosyncratic to the asset. So there is no risk premium for such "diversifiable" risk. Systemic risk cannot be diversified away, and it is reflected in the expected return of the asset.

16.5.1. *A CAPM formula resulting from no arbitrage.* The expected returns in the CAPM are calculated according to some given probability distribution over states (not necessarily the risk free measure). Let $\hat{p} \in \Delta$ be a probability measure on $\{1, \dots, m\}$.

Then

$$q^j = \sum_{i=1}^{n} \bar{\pi}_i a_i^j = \sum_{i=1}^{n} \frac{\bar{\pi}_i}{\hat{p}_i} \hat{p}_i a_i^j = \sum_{i=1}^{n} \theta_i p_i a_i^j = E_{\hat{p}} \tilde{\theta} \tilde{a}^j. \tag{25}$$

We use $\tilde{\theta}$ to denote the random variable $i \mapsto \frac{\bar{\pi}_i}{\hat{p}_i}$.

Equation 25 is important. The random variable $\tilde{\theta}$ is a ***stochastic discount factor***. It says how a payoff in each state should be "discounted" so as to give the correct, arbitrage free, price q^j. We see many instances of such stochastic discount factors in asset pricing, this is perhaps the simplest one.

For any two random variables $\tilde{X}$ and $\tilde{Y}$, we have that $\mathbf{Cov}_{\hat{p}}(\tilde{X}, \tilde{Y}) = \mathbf{E}_{\hat{p}} \tilde{X}\tilde{Y} - \mathbf{E}_{\hat{p}} \tilde{X} \mathbf{E}_{\hat{p}} \tilde{Y}$. So,

$$q^j = \mathbf{E}_{\hat{p}} \tilde{\theta} \tilde{a}^j = \mathbf{Cov}_{\hat{p}}(\tilde{\theta}, \tilde{a}^j) + \mathbf{E}_{\hat{p}} \tilde{\theta} \mathbf{E}_{\hat{p}} \tilde{a}^j.$$

Now, note that $\mathbf{E}_{\hat{p}} \tilde{\theta} = \sum_i \bar{\pi}_i = q^{rf} = 1/R^{rf}$.

Then,

$$\mathbf{E}_{\hat{p}} \tilde{R}^j = \frac{\mathbf{E}_{\hat{p}} \tilde{a}^j}{q^j} = R^{rf} - \frac{R^{rf}}{q^j} \mathbf{Cov}_{\hat{p}}(\tilde{\theta}, \tilde{a}^j) = R^{rf} - \mathbf{Cov}_{\hat{p}}(\tilde{\nu}, \tilde{R}^j) \tag{26}$$

(where $\tilde{\nu} = R^{rf} \tilde{\theta}$).

Equation (26) is a "CAPM-like" formula. It says that the expected return on an asset j equals the risk-free rate of return plus a term that depends on the correlation of the asset returns with a "market variable" $\tilde{\nu}$. The variable $\tilde{\nu}$ is common to all assets, that is why it is a market variable.

16.6. **Consumption CAPM.** Consider a consumer with income I, who can invest in J assets. The (random) rate of return on asset j is $\tilde{R}^j$. Asset 1 is a risk-free asset.

Let x_0 denote consumption (of "money") on date 0. So that the agent uses $I - x_0$ to invest in assets for consumption on date 1. Investment in asset j is

$$z_j = (I - x_o)\eta^j,$$

where $\eta = (\eta^j)_{j=1}^J \in \mathbf{R}_+^J$ with $\sum_j \eta^j = 1$.

Then the random payoff of a portfolio defined by η is

$$\tilde{x}_1 = (I - x_o)\sum_{j=1}^J \eta^j \tilde{R}^j = (I - x_o)\left[R^1 + \sum_{j=2}^J \eta^j(\tilde{R}^j - R^1)\right].$$

The consumer's problem is

$$\max u(x_0) + \delta \mathbf{E} u\left\{(I - x_o)\left[R^1 + \textstyle\sum_{j=2}^J \eta^j(\tilde{R}^J - R^1)\right]\right\}$$
$$\text{st}\begin{cases} 0 \leq x_0 \leq I \\ \eta_j \geq 0 \\ \sum_{j=2}^J \eta^j \leq 1 \end{cases}$$

Suppose that $u : \mathbf{R}_+ \to \mathbf{R}$ is smooth, monotonic and concave. Suppose also that we can focus on interior solutions. Then the first order conditions that characterize a solution are:

$$u'(x_0) = \delta \mathbf{E} u'(\tilde{x}_1)\tilde{R}$$

where $\tilde{R} = R^1 + \sum_j \eta^j(\tilde{R}^j - R^1)$, and

$$\delta \mathbf{E} u'(\tilde{x}_1)(I - x_0)(\tilde{R}^j - R^1) = 0, \quad j = 2, \ldots, J.$$

The first equality, $u'(x_0) = \delta \mathbf{E} u'(\tilde{x}_1)\tilde{R}$ is a so-called *Euler equation,* describing the intertemporal trade-off between consumption and saving.

The second collection of equations serve to give a CAPM formula for the returns of each of the $J - 1$ risky assets. Specifically, since $\delta > 0$ and $(I - x_0) > 0$ we have that

$$\mathbf{E} u'(\tilde{x}_1)(\tilde{R}^j - R^1) = 0, \quad j = 2, \ldots, J.$$

Recall the properties of the covariance:

$$\begin{aligned}0 &= \mathbf{E}u'(\tilde{x}_1)(\tilde{R}^j - R^1)\\ &= \mathbf{Cov}(u'(\tilde{x}_1), \tilde{R}^j - R^1) + \mathbf{E}u'(\tilde{x}_1)\mathbf{E}(\tilde{R}^j - R^1)\\ &= \mathbf{Cov}(u'(\tilde{x}_1), \tilde{R}^j) + \mathbf{E}u'(\tilde{x}_1)\mathbf{E}(\tilde{R}^j - R^1)\end{aligned}$$

Then

$$\mathbf{E}\tilde{R}^j = R^1 - \frac{1}{\mathbf{E}u'(\tilde{x}_1)}\mathbf{Cov}(u'(\tilde{x}_1), \tilde{R}^j).$$

If asset j is positively correlated with consumption then it is negatively correlated with marginal utility (concavity of u). So it commands a positive "risk premium" and $\mathbf{E}\tilde{R}^j > R^1$.

16.7. **Lucas Tree Model.** We turn to a one-agent exchange economy with many goods. Specifically, consumption occurs over time, and it is uncertain. Endowments are also stochastic, and arrive over time. The problem is to characterize prices that support the autarky equilibrium where the agent consumes the endowment.

There is a single good, "fruit," in each period, and a single asset: a "tree." The tree pays off "dividends," a random production of fruit in every period. Time is infinite, and ranges from $t = 0, 1, \ldots$ In period t, the production of fruit is realized, and a spot market opens up in fruit. The consumer can therefore sell and purchase fruit in the spot market. She can also buy trees. We shall normalize the price of fruit in each spot market to be 1, and determine the price q_t of trees in period t.

To sum up, in period t the consumer has an income derived from her holdings of trees, and how much fruit these have produced. If each tree has a dividend d_t and she holds s_t trees, then her income in period t is $w_t = s_t(q_t + d_t)$. This income can be used to purchase fruit (at a price of 1) for consumption, c_t, and trees. If she buys s_{t+1} trees then she spends $a_t = s_{t+1}q_t$ on trees. So her budget constraint for period t is $c_t + a_t \leq w_t$.

The rate of return on trees is

$$R_{t+1} = \frac{q_{t+1} + d_{t+1}}{q_t},$$

composed of a capital gain component q_{t+1}/q_t, and a dividend payoff. Note that $R_{t+1}a_t = s_{t+1}(q_{t+1} + d_{t+1}) = w_{t+1}$

The consumer seeks to maximize the expected discounted sum of per-period utility. The consumer has a utilty function $u : \mathbf{R}_+ \to \mathbf{R}$ over fruit. Suppose that u is C^1, strictly increasing and concave. The consumer has a discount factor $\delta \in (0,1)$.

$$\begin{array}{c} \text{Max}\mathbf{E}\sum_{t=0}^{\infty}\delta^t u(c_t) \\ \text{s.t}\begin{cases} c_t + a_t & \leq w_t \\ w_{t+1} & = R_{t+1}a_t \end{cases} \end{array}$$

Assume that $\{R_t\}$ follows a Markov process.

The Bellman equation is

$$v(w,R) = \sup\{u(c) + \delta\mathbf{E}[v(\tilde{R}a,\tilde{R})|R] : c + a = w\}.$$

Consider the problem on the right-hand-side of the Bellman equation

$$\begin{array}{c} \max u(w-a) + \delta\mathbf{E}[v(\tilde{R}a,\tilde{R})|R] \\ \text{s.t. } 0 \leq a \leq w \end{array}$$

Suppose that the solution is interior and that the value function v is differentiable. Use v_1' to denote $\frac{\partial v(w,R)}{\partial w}$. Then the first-order condition for a maximum in the right-hand side of the Bellman equation is

$$u'(w-a) = \delta\mathbf{E}[v_1'(\tilde{R}a,\tilde{R})\tilde{R}|R].$$

Moreover, the the envelope theorem gives us that

$$v_1'(w,R) = u'(w-a).$$

Thus, along an optimal solution we must have that

$$u'(c) = \delta\mathbf{E}[u'(\tilde{c})\tilde{R}|R].$$

Specifically, if $\{c_t\}$ is an optimal path we must have that

$$u'(c_t) = \delta\mathbf{E}_t[u'(c_{t+1})R_{t+1}], \tag{27}$$

where $\mathbf{E}_t$ means expectation conditional of R_t. Equation (27) is an *Euler equation.*

We can write the Euler equation as

$$1 = \delta\mathbf{E}_t\frac{u'(c_{t+1})}{u'(c_t)}R_{t+1},$$

and the Euler equation implies that

$$q_t = \mathbf{E}_t \underbrace{\delta \frac{u'(c_{t+1})}{u'(c_t)}}_{M_t^{t+1}} (q_{t+1} + d_{t+1}).$$

Here M_t^{t+1} is the ***stochastic discount factor*** for consumption in period $t+1$ at time t.

Then

$$\mathbf{E}_t q_{t+1} = \mathbf{E}_t \mathbf{E}_{t+1} M_{t+1}^{t+2}(q_{t+2} + d_{t+2}) = \mathbf{E}_t M_{t+1}^{t+2}(q_{t+2} + d_{t+2})$$

by the "law of iterated expectations." Note also that if we define $M_t^{t+\tau} = \delta^\tau u'(c_{t+\tau})/u'(c_t)$ then $M_t^{t+1} M_{t+1}^{t+2} = M_t^{t+2}$. Hence we obtain that

$$\begin{aligned} q_t &= \mathbf{E}_t M_t^{t+1} M_{t+1}^{t+2}(q_{t+2} + d_{t+2}) + \mathbf{E}_t M_t^{t+1} d_{t+1}) \\ &= \mathbf{E}_t M_t^{t+2}(q_{t+2} + d_{t+2}) + \mathbf{E}_t M_t^{t+1} d_{t+1}) \\ &= \mathbf{E}_t M_t^{t+2} q_{t+2} + \mathbf{E}_t M_t^{t+2} d_{t+2} + \mathbf{E}_t M_t^{t+1} d_{t+1}). \end{aligned}$$

Continuing in this fashion we obtain that

$$q_t = \mathbf{E}_t \sum_{\tau=1}^{T} M_t^{t+\tau} d_{t+\tau} + \mathbf{E}_t M_t^{t+T} q_{t+T}.$$

If we assume that $\lim_{T\to\infty} \mathbf{E}_t M_t^{t+T} q_{t+T} = 0$, which means ruling out "bubbles" where agents do not expect at time t that the process $\{M_t^{t+\tau} q_{t+\tau}\}$ converges to zero, then we obtain that

$$q_t = \lim_{T\to\infty} \mathbf{E}_t \sum_{\tau=1}^{T} M_t^{t+\tau} d_{t+\tau} = \mathbf{E}_t \sum_{\tau=1}^{\infty} M_t^{t+\tau} d_{t+\tau},$$

assuming that we can pass the limit inside the expectation. Thus the price of the tree in period t is the discounted expected sum of future dividends. The discount factor is a ***subjective stochastic discount factor***; it depends on marginal utility and pure time discounting.

Consider now an equilibrium of this economy. There is a single tree and a single consumer. So equilibrium (market clearing) in the period t spot market requires that $c_t = d_t$ and $s_t = 1$. Hence we have that

$$q_t = \mathbf{E}_t \sum_{\tau=1}^{\infty} \delta^\tau \frac{u'(d_{t+\tau})}{u'(d_t)} d_{t+\tau}.$$

16.8. **Risk-neutral consumer and the martingale property.** The consumer is risk-neutral when u is linear, so that $u'(c_{t+1})/u'(c_t) = 1$. Then we have

$$q_t = \mathbf{E}_t \delta(q_{t+1} + d_{t+1}). \tag{28}$$

So adjusting for dividends and time-discounting, the price of trees follows a martingale process. This is the efficient markets hypothesis.

As we shall see next, the martingale property holds in the absence of risk neutrality, using the risk-free equivalent measure to calculate expectations (or the equivalent martingale measure).

16.9. **Finite state space.** We can simplify the calculations a bit, and gain some intuition from considering the model with a finite set of possible dividends. Suppose that d_t takes values in the set $D = \{d^1, \ldots, d^L\}$. When $d_t = d^j$ then d_{t+1} is drawn from D according to the probability distribution $P_j = (P_{j,1}, \ldots, P_{j,L})$. So $\Pr(d_{t+1} = d^l | d + t = d^j) = P_{i,l}$.

In this case we can write the price of trees as a function of the value of d_t, and we obtain that

$$q(d^j) = \sum_{l=1}^{L} P_{j,l} \delta \frac{u'(d^l)}{u'(d^j)} (q(d^l) + d^l).$$

Define a new probability $\bar{P}_j$ by

$$\bar{P}_{j,l} = \frac{P_{j,l}\delta u'(d^l)/u'(d^j)}{\sum_k P_{j,k}\delta u'(d^k)/u'(d^j)} = \frac{P_{j,l} u'(d^l)}{\sum_k P_{j,k} u'(d^k)}$$

and denote expectation with respect to this probability by $\bar{\mathbf{E}}_j$. Let $R_j = (\sum_k P_{j,k}\delta u'(d^k)/u'(d^j))^{-1}$: think of R_j as the rate of return on a risk-less asset (one that pays one unit of fruit in the next period, regardless of the state).

Then we have

$$q(d^j) = R_j^{-1} \bar{\mathbf{E}}_j (q(d^{\tilde{l}}) + d^{\tilde{l}}).$$

Note that this is an equation like (28), expressing that the dividend-adjusted price follows a martingale-type property.

Now the probabilities $\bar{P}_{j,l}$ define a probability measure over sequences of dividends. For example the probability that $d_{t+2} = d^h$ and $d_{t+1} = d^l$

given that $d_t = d^j$ is $\bar{P}_{j,l}\bar{P}_{l,h}$. And the marginal probability that $d_{t+2} = d^h$ given that $d_t = d^j$ is

$$\bar{P}^2_{j,h} = \sum_l \bar{P}_{j,l}\bar{P}_{l,h} = \sum_l \frac{P_{j,l}u'(d^l)}{\sum_k P_{j,k}u'(d^k)} \frac{P_{l,h}u'(d^h)}{\sum_k P_{l,k}u'(d^k)}$$

Let P^*_j be the resulting probability measure on sequences of dividends $d_{t+1}, d_{t+2}, \ldots$, given that $d_t = d^j$; and denote its expectation operator by $\mathbf{E}^*_j$.

Then if $d_t = d^j$ we have that the "cum dividend" price of the tree is

$$\hat{q}_t = q(d_t) + d_t = d^j + q(d^j) = \bar{\mathbf{E}}_j R_j^{-1}\hat{q}_{t+1} = \mathbf{E}^*_t R_t^{-1} R_{t+1}^{-1}\hat{q}_{t+2}.$$

Hence, for any $\tau > 1$,

$$\hat{q}_t - d_t = \mathbf{E}^*_t \frac{\hat{q}_{t+\tau}}{\Pi_{t'=t}^{\tau-1} R_{t'}}.$$

So that the price of the tree is equal to the expected discounted future price at $t + \tau$, when the expectation is calculated according to the equivalent martingale measure. In other words, using the equivalent martingale probability measure, discounted prices follow a martingale process.

16.10. **Logarithmic utility.** Macroeconomists like to consider the case when $u(c) = \ln c$ so that

$$\frac{u'(d_{t+\tau})}{u'(d_t)} d_{t+\tau} = \frac{d_t}{d_{t+\tau}} d_{t+\tau} = d_t.$$

Hence,

$$q_t = d_t \mathbf{E}_t \sum_{\tau=1}^{\infty} \delta^\tau = \frac{\delta}{1-\delta} d_t$$

So the ***dividend-price ratio*** $d_t/q_t = (1-\delta)/\delta$ is constant.

16.11. **CRRA utility.** Another common assumption is that u is of the CRRA form:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}.$$

In this case, $u'(c_{t+\tau})/u'(c_t) = (c_t/c_{t+\tau})^\sigma$. Therefore,

$$q_t = \mathbf{E}_t \sum_{\tau=1}^{\infty} \delta^\tau (\frac{d_t}{d_{t+\tau}})^\sigma d_{t+\tau}. \tag{29}$$

Suppose that $\{d_t\}$ is iid and that $\bar{d} = \mathbf{E}_t d_{t+\tau}^{1-\sigma}$. Then,

$$\frac{q_t}{d_t^\sigma} = \mathbf{E}_t \sum_{\tau=1}^{\infty} \delta^\tau d_{t+\tau}^{1-\sigma} = \frac{\delta}{1-\delta}\bar{d}.$$

Suppose that d_t is iid log-normal with mean $= 1$. Then $\mathbf{E}_t d_{t+\tau}^{1-\sigma} = \mathbf{E}(e^X)^{1-\sigma} = \mathbf{E}e^{(1-\sigma)X}$, where $X \sim N(\mu, \nu^2)$ with $\mu = -\nu^2/2$ so that d_t has mean $= 1$. Hence,

$$\begin{aligned}\mathbf{E}_t d_{t+\tau}^{1-\sigma} &= \exp((1-\sigma)\mu + (1/2)(1-\sigma)^2\nu^2) \\ &= \exp(-(1-\sigma)\nu^2/2 + (1/2)(1-\sigma)^2\nu^2) \\ &= \exp(-\sigma(1-\sigma)\nu^2/2).\end{aligned}$$

Thus

$$\frac{q_t}{d_t^\sigma} = \frac{\delta}{1-\delta} e^{\sigma(\sigma-1)\nu^2/2},$$

and therefore

$$\log q_t = \sigma \log d_t + \sigma(\sigma - 1)\nu^2/2 + \log(\delta/(1-\delta))$$

and we have

$$\mathbf{V} \log q_t = \sigma^2 \mathbf{V} \log d_t.$$

This means that (log) prices will be more volatile than (log) dividends if and only if $\sigma > 1$.

Now suppose instead that dividends follow a random walk. Specifically suppose that the $\log d_{t+1} = \log d_t + \varepsilon_t$, where $\{\varepsilon_t\}$ are iid normally distributed with mean zero and variance ν^2. Then d_{t+1}/d_t is iid log-normal with mean 1. Hence $\bar{d} = \mathbf{E}d_{t+1}/d_t$.

From the Euler equation we have that

$$q_t = \delta \mathbf{E}_t (\frac{d_t}{d_{t+1}})^\sigma (d_{t+1} + q_{t+1}),$$

So the price-dividend ratio obeys

$$\frac{q_t}{d_t} = \delta \mathbf{E}_t (\frac{d_t}{d_{t+1}})^{\sigma-1} + \delta \mathbf{E}_t (\frac{d_t}{d_{t+1}})^{\sigma-1} \frac{q_{t+1}}{d_{t+1}}. \tag{30}$$

Now, as before, we have that

$$\mathbf{E}(\frac{d_t}{d_{t+1}})^{\sigma-1} = \mathbf{E}(\frac{d_{t+1}}{d_t})^{1-\sigma} = \delta \exp(-\sigma(1-\sigma)\nu^2/2)$$

We conjecture that there is an equilibrium in which the price-dividend ratio is a constant γ. Then Equation (30) implies that

$$\gamma = \delta e^{\sigma(\sigma-1)\nu^2/2}(1+\gamma) \Rightarrow \gamma = \frac{\delta e^{\sigma(\sigma-1)\nu^2/2}}{1+\delta e^{\sigma(\sigma-1)\nu^2/2}}.$$

So we get a similar relation between the log of price and dividend:

$$\log q_t = \log d_t + \sigma(\sigma-1)\nu^2/2 + \log\delta - \log(1+\delta e^{\sigma(\sigma-1)\nu^2/2})$$

16.11.1. *Government expenditure and taxes.* Suppose that goverment expenditure is a fraction of dividends, so that $g_t = \varepsilon_t d_t$, and $\{\varepsilon_t\}$ follows a Markov process with $\varepsilon_t \in (0,1)$. Government expenditure is financed with a lump-sum tax T_t levied on the cosumer, so that the consumer's budget constraint becomes $c_t + a_t \leq w_t - T_t$. Let q_t^L be the resulting price on trees. Now market clearing demands that $c_t + g_t = d_t$, and thus $c_t = (1-\varepsilon_t)d_t$. Then the Euler equation determines that

$$q_t^L = \delta \mathbf{E}_t \frac{u'((1-\varepsilon_{t+1})d_{t+1})}{u'((1-\varepsilon_t)d_t)}(q_{t+1}^L + d_{t+1}) = \sum_{\tau=1}^{\infty} \mathbf{E}_t \delta^\tau \frac{u'((1-\varepsilon_{t+\tau})d_{t+\tau})}{u'((1-\varepsilon_t)d_t)} d_{t+\tau}$$

Suppose that the consumer has logarithmic utility $u(c) = \log c$. Then we obtain that

$$q_t^L = \sum_{\tau=1}^{\infty} \mathbf{E}_t \delta^\tau \frac{(1-\varepsilon_t)d_t}{(1-\varepsilon_{t+\tau})d_{t+\tau}} d_{t+\tau} = (1-\varepsilon_t)d_t \sum_{\tau=1}^{\infty} \mathbf{E}_t \delta^\tau \frac{1}{(1-\varepsilon_{t+\tau})}$$

Now suppose in contrast that government expenditure is financed through a tax on trees. Let the resulting price on trees be q_t^A Then the consumer's income becomes $w_t = s_t((q_t^A - T_t) + d_t)$ and the budget constraint $c_t + a_t \leq w_t$. The rate of return on trees, however, must factor in the tax payments for owners of trees. Hence,

$$R_{t+1} = \frac{q_{t+1}^A - T_{t+1} + d_{t+1}}{q_t^A}.$$

We need $T_t = g_t = \varepsilon_t d_t$. So the asset pricing formula (with logarithmic utility) becomes

$$\begin{aligned} q_t^A &= \delta \mathbf{E}_t \frac{c_t}{c_{t+1}} (q_{t+1}^A - T_{t+1} + d_{t+1}) \\ &= \delta \mathbf{E}_t \frac{c_t}{c_{t+1}} (q_{t+1}^A + (1-\varepsilon_{t+1}) d_{t+1}) \\ &= \delta \mathbf{E}_t \frac{(1-\varepsilon_t) d_t}{(1-\varepsilon_{t+1}) d_{t+1}} (q_{t+1}^A + (1-\varepsilon_{t+1}) d_{t+1}) \\ &= \sum_{\tau=1}^{\infty} \mathbf{E}_t \delta^\tau \frac{(1-\varepsilon_t) d_t}{(1-\varepsilon_{t+\tau}) d_{t+\tau}} (1-\varepsilon_{t+\tau}) d_{t+\tau} \\ &= \frac{\delta}{1-\delta} (1-\varepsilon_t) d_t \end{aligned}$$

Note that $\varepsilon_t \in (0,1)$ implies that

$$\sum_\tau \delta^\tau \mathbf{E}_t \frac{1}{1-\varepsilon_t} > \sum_\tau \delta^\tau = \frac{\delta}{1-\delta}.$$

Hence, $q_t^L > q_t^A$.

This difference in price is purely an accounting exercise. In either case, consumption is $(1-\varepsilon_t)d_t$ as government expenditure "crowds out" private consumption. For the lump sum tax, income is directly adjusted by taxes. For the tax on trees, income has to be adjusted by making trees cheaper.

17. Large economies and approximate equilibrium

17.1. **The Shapley-Folkman theorem.** Suppose a set of sheep and goats are out to pasture on a field. If, for every set of four animals, there is a (straight) fence separating the goats from the sheep, then there is a fence that separates all the sheep from all the goats.

This bucolic theorem about separating sheep and goats is an example of a "combinatorial" separation property. The next two results are general combinatorial separation results. The first (which we are not going to prove) is called Kirchberger's theorem. The second is called Radon's theorem: it is very simple and highlights the main idea behind these results.

Theorem 98. *Kirchberger's theorem] Let A and B be finite subsets of $\mathbf{R}^n$. If, for any set C of cardinality at most $n+2$, $A \cap C$ and $B \cap C$*

can be strictly separated by a hyperplane, then A and B can be strictly separated by a hyperplane.

Theorem 99 (Radon's theorem)**.** *Let $A \subseteq \mathbf{R}^n$ be a finite set with cardinality at least $n+2$. Then A can be partitioned into A_1 and A_2 such that*

$$cvh(A_1) \cap cvh(A_2) \neq \emptyset$$

Proof. Suppose wlog that A has cardinality $n+2$ and let $A = \{a_0, \ldots, a_{n+1}\}$. Consider the set $\{a_1 - a_0, \ldots, a_{n+1} - a_0\}$. Such a set cannot be linearly independent, so there exists $\lambda_1, \ldots, \lambda_{n+1}$, not all zero, for which $\sum_{i=1}^{n+1} \lambda_i(a_i - a_0) = 0$. Define $\lambda_0 = -\sum_{i=1}^{n+1} \lambda_i$, $I = \{i \in \{0, \ldots, n+1\} : \lambda_i > 0\}$ and $J = \{i \in \{0, \ldots, n+1\} : \lambda_i < 0\}$. These two sets are nonempty because not all λ_i are zero, and $\sum_{i=0}^{n+1} \lambda_i = 0$. Moreoever, $\sum_{i=0}^{n+1} \lambda_i = 0$ implies that $\sum_{i \in I} \lambda_i = -\sum_{i \in J} \lambda_i$, and $\sum_{i=1}^{n+1} \lambda_i(a_i - a_0) = 0$ that $\sum_{i \in I} \lambda_i a_i = -\sum_{i \in J} \lambda_i a_i$.

Now let $A_1 = \{a_i : i \in I \text{ or } \lambda_i = 0\}$ and $A_2 = \{a_i : i \in J\}$. Thus

$$\sum_{i \in I} \frac{\lambda_i}{\sum_{j \in I} \lambda_j} a_i = \sum_{i \in J} \frac{-\lambda_i}{-\sum_{j \in j} \lambda_j} a_i = \sum_{i \in J} \frac{\lambda_i}{\sum_{j \in j} \lambda_j} a_i \in \text{cvh}(A_1) \cap \text{cvh}(A_2).$$

□

Theorem 100 (Caratheodory)**.** *Let $A \subseteq \mathbf{R}^n$ and $x \in cvh(A)$. Then there are $a_1, \ldots, a_{n+1} \in A$ such that $x \in cvh(\{a_1, \ldots, a_{n+1}\})$.*

Theorem 101 (Shapley-Folkman)**.** *Let $A_1, \ldots, A_K \subseteq \mathbf{R}^n$ and*

$$x \in cvh(A_1 + \ldots + A_K).$$

Then there is $a_i \in cvh(A_i)$, $i = 1, \ldots, K$, such that $x = a_1 + \ldots + a_K$ and $a_i \in A_i$ for all but at most n values of i.

The next lemma and its use in proving these theorems is taken from Bob Anderson's lecture notes.

Lemma 102. *Let $A_1, \ldots, A_K \subseteq \mathbf{R}^n$ and*

$$x \in cvh(A_1 + \ldots + A_K).$$

Then

$$x = \sum_{i=1}^{K} \sum_{j=0}^{m_i} \lambda_{i,j} a_{i,j},$$

where $a_{i,j} \in A_i$ for all i, j, $\lambda_{i,j} > 0$, $\sum_{j=0}^{m_i} \lambda_{i,j} = 1$, and $\sum_{i=1}^{K} m_i \leq n$.

Observe that Lemma 102 says that $x = \sum_i \bar{a}_i$, where $\bar{a}_i \in \text{cvh}(A_i)$, but it bounds the number of points in the *support* of the K convex combinations $(\lambda_{i,j})_{j=0}^{m_i}$.

Proof of Caratheodory's theorem. Let $x \in \text{cvh}(A)$. By Lemma 102 with $K = 1$ there is $a_0, a_1, \ldots, a_m \in A$ and $\lambda_0, \lambda_1, \ldots, \lambda_m$ with $x = \sum_{j=0}^{m} \lambda_j a_j$ and $m \leq n$. □

Proof of the Shapley-Folkman theorem. Let $a_{i,j}, \lambda_{i,j}$ $i = 1, \ldots, K$, $j = 1, \ldots, m_i$ be as in the statement of Lemma 102. m_i is an integer. So $m_i > 0$ means that $m_i \geq 1$. Thus, $\sum_{i=1}^{K} m_i \leq n$ means that $m_i = 0$ for all but at most n values of $i \in \{1, \ldots, K\}$. In $m_i = 0$ then $\lambda_0 = 1$. Thus

$$x = \sum_{i:m_i=0} a_{i,0} + \sum_{i:m_i \geq 1} \underbrace{\sum_{j=0}^{m_i} \lambda_{i,j} a_{i,j}}_{\in \text{cvh}(A_i)} .$$

□

The proof of Lemma 102 relies on familiar ideas. A large number of vectors in $\mathbf{R}^n$ cannot be linearly independent, so any linear combination of a large number of vector can be simplified. In the lemma, we talk about convex combinations, so the relevant concept is that of "affine independence:" we translate the vectors $a_{i,j}$ and work with the fact that the collection $\{a_{i,j} - a_{i,0} : i = 1, \ldots, K, j = 1, \ldots, m_i\}$ cannot be linearly independent when $\sum_i m_i > n$.

Proof. Let $x \in \text{cvh}(\sum_i A_i)$. Then there is $\lambda_j \in (0, 1)$ and $x_j \in \sum_i A_i$, with $j = 0, \ldots, m$, such that $x = \sum_{j=0}^{m} \lambda_j x_j$ and $\sum_j \lambda_j = 1$. Moreover, for each j there is $a_{i,j} \in A_i$, $i = 1, \ldots, K$, and $\sum_{i=1}^{K} a_{i,j} = x_j$. Thus,

$$x = \sum_{i=1}^{K} \sum_{j=0}^{m_i} \lambda_{i,j} a_{i,j} \tag{31}$$

where $m_i = m$ and $\lambda_{i,j} = \lambda_j$ for all i, j. Now, we shall prove that for each representation of x in the form of (31) with $\sum_i m_i > n$ there exists another representation of x in the form of (31) with a strictly smaller value of $\sum_i m_i$.

Suppose then that $\sum_i m_i > n$. Consider the set of vectors

$$\{a_{i,j} - a_{i,0} : 1 \leq i \leq K \text{ and } 1 \leq j \leq m_i\}.$$

This subset of $\mathbf{R}^n$ has cardinality $\sum_i m_i > n$. Therefore, it is linearly dependent.

Let $\beta_{i,j}$, not all zero, be such that

$$0 = \sum_{i=1}^{K} \sum_{j=1}^{m_i} \beta_{i,j}(a_{i,j} - a_{i,0}).$$

Then, for any t,

$$\begin{aligned} x &= \sum_{i=1}^{K} \sum_{j=0}^{m_i} \lambda_{i,j} a_{i,j} + t \sum_{i=1}^{K} \sum_{j=1}^{m_i} \beta_{i,j}(a_{i,j} - a_{i,0}) \\ &= \sum_{i=1}^{K} \left[\sum_{j=1}^{m_i} (\lambda_{i,j} + t\beta_{i,j}) a_{i,j} + (\lambda_{i,0} - t \sum_{j=1}^{m_i} \beta_{i,j}) a_{i,0} \right] \end{aligned}$$

Let $\lambda^*_{i,j}(t) = \lambda_{i,j} + t\beta_{i,j}$ and $\lambda^*_{i,0}(t) = \lambda_{i,0} - t \sum_{j=1}^{m_i} \beta_{i,j}$, and observe that $\sum_{j=0}^{m_i} \lambda^*_{i,j}(t) = \sum_{j=1}^{m_i}(\lambda_{i,j} + t\beta_{i,j}) + \lambda_{i,0} - t\sum_{j=1}^{m_i} \beta_{i,j} = 1$, and that $\lambda^*_{i,j}(0) = \lambda_{i,j} > 0$.

Now note that the $\beta_{i,j}$ are not all zero, so that there must exist at least one $\lambda^*_{i,j}(t)$ that is strictly monotone (increasing or decreasing) as a linear function of t. Since $\sum_{j=0}^{m_i} \lambda^*_{i,j}(t) = 1$ for all t, there must in fact exist (i,j) for which $\lambda^*_{i,j}(t)$ is strictly monotone decreasing: $\lambda^*_{i,j}(t)$ is a linear function of t, so there is t for which $\lambda^*_{i,j}(t) = 0$. Finally, $\lambda^*_{i,j}(0) > 0$ for all (i,j) means that there exists $\hat{t}$ and some (i',j') for which $\lambda^*_{i',j'}(\hat{t}) = 0$ and $\lambda^*_{i,j}(\hat{t}) \geq 0$ for all (i,j).

Given such $\hat{t}$, define now new coefficients $(\hat{\lambda}_{i,j})_{i=1,j=0}^{i=K,j=\hat{m}_i}$ from $\lambda^*_{i,j}(\hat{t})$ by keeping only the ones that are strictly positive. This gives us $\sum_{i=1}^{K} \hat{m}_i < \sum_{i=1}^{k} m_i$. □

Finally, I want to state the following approximate version of Caratheodory due to Sid Barman. Remarkably, this approximation does not depend on n.

Theorem 103. *Let $x \in cvh(\{x_1, \ldots, x_K\}) \subseteq \mathbf{R}^n$, $\varepsilon > 0$ and p an integer with $2 \leq p < \infty$. Let $\gamma = \max\{\|x_k\|_p : 1 \leq k \leq K\}$. Then there is a vector x' that is a convex combination of at most*

$$\frac{4p\gamma^2}{\varepsilon}$$

of the vectors $x_1, \ldots, x_K$ such that $\|x - x'\|_p < \varepsilon$.

17.2. **Approximate equilibrium.** As a simple consequence of the Shapley-Folkman theorem we obtain a result on the existence of approximate Walrasian equilibria. This says that, when preferences are not necessarily convex, we can obtain an approximate equilibrium where demand equals supply and "most" agents (when I is large relative to L) are optimizing.

Theorem 104. *Let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy with L goods. Suppose that each preference $\succeq_i$ is continuous and strictly monotone, and that $\bar{\omega} \gg 0$. Then there exists $p \in \mathbf{R}_{++}^L$ and $(x_i)_{i=1}^I \in \mathbf{R}_+^{IL}$ such that:*

(1) $p \cdot x_i \leq p \cdot \omega_i$ *for all i.*
(2) $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i$.
(3) *The property*

$$x_i' \succ_i x_i \Rightarrow p \cdot x_i' > p \cdot \omega_i$$

holds for all but at most L consumers.

Proof. Under the assumptions we have made, the aggregate excess demand correspondence of each consumer i, z_i^*, is well defined. For each $p \in \mathbf{R}_{++}^L$, $z_i^*(p) \subseteq \mathbf{R}^L$ is nonempty and compact, and $p \mapsto z_i^*(p)$ is upper hemi-continuous. Moreover, $p \cdot \zeta_i = 0$ for all $\zeta_i \in z_i^*(p)$ (Walras' Law). The sets $z_i^*(p)$ are bounded below by $-\mathbf{1}\|\bar{\omega}\|_\infty$.

Let $z^* = \sum_i z_i^*$ be the aggregate excess demand correspondence of the economy $\mathcal{E}$. It follows from the standard argument that if $p^n \to \bar{p} \neq 0$ with $\bar{p}_l = 0$ for some l, then

$$\infty = \lim_{n \to \infty} \inf\{\|\zeta_i\|_\infty : \zeta_i \in z_i^*(p^n)\}.$$

Thus, the assumptions needed for existence of a competitive equilibrium are satisfied by z^*, with the exception of the convexity of the set $z^*(p)$.

The correspondence $p \mapsto \mathrm{cvh} z^*(p)$ is obviously convex-valued. It turns out that it continues to satisfy the properties mentioned above (I omit the proof, which is not difficult), So, by the standard existence theorem (in its correspondence form) the exists p^* with $0 \in \mathrm{cvh}(z^*(p^*))$.

By the Shapley-Folkman theorem, there exists $\zeta_i \in \mathrm{cvh}(z_i^*(p^*))$, $i = 1, \ldots, I$, with $\zeta_i \in z_i^*(p^*)$ for all but at most L values of i, such that $0 = \sum_i \zeta_i$. Define $x_i^* = \zeta_i + \omega_i$. □

We could get a variation on this result by taking the agents who are not optimizing and instead giving them a bundle in their demand correspondences. If we can bound the difference between any point in the demand correspondence of agent i, and in its convex hull, then we can obtain a result on the existence of an approximate equilibrium where all agents are optimizing, but where demand may not be exactly equal to supply. There will instead be a "small" gap between supply and demand. However, to prove such a result we need a lower bound on prices. This is done, for example in a theorem due to (Hildenbrand et al., 1973).[24]

Instead, I present a theorem due to Anderson et al. (1982). It establishes the existence of an approximate equilibrium under weak assumptions on preferences: not only avoiding convexity, but even monotonicity. The equilibrium is approximate in the sense that excess demands are bounded above. The main idea in this theorem is to work with an unusual domain of prices, where prices are bounded below by $1/\sqrt{I}$, and then bound the value of excess demand. Since prices cannot be too low, an upper bound on the value of excess demand implies a bound on (physical) excess demand. The focus on the value of excess demand also means that we shall not require monotonicity of preferences, or Walras Law.

Theorem 105. *Let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy with L goods. Suppose that each preference $\succeq_i$ is continuous. There exists $p \in \mathbf{R}^L_{++}$ and $(x_i)_{i=1}^I \in \mathbf{R}^{IL}_+$ such that*

- $p \cdot x_i \leq p \cdot \omega_i$;
- $y_i \succ_i x_i$ *implies that* $p \cdot y_i > p \cdot \omega_i$; *and*
- *Aggregate excess demand is bounded above in the following sense:*
$$\frac{1}{I}\sum_{l=1}^{L}[\sum_{i=1}^{I}(x_{i,l} - \omega_{i,l})]^+ \leq \frac{L+1}{\sqrt{I}} \max\{\|\omega_i\|_1 : 1 \leq i \leq I\}.$$

Proof. Let
$$M = \{p \in \mathbf{R}^L_+ : \frac{1}{\sqrt{I}} \leq p_l \leq 1, l = 1, \ldots, L\}.$$
For each $p \in M$, let $z_i^*(p) \subseteq \mathbf{R}^L$ be consumer i's excess demand at p. Note that $z_i^*(p)$ is a nonempty and compact set, but that it may not be convex. Note that $p \mapsto z_i^*(p)$ is upper hemicontinous. Let

[24] Starr (1969), who introduced many of these ideas and where the first statement of the Shapley-Folkman theorem appears.

$z^* = \sum_{i=1}^I z_i^*(p)$. Let $K \subseteq \mathbf{R}^L$ be a compact and convex set such that $z^*(M) \subseteq K$.

Define a correspondence $\phi : M \times K \mapsto M \times K$ by

$$\phi(p, \zeta) = \text{argmax}\{q \cdot \zeta : q \in M\} \times \text{cvh}(z^*(q)).$$

The correspondence ϕ is in the hypotheses of Kakutani's fixed point theorem. Let (p, ζ) be a fixed point of ϕ, so $(p, \zeta) \in \phi(p, \zeta)$.

The idea here is to calculate the value of excess demand at a fixed point. That explains the choice of M. The bound on value will imply a bound on physical quantities.

By the Shapley-Folkman theorem, since $\zeta \in \text{cvh}(\sum_{i=1}^I z_i^*(p))$, there exists $(\zeta_1, \ldots, \zeta_I)$ and $C \subseteq \{1, \ldots, I\}$ with $|C| = I - L$ such that

- $\zeta_i \in z_i^*(p)$ for all $i \in C$;
- $\zeta_i \in \text{cvh}(z_i^*(p))$ for all $i \notin C$ and
- $\zeta = \sum_i \zeta_i$.

Choose $\zeta_i' \in z_i^*(p)$ for $i \notin C$ and let $x_i' = \omega_i + \zeta_i'$. Observe that

$$\zeta_{i,l}' - \zeta_{i,l} = x_{i,l}' - (\zeta_{i,l} + \omega_{i,l}) \leq x_{i,l}'.$$

Therefore,

$$\sum_{l=1}^L \max\{\zeta_{i,l}' - \zeta_{i,l}, 0\} \leq \sum_{l=1}^L x_{i,l}' \leq \frac{p \cdot \omega_i}{\min\{p_l : 1 \leq l \leq L\}} \leq \sqrt{I} \|\omega_i\|_1,$$

where the second inequality follows as $\sum_{l=1}^L x_{i,l}' \leq \frac{p \cdot x_i}{\min\{p_l : 1 \leq l \leq L\}}$ and $p \cdot x_i \leq p \cdot \omega_i$. The third inequality follows from the definition of M.

This implies that, for any $q \in M$,

$$q \cdot \sum_{i \notin C} (\zeta_i' - \zeta_i) \leq \sum_{i \notin C} \sum_{l=1}^L \max\{\zeta_{i,l}' - \zeta_{i,l}, 0\} \leq L\sqrt{I} \max\{\|\omega_i\|_1 : i \notin C\}.$$

Note that $p \cdot \zeta_i \leq 0$ for all i. By definition of ϕ, and since $(p, \zeta) \in \phi(p, \zeta)$, $q \cdot \zeta \leq p \cdot \zeta \leq 0$ for any $q \in M$. So we have that, for any $q \in M$,

$$q \cdot (\sum_{i \in C} \zeta_i + \sum_{i \notin C} \zeta_i') = \underbrace{q \cdot \sum_{i=1}^I \zeta_i}_{\leq 0} + q \cdot \sum_{i \notin C} (\zeta_i' - \zeta_i) \leq L\sqrt{I} \max\{\|\omega_i\|_1 : 1 \leq i \leq I\}.$$

Choose in particular $q \in M$ such that $q_l = 1$ if $\zeta_l > 0$ and $q_l = 1/\sqrt{I}$ if $\zeta_l \leq 0$. Then we have that

$$\begin{aligned}\sum_{l=1}^{L} \max\{\zeta_l, 0\} &= \sum_l q_l \max\{\zeta_l, 0\} \\ &= q \cdot \zeta - \frac{1}{\sqrt{I}} \sum_{l:\zeta_l \leq 0} \zeta_l \\ &\leq L\sqrt{I} \max\{\|\omega_i\|_1 : 1 \leq i \leq I\} + \frac{1}{\sqrt{I}} \sum_{l:\zeta_j \leq 0} \bar{\omega}_l \\ &\leq L\sqrt{I} \max\{\|\omega_i\|_1 : 1 \leq i \leq I\} + \frac{I}{\sqrt{I}} \max\{\|\omega_i\|_1 : 1 \leq i \leq I\} \\ &= (L+1)\sqrt{I} \max\{\|\omega_i\|_1 : 1 \leq i \leq I\}.\end{aligned}$$

The first inequality follows from the bound on $q \cdot \zeta$ we established above, and the fact that if $\zeta_l < 0$ then $-\zeta_l \leq \bar{\omega}_l$, so

$$-\sum_{l:\zeta_l \leq 0} \zeta_l \leq \|\bar{\omega}\|_1.$$

The second inequality holds because $\|\bar{\omega}\|_1 \leq I \max\{\|\omega_i\|_1 : 1 \leq i \leq I\}$. □

The idea in this proof is that if (p, ζ) is a fixed point of ϕ, and if we could show that $\zeta = \sum_i \zeta_i$ with $\zeta_i \in z_i^*(p)$ for all i, then $p \cdot \zeta \leq 0$ (compliance with budget constraints) and a lower bound on p_l provides an upper bound on ζ_l.

17.3. **Core convergence revisited.** The Debreu-Scarf theorem shows that core allocations of large replica economies converge to Walrasian equilibrium allocations. Using the Shapley-Folkman theorem, it is possible to show a version of this result that does not need the machinery of replica economies, and can also dispense with convexity of preferences. The theorem, which is due to Anderson (1978), ensures the existence of a price vector at which agents are on average close to minimizing expenditure.[25]

Theorem 106. *Let $\mathcal{E} = (\succeq_i, \omega_i)_{i=1}^I$ be an exchange economy with L goods in which each preference $\succeq_i$ is monotonic. If x^* is in the core of $\mathcal{E}$ then here exists a price vector $p \in \mathbf{R}^L_+$, $p > 0$, such that*

[25] See also Dierker (1975).

(1)
$$\frac{1}{I}\sum_{i=1}^{I}|p\cdot(x_i^*-\omega_i|\leq\frac{2M}{I}$$
(2)
$$\frac{1}{I}\sum_{i=1}^{I}|\inf\{p\cdot(y_i-\omega_i):y_i\succ_i x_i^*\}|\leq\frac{2M}{I},$$
where
$$M=L\max\{\omega_{i,l}:1\leq i\leq I,1\leq l\leq L\}$$

The second statement in the theorem deserves an explanation. It says that if one were to make (on average) agents' incomes a bit smaller, then there would be no bundle affordable that would be preferred to the agent's consumption bundle in x^*. For an exact Walrasian equilibrium, note that we would have $|\inf\{p\cdot(y_i-\omega_i):y_i\succ_i x_i^*\}|=0$. Incidentally, the theorem does not explicitly say that demand equals supply but note that x^* is an allocation, so $\sum_i x_i^*=\sum_i\omega_i$.

17.3.1. *Proof of Theorem 106.* Let $\theta=\max\{\|\omega_i\|_\infty:i=1,\ldots,I\}$, so $M=L\theta$. Let
$$z=-L\theta\mathbf{1}=-M\mathbf{1}.$$

Let
$$B_i=\{y_i-\omega_i:y_i\succ_i x_i^*\}\cup\{0\}\text{ and }B=\sum_i B_i$$
Note that B is a hybrid of the approaches we used for the second welfare theorem and the Debreu-Scarf core convergence theorem. It is an "aggregate" object in the sense that it adds up the translated upper contour sets, but by including the null vector we can omit agents in the sum and obtain an aggregate for any coalition of agents. Note also that B_i may not be convex even if preferences are convex.

Since x^* is in the core, we have:

Lemma 107. $B\cap(-\mathbf{R}^L_{++})=\emptyset$.

We shall first prove that
$$\text{cvh}(B)\cap(z-\mathbf{R}^L_{++})=\emptyset.$$
Suppose, towards a contradiction, that there exists $b\in\text{cvh}(B)$ and $z'\ll 0$ such that $z+z'=b$. By the Shapley-Folkman theorem there is

(b_i) and $I' \subseteq I$ with $b = \sum_i b_i$, $b_i \in B_i$ for all $i \in I'$ and $b_i \in \text{cvh}(B_i)$ for all $i \in I \setminus I'$, and $|I \setminus I'| \leq L$. Then

$$\sum_{i \in I'} b_i = z + z' - \sum_{i \notin I'} b_i \ll z - \sum_{i \notin I'} b_i.$$

Now, $b_i \in B_i$ means that either $b_i = 0$ or $b_i = y_i - \omega_i$ with $y_i \succ_i x_i^*$. Either way, $b_i \geq -\omega_i$. So for any $b_i \in \text{cvh}(B_i)$, $b_i \geq -\omega_i$. Therefore, using the definition of θ, we obtain that

$$-\sum_{i \notin I'} b_i \leq |I \setminus I'|\, \theta \mathbf{1} \leq L\theta\mathbf{1}.$$

Thus $\sum_{i \in I'} b_i \ll z + L\theta\mathbf{1} = 0$

Define (b_i^*) by setting $b_i^* = b_i$ for all $i \in I'$ and $b_i^* = 0$ for all $i \notin I'$. Thus $\sum_i b_i^* \in B$, but $\sum_i b_i^* = \sum_{i \in I'} b_i \ll 0$. A contradiction of Lemma 107.

Now, by the separating hyperplane theorem, $\text{cvh}(B) \cap (z - \mathbf{R}^L_{++}) = \emptyset$ implies that there is $p \in \mathbf{R}^L$, $p \neq 0$, such that

$$\sup\{p \cdot \tilde{z} : \tilde{z} \ll z\} \leq \inf\{p \cdot b : b \in \text{cvh}(B)\}$$

In fact, by standard arguments, $p > 0$; and, using a normalization, we can take p to satisfy $p \cdot \mathbf{1} = 1$.[26]

By monotonicity, for all $k \geq 1$, $x_i^* + \mathbf{1}(1/k) - \omega_i \in B_i$, so

$$p \cdot (x_i^* + \mathbf{1}(1/k) - \omega_i) \geq \inf\{p \cdot B_i\}.$$

Hence

$$p \cdot (x_i^* - \omega_i) \geq \inf\{p \cdot B_i\}.$$

Now, $B_i \subseteq B$ (as $0 \in B_j$ for all $j \neq i$). This implies that

$$\begin{aligned} p \cdot (x_i^* - \omega_i) &\geq \inf\{p \cdot B_i\} \\ &\geq \inf\{p \cdot B\} \\ &\geq \sup\{p \cdot \tilde{z} : \tilde{z} \ll z\} \\ &= p \cdot z = -M, \end{aligned}$$

where the last equality uses the normalization imposed on p.

Let $S = \{i \in I : p \cdot (x_i - \omega_i) < 0\}$. Then $\sum_i p \cdot (x_i^* - \omega_i) = 0$ implies that

$$\frac{1}{I}\sum_i |p\cdot(x_i - \omega_i)| = \frac{1}{I}\sum_{i \notin S} p\cdot(x_i - \omega_i) - \frac{1}{I}\sum_{i \in S} p\cdot(x_i - \omega_i) = \frac{-2}{I}\sum_{i \in S} p\cdot(x_i - \omega_i).$$

26 We have $p > 0$ because if $p_l < 0$ then we may choose $\tilde{z} \ll z$ to make $p \cdot \tilde{z}$ arbitrarily large. So $p \geq 0$, but we also know that $p \neq 0$.

Lemma 108.

$$\sum_{i\in S} p\cdot(x_i-\omega_i)\geq -M.$$

Proof. For all k, $x_i^*+\mathbf{1}(1/k)-\omega_i\in B_i$. So $\sum_{i\in S}(x_i^*+\mathbf{1}(1/k)-\omega_i)\in B$. Then

$$p\cdot\sum_{i\in S}(x_i^*+\mathbf{1}(1/k)-\omega_i)\geq p\cdot z=-M.$$

The result follows as k is arbitrary. □

By Lemma 108,

$$\frac{1}{I}\sum_{i}|p\cdot(x_i-\omega_i)|=\frac{-2}{I}\sum_{i\in S}p\cdot(x_i-\omega_i)\leq\frac{2M}{I}$$

Now lets turn to the issue of expenditure minimization. Let

$$A=\frac{1}{I}\sum_{i=1}^{I}|\inf\{p\cdot(y_i-\omega_i):y_i\succ_i x_i^*\}|$$

Let $L_i=\{y_i-\omega_i:y_i\succ_i x_i^*\}$. By monotonicity, $p\cdot(x_i^*+\mathbf{1}(1/k)-\omega_i)\in p\cdot L_i$ for all k. So $p\cdot(x_i^*-\omega_i)\geq\inf p\cdot L_i$. Note that if $i\in S$ then $0\geq p\cdot(x_i^*-\omega_i)\geq p\cdot L_i$. So $\inf p\cdot L_i=\inf p\cdot B_i$. Hence

$$\begin{aligned}A&=\frac{-1}{I}\sum_{i\in S}\inf p\cdot B_i+\frac{1}{I}\sum_{i\notin S}|\inf p\cdot L_i|\\&=\frac{-1}{I}\sum_{i=1}^{I}\inf p\cdot B_i+\frac{1}{I}\sum_{i\notin S}\{\inf p\cdot B_i+|\inf p\cdot L_i|\},\end{aligned}$$

where the second equality results from adding and subtracting $\frac{1}{I}\sum_{i\notin S}\inf p\cdot B_i$.

For $i\notin S$ there are two possibilities. The first is that $\inf p\cdot L_i>0$. Then $\inf p\cdot B_i=0$ and thus

$$\inf p\cdot B_i+|\inf p\cdot L_i|=\inf p\cdot L_i\leq p\cdot(x_i^*-\omega_i)$$

The second is that $\inf p\cdot L_i\leq 0$, so that $\inf p\cdot B_i=\inf p\cdot L_i$ and therefore

$$\inf p\cdot B_i+|\inf p\cdot L_i|=\inf p\cdot B_i-\inf p\cdot L_i=0\leq p\cdot(x_i^*-\omega_i),$$

where the last inequality follows from $i\notin S$.

As a consequence we obtain that

$$A \leq \frac{-1}{I} \sum_{i \in I} \inf p \cdot B_i + \frac{1}{I} \sum_{i \notin S} p \cdot (x_i^* - \omega_i).$$

Now, $\sum_i \inf p \cdot B_i = \inf p \cdot B \geq p \cdot z = -M$. So that

$$\frac{-1}{I} \sum_{i} \inf p \cdot B_i \leq \frac{M}{I}.$$

On the other hand, $\sum_{i \notin S} p \cdot (x_i^* - \omega_i) = -\sum_{i \in S} p \cdot (x_i^* - \omega_i) \leq M$, by Lemma 108. Thus $A \leq M/I + M/I$.

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