6 Oct, 2020 **Ph 136a** E.S. Phinney

Applications of Classical Physics

Problem Set 1

Due **before 1:00pm** Wednesday 14 Oct, 2020

Submitting your homework Please upload your completed homework solutions as a pdf file to Canvas (you can output tablet writing to pdf, scan handwriting to pdf, or convert images of handwriting to pdf, or use pdflatex or other typesetting software with pdf output). If that fails to work, you may instead email the file to the TA, `twang3@caltech.edu`, with the subject line `ph136 homework 1` in the subject line. Note that Caltech email will reject attachment sizes larger than 10Mbyte, so be conscious of scanning parameters!

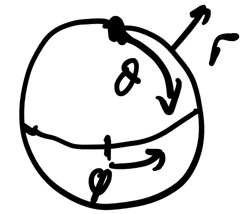
Readings: Reading for this week is Chapters 1 and 2 of Thorne & Blandford *Modern Classical Physics*, hereafter called TBMCP. If you have already seen the 4-vector approach to special relativity, you can just skim those sections of Chapter 2.

Non-Collaboration Problem Notice that problem 5 is a non-collaboration problem: you must solve this problem on your own (as if it were an untimed quiz), without discussing it with other students, or with the TA (see the collaboration and grading policy for how these will be graded and used: `http://www.its.caltech.edu/~esp/ph136a/Policies.pdf`)

Homework Problems: (55 points total)

1. **TBMCP Problem 1.8 (9 points: 3+3+3)** *Vectorial Identities for the Cross Product and Curl*
2. **TBMCP Problem 1.11 (10 points: 3+3+2+2)** *Integral of a Vector Field over a Sphere*
3. **TBMCP Problem 2.6 (6 points: 2+2+2)** *Numerics of Component Manipulations*
4. **TBMCP Problem 2.10 (10 points: (2+2+2)+4)** *3-metric as a projection tensor*
5. **TBMCP Problem 2.9 NONCOLLABORATION (10 points: 2+2+2+4)** *Frame-Independent Expressions for Energy, Momentum and Velocity*
6. **Handwritten problem following (10 points: 2+2+2+2+1+1)** *The dual vector to the gradient one-form in spherical polar coordinates*

In spherical polar coordinates, the metric tensor for flat Euclidean 3-space is



$$g(\text{--},\text{--}) = \mathbf{d}r \otimes \mathbf{d}r + r^2\, \mathbf{d}\theta \otimes \mathbf{d}\theta + r^2 \sin^2\theta\, \mathbf{d}\varphi \otimes \mathbf{d}\varphi$$

↑ ↑
slots for <u>vectors</u>

Thus, the components of the metric tensor in this coordinate chart are

$$g_{rr} = 1 \;,\quad g_{\theta\theta} = r^2 \;,\quad g_{\varphi\varphi} = r^2 \sin^2\theta \;,$$

and all other components (e.g. $g_{r\theta}$, $g_{r\theta}$) vanish.

As discussed in lecture 2, the gradient of a function is a 1-form, not a vector. Let the function be $f(r, \theta, \varphi)$. Then the gradient 1-form is

$$\mathbf{d}f = \frac{\partial f}{\partial r}\, \mathbf{d}r + \frac{\partial f}{\partial \theta}\, \mathbf{d}\theta + \frac{\partial f}{\partial \varphi}\, \mathbf{d}\varphi \tag{1}$$

$$\equiv f_{,i}\, \mathbf{d}x^i \qquad \text{where } \mathbf{d}x^1 = \mathbf{d}r,\ \mathbf{d}x^2 = \mathbf{d}\theta,\ \mathbf{d}x^3 = \mathbf{d}\varphi$$

Also as discussed in lecture 2, we can use the metric to associate a <u>dual vector</u>

to the gradient 1-form;

$$\underbrace{g(\vec{\nabla} f, \ldots)}_{\text{dual vector}} = \underbrace{d\!\!\!\;|f}_{\text{gradient 1-form}}$$

a) In component notation show that this is

$$g_{ij}(\vec{\nabla} f)^j = (d\!\!\!\;|f)_i$$

Einstein summation convention

b) Considering the matrix of components

$$\| g_{ij} \|,$$

define the inverse matrix

$$\| g^{kl} \| = \| g_{ij} \|^{-1}$$

show that the components of the dual grad vector are $(\vec{\nabla} f)^k = g^{ki} (d\!\!\!\;|f)_i$

Einstein summation convention

c) Thus show that

(2) $$\vec{\nabla} f = \frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial f}{\partial \varphi}\frac{\partial}{\partial \varphi}$$

where the basis vectors are

$$\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$$

d) Show that the dual metric is

$$g(\text{---}, \text{---}) = \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta} \otimes \frac{\partial}{\partial \theta}$$

slots for 1-forms

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \varphi} \otimes \frac{\partial}{\partial \varphi}$$

$$= \hat{\underset{\sim}{e}}_r \otimes \hat{\underset{\sim}{e}}_r + \hat{\underset{\sim}{e}}_\theta \otimes \hat{\underset{\sim}{e}}_\theta + \hat{\underset{\sim}{e}}_\varphi \otimes \hat{\underset{\sim}{e}}_\varphi$$

metric components in this orthonormal basis are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\hat{\underset{\sim}{e}}_r = \frac{\partial}{\partial r}$, $\hat{\underset{\sim}{e}}_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}$, $\hat{\underset{\sim}{e}}_\varphi = \frac{1}{r \sin\theta}\frac{\partial}{\partial \varphi}$

e) Using part (d), show that

(3) $$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{\underset{\sim}{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\underset{\sim}{e}}_\theta + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \varphi}\hat{\underset{\sim}{e}}_\varphi$$

f) Compare your result (3) in part (e), which uses an orthonormal basis, to the 1-form (covariant) and vector (contravariant) expressions (1) and (2) respectively for the gradient and its dual vector in coordinate bases $(dr, d\theta, d\varphi)$ and $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}\right)$.

Which of the three is the vector form for the gradient in spherical coordinates which you learned in elementary physics & math classes?