Problem Set 6 - Solution

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1. C& O equation 29.101 expresses the Saha Equation for our universe with only hydrogen.

$$\frac{f}{1-f} = \frac{m_H a^3}{f\rho_{b,0}} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT} \ , \tag{1}$$

where a is the scale factor and $T = T_0/a$ and $T_0 = 2.728$, the CMB temperature. $\rho_{b,0} = \Omega_b \rho_{0,c}$ is given as in the last problem set with $\Omega_b = 0.045$ and $\rho_{0,c} = 3H_0^2/8\pi G$. We also take $H_0 = 71$ km s^{-1} Mpc^{-1} and $\chi_I = 13.6$ eV for the ionization energy. We want to find the temperature at which half of the protons and electrons have recombined to form neutral hydrogen, this corresponds to $f = 0.5$. Therefore

$$\frac{f^2}{1-f} = \frac{8\pi G m_H T_0^3}{3\Omega_b H_0^2 T^3} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT} \tag{2}$$

Plugging all the numbers in and solving numerically (Mathematica) gives a temperature of $T = 3758$ K.

2. C&O 29.23

We're considering the SZ effect where CMB photons scatter off high energy electrons and gain energy on average.

a) Let's consider the four different cases in question diagrammed in the C&O fig 29.13. In 1. and 3. the photon continues in the direction of propagation and in cases 2. and 4. the photon gets turned around by scattering off of the electron. If we move into the frame of the electron we can think of the Compton effect.

$$\Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta) \ , \tag{3}$$

where θ is the scattering angle, 0 for cases 1. and 3. and π for cases 2 and 4. (there is no angular aberration since we are on axis) Clearly for $\theta = 0$ there will be no change in the wavelength, so for cases 1. and 3. when we shift back into the original frame the frequency will not have changed in the interaction.

$$\frac{\Delta\nu}{\nu} = 0 \text{ for cases 1. and 3. .} \tag{4}$$

Now we need to consider the other two cases. These will have a change in the frequency. It's straight forward to consider the relativistic particle collision starting with momentum and energy conservation and making use of Lorentz invariants. So momentum conservation,

$$-\frac{h\nu_0}{c} + p_0 = \frac{h\nu_1}{c} + p_1 \text{ for case 2.} \tag{5}$$

$$-\frac{h\nu_0}{c} - p_0 = \frac{h\nu_1}{c} - p_1 \text{ for case 4. ,} \tag{6}$$

where p_i indicates the momentum of the electron before (0) and after the collision (1). Energy conservation for both cases,

$$h\nu_0 + E_0 = h\nu_1 + E_1 \ . \tag{7}$$

This is a relativistic collision because the electrons are high energy, so we make use of the relativistic invariant rest-mass of the electron $m^2c^4 = E^2 - p^2c^2$. Therefore,

$$E_0^2 - p_0^2c^2 = E_1^2 - p_1^2c^2 \tag{8}$$

We can rearrange the above equations as

$$(E_1 - E_0)(E_1 + E_0) = c^2(p_1 - p_0)(p_1 + p_0) \tag{9}$$

$$h\Delta\nu = -(E_1 - E_0) \tag{10}$$

$$p_1 - p_0 = -\frac{h}{c}(\nu_1 + \nu_0) \text{ for case 2} \tag{11}$$

$$p_1 - p_0 = \frac{h}{c}(\nu_1 + \nu_0) \text{ for case 4 .} \tag{12}$$

The electron is moving rapidly and hitting a low energy photon, so it won't change in energy much. Although we consider the difference in energy before and after to be small but nonzero we can approximate the sum of the momenta and energies as $p_1 + p_0 \approx 2p_0$ and $E_1 + E_0 \approx 2E_0$. So we can combine the equations as

$$\pm\,\Delta\nu = \frac{p_0c}{E_0}(\nu_1 + \nu_0) \ , \tag{13}$$

where the positive sign is for case 2 and the negative sign is for case 4. Since $p_0 = \gamma m_e v_e$ and $E_0 = \gamma m_e c^2$, $p_0c/E_0 = v_e/c = \beta$. By replacing $\nu_1 = \Delta\nu + \nu_0$ and solving for $\Delta\nu$ (and making an approximation for beta small) we have

$$\frac{\Delta\nu}{\nu} = 0 \quad \text{for case 1} \tag{14}$$

$$= 2\beta + 2\beta^2 \quad \text{for case 2} \tag{15}$$

$$= 0 \quad \text{for case 3} \tag{16}$$

$$= -2\beta + 2\beta^2 \quad \text{for case 4} \tag{17}$$

Averaging these four quantities gives

$$\frac{\Delta\nu}{\nu} \approx \beta^2 = \frac{v_e^2}{c^2} \ . \tag{18}$$

Since we characterize the photon gas with this mean energy via a temperature, $m_e v_e^2/2 = 3kT_e/2$, we can also write that last expression

$$\frac{\Delta\nu}{\nu} \approx 3\frac{kT_e}{m_e c^2} \tag{19}$$

For $T_e = 10$ K, $\Delta\nu/\nu = 0.0505$. Even at these high temperatures the change isn't substantial but it is nonetheless non-negligible. The low energy CMB photons that are scattered in the SZ effect have long wavelengths. The Compton wavelength, λ_C, is pretty small compared to these photon wavelengths so the change is rather small.

b) The fraction that will be scattered is equal to the probability of scattering for optical depths less than unity.

$$f = \sigma_T n_e d \ , \tag{20}$$

where $d = 6$ Mpc is the distance traveled through the gas (assumed spherical), $n_e = 10^4$ m^{-3} is the electron number density and $\sigma_T = 6.625 \times 10^{-25}$ cm^2 is the scattering cross section. So, plugging in, $f = 0.122668$, which is less than 1 so the expression is valid.

c) Wien's Law (in frequency), C&O eq. 29.59,

$$\frac{\nu_{max}}{T} = a = 5.88 \times 10^{10}\text{Hz K}^{-1} \ . \tag{21}$$

Taking differentials, we have

$$\frac{\Delta T}{T_0} = \frac{1}{a}\frac{\Delta\nu}{\nu} = 3\frac{kT_e}{am_e c^2} \tag{22}$$

3. C&O 29.17

a) The Binding energy is equivalent to the mass-energy difference between deuterium and it's constituent particles.

$$E = (m_D - m_H - m_n)uc^2 = -0.00238uc^2 = -2.23052 \text{ MeV} \ , \tag{23}$$

where u is the atomic mass unit, $u = 1.66053886 * 10 - 24$ g and the energy is negative since it's binding energy.

b) To get the wavelength

$$\lambda = \frac{hc}{|E|} = 5.56989 * 10^{-11}\text{cm} \ . \tag{24}$$

This is about 0.57 angstroms, pretty small.

c) Wien's law:

$$\lambda T = 2.8977685 \times 10^{-1} \text{ cm K} \tag{25}$$

So using the wavelength above, $T = 5.2 * 10^9$ K. In the early universe before the temperature dropped below this value the photons typically had enough energy to break up deuterium.

4. We are given the number densities per frequency on equations 1 and 2 of the problem set. So we want to integrate these over all frequencies.

a)

$$n = \frac{8\pi}{c^3} \int_0^\infty \frac{\nu^2 d\nu}{\exp\left(h\nu/kT\right) - 1} \tag{26}$$

We substitute $x = h\nu/kT$ To rewrite the integral and pull out the physical relationships.

$$n = \frac{8\pi}{c^3} \frac{k^3T^3}{h^3} \int_0^\infty \frac{x^2 dx}{\exp x - 1} = 2.404 \frac{8\pi}{c^3} \frac{k^3T^3}{h^3} \tag{27}$$

For the energy density we multiply be the energy of a photon $h\nu$ and then integrate.

$$u = \frac{8\pi}{c^3} \int_0^\infty \frac{h\nu^3 d\nu}{\exp\left(h\nu/kT\right) - 1} \tag{28}$$

Making the same substitution yields

$$u = \frac{8\pi}{c^3} \frac{k^4T^4}{h^3} \int_0^\infty \frac{x^3 dx}{\exp x - 1} = \frac{8\pi^5}{15c^3} \frac{k^4T^4}{h^3} \tag{29}$$

where the integral was evaluated in Mathematica as $\pi^4/15$ a result that can be determined using contour integration in the complex plain. Notice that the collection of constants in front of the T^4 are equivalent to the radiation constant $a = 4\sigma/c$ where σ is the stefan-Boltzmann constant.

b) For highly relativistic fermions we repeat the procedure above but with the distribution function describing fermions.

$$n = g\frac{4\pi}{c^3} \int_0^\infty \frac{\nu^2 d\nu}{\exp\left(h\nu/kT\right) + 1} \tag{30}$$

$$n = g\frac{2\pi}{c^3} \frac{k^3T^3}{h^3} \int_0^\infty \frac{x^2 dx}{\exp x + 1} = 1.803 g \frac{4\pi}{c^3} \frac{k^3T^3}{h^3} \tag{31}$$

$$u = g\frac{4\pi}{c^3} \int_0^\infty \frac{h\nu^3 d\nu}{\exp\left(h\nu/kT\right) + 1} \tag{32}$$

$$u = g\frac{4\pi}{c^3} \frac{k^4T^4}{h^3} \int_0^\infty \frac{x^3 dx}{\exp x + 1} = g\frac{28\pi^5}{120c^3} \frac{k^4T^4}{h^3} \tag{33}$$