Problem Set 5 - Solution

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1. a) We consider a source at rest in comoving coordinates at redshift $z = a(t_0)/a(t_e) - 1$. We want to find out how this redshift changes over the present time, so we differentiate z with respect to t_0.

$$\frac{dz}{dt_0} = \frac{1}{a(t_e)} \frac{da(t_0)}{dt_0} - \frac{a(t_0)}{a^2(t_e)} \frac{da(t_e)}{dt_e} \frac{dt_e}{dt_0} \tag{1}$$

It is important to note that emission time does not change at the same rate as the observation time. We now use the definition of the Hubble constant to clean up the expression, $H = \dot{a}/a$

$$\frac{dz}{dt_0} = \frac{a(t_0)}{a(t_e)} H_0 - \frac{a(t_0)}{a(t_e)} H(z) \frac{dt_e}{dt_0} , \tag{2}$$

where $H_0 = a(\dot{t_0})/a(t_0)$. In order to figure out $dt_e dt_0$ we have to consider the fact that the object is at rest in comoving coordinates. The comoving coordinate is

$$r = \int_{t_e}^{t_0} \frac{a(t_0)c}{a(t)} dt . \tag{3}$$

We also know that the comoving coordinate is a constant so differentiating this with respect to t_0 gives (using Leibniz Integral Rule with variable limits)

$$0 = \frac{a(t_0)c}{a(t_0)} - \frac{a(t_0)c}{a(t_e)} \frac{dt_e}{dt_0} . \tag{4}$$

rearranging gives

$$\frac{dt_e}{dt_0} = \frac{a(t_e)}{a(t_0)} . \tag{5}$$

This matches C&O equation 29.143 for cosmological time dilation. We can plug this into equation 1 and use our definition for the redshift to get

$$\frac{dz}{dt} = H_0(1 + z) - H(z) , \tag{6}$$

b) We calculate the change in redshift by integrating 6 from $z_0 = 0.5$ to z' and from $t = 0$ to $t = t_1 = 10$ yrs.

$$t_1 = \int_{z_0}^{z'} \frac{dz}{H_0(1+z) - H(z)} \tag{7}$$

In our universe $H(z) = H_0 E(z)$ with $E(z) = \sqrt{0.73 + 0.27(1+z)^3}$ and $H_0 = 71$ km s^{-1} Mpc^{-1}. You can evaluate the integral numerical plugging in for z' but we don't expect the redshift to change much over 10 years, so we can approximate,

$$\Delta z \approx t_1 H_0(1 + z - \sqrt{073 + 0.27(1+z)^3}) \ . \tag{8}$$

Plugging in at $z = 0.5$ gives $\Delta z \approx 1.5892 \times 10^{-10}$ which is indeed a small change. This corresponds to a velocity difference of $v = c\Delta z \approx 4.7677$ cm s^{-1}. This velocity difference is too small to be detected with current high resolution spectroscopy.

2. Imagine the universe as if it didn't recombine at $z \sim 1088$ and instead the purely hydrogen gas remained ionized. The gas is opaque to light because of the photons scattering off the sea of electrons. As the universe expands the gas becomes more dilute and the density drops. Once the density is low enough, some photons will escape the gas without scattering, we define the redshift at which this happens to be, z_t, the redshift of transparency. In this universe this also the redshift of last scattering.

   We consider a photon traveling back through time to the redshift at which it last scattered. During the journey it will traverse an optical depth (i.e. scattering probability) of

$$\tau = \int n_e \sigma_T dl = \int n_e \sigma_T c dt \ , \tag{9}$$

   where we only considering Thomson scattering and the integral in time goes from now to the time of last scattering. n_e is the electron density and σ_T is the scattering cross-section. Using $1 + z = 1/a$ and $H = da/dt$ we convert the differential in time to redshift and introduce the redshift dependence of the density.

$$\tau = \frac{\Omega_b \rho_{c,0} \sigma_T c}{m_H} \int_0^{z_t} \frac{(1+z)^3}{H(z)(1+z)} dz \ , \tag{10}$$

   where $\rho_{c,0}$ is the present critical density, $\Omega_b = 0.045$ and m_H is the mass of a hydrogen atom. We have the critical density, $\rho_{c,0} = 3H_0^2/8\pi G$. Again, in our universe $H(z) = H_0 E(z)$ with $E(z) = \sqrt{0.73 + 0.27(1+z)^3}$ and $H_0 = 71$ km s^{-1} Mpc^{-1}. So,

$$\tau = \frac{3H_0 \Omega_b \sigma_T c}{8\pi G m_H} \int_0^{z_t} \frac{(1+z)^2}{\sqrt{0.73 + 0.27(1+z)^3}} dz \ , \tag{11}$$

   The last scattering occur at $\tau \sim 1$. We can then integrate and solve for z_t at $\tau = 1$

$$\frac{8\pi G m_H \tau}{3H_0 \Omega_b \sigma_T c} = \frac{2}{3 * 0.27} \left([0.73 + 0.27(1+z_t)^3]^{\frac{1}{2}} - 1 \right) \ . \tag{12}$$

   Solving for the transparency redshift z_t,

$$1 + z_t = \left[\frac{100}{27} \left(\frac{108\pi G m_H \tau}{100 H_0 \Omega_b \sigma_T c} + 1 \right)^2 - \frac{73}{27} \right]^{1/3} \ . \tag{13}$$

Plugging in gives a value of $z_t = 47.65$

For equation 11 the denominator can be approximated to be dominated by the mass term over the redshifts in question since matter was much more dominant than the constant term in the past. That would give

$$\tau = \frac{3H_0\Omega_b\sigma_T c}{8\pi G m_H\sqrt{\Omega_{m,0}}}\int_0^{z_t}(1+z)^{1/2}dz \ , \tag{14}$$

The solution here would be

$$1+z_t = \left(\frac{4\pi G m_H \tau\sqrt{\Omega_{m,0}}}{H_0\Omega_b\sigma_T c}+1\right)^{2/3} \tag{15}$$

Plugging in would give $z_t = 47.55$, really close to our previous result; the approximation does not change the result substantially. In a complete matter dominated universe with $\Omega_{m,0} = 1$, $z_t = 75$

3. a) To keep the universe ionized after the redshift of recombination requires enough energy to ionize all the hydrogen in the universe and keep it ionized. In an equilibrium where the gas i just kept ionized, the rate of ionization per volume must be equal to the rate of recombination. The recombination rate is given by $\alpha(T)n_e n_p$. Since we consider pure hydrogen, $n_e = n_p$. For a given energy density u you can divide by the energy of ionization χ_H to get ionizations per volume and divide again by dt to get the ionization rate per volume. Thus,

$$\frac{u}{\chi_H dt} = \alpha(T)n_e n_p \ . \tag{16}$$

We are looking for the energy per comoving volume, u_c which is related to the energy density by the scale factor cubed. With $1+z = 1/a$, $u = u_c(1+z)^3$. From the previous question we also know how the electron density changes with redshift, $n_e = (1+z)^3\Omega_b\rho_{0,c}/m_H$. Using the definition of $H(z)$ and z we can convert the differential in time to one in redshift, $dt = dz/[H(z)(1+z)]$. Plugging these expressions into equation 16 we get,

$$\frac{u_c(1+z)^3}{\chi_H} = \alpha(T)\left(\frac{\Omega_b\rho_{0,c}}{m_H}\right)^2(1+z)^6\frac{dz}{H(z)(1+z)} \ . \tag{17}$$

Simplifying and integrating with from the time of recombination to the transparency time gives the required energy per comoving volume.

$$u_c = \left(\frac{\Omega_b\rho_{0,c}}{m_H}\right)^2\chi_H\int_{z_t}^{z_{rec}}(1+z)^3\alpha(T(z))\frac{1}{H_0E(z)}\frac{dz}{1+z} \ , \tag{18}$$

where $E(z) = \sqrt{0.73+0.27(1+z)^3}$ and the limits are switched going from time to redshift .

b) To evaluate numerically, we have $z_{rec} = 1088$, $z_t = 47.65$, $\Omega_b = .045$ and a constant temperature $T = 3000$ K, therefore

$$u_c = \left(\frac{\Omega_b 3H_0^2}{8\pi G m_H}\right)^2 \frac{\chi_H}{H_0}\frac{2.6\times10^{-13}\mathrm{cm}^3\mathrm{s}^{-1}}{\sqrt{0.3}}\int_{47.65}^{1088}(1+z)^2[0.73+0.27(1+z)^3]^{-1/2}dz\ . \quad (19)$$

The integral is the same as in problem 2 and can be done analytically or just evaluated numerically (using say Mathematica). Either way, we get an answer of 1.467×10^{-14} erg cm^{-3}.

c) The baryonic rest mass energy density is $u_b = \Omega_b \rho_{0,c}c^2 = 3.84\times10^{-10}$ erg cm^{-3}. So that means $u_c/u_b \approx 3.823\times10^{-5}$

Could this have come from stars. The energy available for stars relative to the baryonic rest mass energy density is the portion of the mass that is available for fuel, 0.1, times the fraction of energy released in the fusion reaction, 0.007 for hydrogen fusion, times the relative mass in stars.

$$\frac{u_*}{u_b} = 0.1*0.007*\frac{\Omega_{*,0}\rho_{0,c}c^2}{\Omega_b\rho_{0,c}c^2} = 0.1*0.007*\frac{0.0024}{0.045} \approx 3.7\times10^{-5} \quad (20)$$

Comparing to 3.823×10^{-5} this is very similar. So almost all starlight would have to be ionizing. Almost all stars would have to be of very high mass $\sim 50M_\odot$. Since these die quickly into mostly black holes they'll need to be reproduced somehow to maintain the radiation. The IMF does not produce such high mass stars in this great a number. It requires very large number of high mass stars and a large number of stellar black holes and thus very unlikely.

4. We consider the steady-state universe with constant H. The differential element of distance is related to the comoving distance by the scale factor. We start with that relation and convert the differential in space to one in time and then switch to redshifts.

$$\begin{aligned} dr &= a\,dD_c \\ cdt &= a\,dD_c\ . \end{aligned} \quad (21)$$

Since $1+z = 1/a$ and $H = a^{-1}da/dt$, we can relate the differential time to one in redshift.

$$dt = \frac{-1}{H}adz \quad (22)$$

Replacing and integrating with H constant yields

$$D_c = \frac{c}{H}\int_0^z dz = \frac{c}{H}z \quad (23)$$

The easiest way to get the luminosity distance is to be aware that the luminosity distance is related to the comoving distance by the scale factor as in C & O eq. 29.183.

$$D_L(z) = D_c(1+z) = \frac{c}{H}z(1+z)\ , \quad (24)$$

Consider objects with a constant proper density n_p. This density is related to the comoving number density as $n_p = n_c(1+z)^3$. In order to get the number of objects it's easiest to consider comoving coordinates so

$$dN = 4\pi n_c D_c^2 dD_c \tag{25}$$

To get the number with plug in to express everything in terms of redshift and integrate with n_p constant.

$$N = 4\pi n_p \frac{c^3}{H^3}\int_0^z \frac{x^2}{(1+x)^3}dx \ , \tag{26}$$

where x is the dummy variable. The integral can be done readily by integrating by parts twice leading to the result

$$N(<z) = 4\pi n_p (c/H)^3 \left(\ln(1+z) - \frac{3z^2+2z}{2(1+z)^2}\right) \ . \tag{27}$$

5. We need to compare the absolute magnitude to the apparent magnitude of a distant object with a given spectrum through a given passband, say between ν_0 and ν_1.

$$M_B - B = -2.5\log_{10}\frac{F_{10}}{F_D} = -2.5\log_{10}\frac{L_{10}D_L^2}{L_D(10\text{pc})^2} \ , \tag{28}$$

where F_{10} is the received flux from 10 pc and F_D is the received flux from the luminosity distance D_L. L_{10} and L_D are the source luminosities from the respective distances in our frequency range. These luminosities are different because the distant object is redshifted so light that is observed within the passband comes from a different part of the spectrum in the two cases. We determine the luminosity by integrating the frequency over the spectrum and multiplying by the area of the source.

$$L = A\int_{\nu_a}^{\nu_b} a\nu^{-\alpha}\, d\nu \tag{29}$$

where we let a be the proportionality constant in the spectrum. For the source at 10 pc $\nu_a = \nu_0$ and $\nu_b = \nu_1$. For the source at D_L pc $\nu_a = \nu_0(1+z)$ and $\nu_b = \nu_1(1+z)$. The source emits at these higher frequencies, the light gets redshifted into the passband and so that's what is measured.

$$\begin{aligned} L_{10} = A\int_{\nu_0}^{\nu_1} a\nu^{-\alpha}\, d\nu &= \frac{aA}{1-\alpha}(\nu_1^{1-\alpha} - \nu_0^{1-\alpha}) \\ L_D = A\int_{\nu_0(1+z)}^{\nu_1(1+z)} a\nu^{-\alpha}\, d\nu &= \frac{aA}{1-\alpha}(1+z)^{1-\alpha}(\nu_1^{1-\alpha} - \nu_0^{1-\alpha}) \end{aligned} \tag{30}$$

Plugging these expressions into equation 28 and simplifying we get

$$M_B - B = -2.5\log_{10}(1+z)^{\alpha-1} - 5\log_{10}\frac{D_L}{10\text{pc}} \tag{31}$$

We use the luminosity distance of equation 24 for the steady state universe to simplify further, taking $H = 71$ km s^{-1} Mpc^{-1}.

$$M_B = B + 2.5(1-\alpha)\log_{10}(1+z) - 5\log_{10} z(1+z) - 5\log_{10}\frac{c}{H * 10\text{pc}} \quad (32)$$
$$M_B = B - 5\log_{10} A(z) + K(z) - 43.13 \quad (33)$$

6. a) We're given the cumulative distribution for the number of quasars as a function of the apparent magnitude B valid for a given range of B. By differentiating this equation with respect to B we get the number density function.

$$N(<B) = \exp[0.84 \ln 10(B - 18.3)] \quad (34)$$
$$p(B) = \frac{dN}{dB} = 0.84 \ln 10 \exp[0.84 \ln 10(B - 18.3)]\ . \quad (35)$$

$p(B)dB$ describes the probability to find a quasar of apparent magnitude B in range dB. This is independent of redshift. So, for redshift we're told that the number of quasars in a logarithmic bin of redshift is constant within a given redshift range $.1 < z < 3$. This means that $dN/d(\ln z) = const.$ Let that constant be a. Therefore,

$$p(z) = \frac{dN}{dz} = \frac{a}{z}\ .. \quad (36)$$

$p(z)dz$ describes the probability to find a quasar of redshift z in range dz. We find the constant a by normalizing over the redshift range.

$$a = \left[\int_{0.1}^{3} z^{-1}dz\right]^{-1} = \frac{1}{\ln 30} \quad (37)$$

These are two independent probability density functions, so we take their product to get the joint probability density functions.

$$n(B,z)dBdz \simeq p(B)p(z)dBdz = \frac{0.84 \ln 10}{\ln 30} e^{0.84 \ln 10(B-18.3)} dB\frac{dz}{z} \quad (38)$$
$$n(B,z)dBdz \simeq 0.56e^{1.93(B-18.3)} dB\frac{dz}{z}\ . \quad (39)$$

b) Quasars have spectrum $F_\nu \propto \nu^{-1}$, say $F_\nu = b\nu^{-1}$. In order to consider the number of quasars we need to know how the apparent magnitude is related to the absolute magnitude. This follows equation 28 and problem 5, however now $\alpha = 1$. Going through that derivation again changes equations 30 to

$$L_{10} = A\int_{\nu_0}^{\nu_1} a\nu^{-\alpha}\, d\nu = aA \ln\frac{\nu_1}{\nu_0}$$
$$L_D = A\int_{\nu_0(1+z)}^{\nu_1(1+z)} a\nu^{-\alpha}\, d\nu = aA \ln\frac{\nu_1}{\nu_0} \quad (40)$$

Thus the $K(z)$ term drops in equation 33 leaving

$$M_B = B - 5\log_{10} z(1+z) - 43.13\ , \tag{41}$$

for the steady state universe. For quasar at $z = 0.5$, $B = 16.51$. For quasar at $z = 2$, $B = 21$.

To get the the predicted number of sources with a constant proper density n_p in a solid angle ω, per log redshift interval in the steady-state universe, we make use of the result of problem 4 (equation 27) and differentiate. However we consider only a solid angle ω instead of the whole sky. $4\pi \to \omega$

$$\frac{dN}{dz} = \omega n_p \left(\frac{c}{H}\right)^3 \frac{z^2}{(1+z)^3} \tag{42}$$

$$\frac{dN}{d\ln z} = z\frac{dN}{dz} = \omega n_p \left(\frac{c}{H}\right)^3 \left(\frac{z}{1+z}\right)^3 \tag{43}$$

We can predict the ratio of $n(z) = dN/d\ln z$ from one redshift to another using equation 43 for the steady-state universe. It predicts $r_{ss} = n(z=2)/n(z=0.5) = 8$. We now need to consider equation 39 to determine what the observed ratio is. From equation 39 the observed number per log redshift interval is

$$n' = \frac{dN'}{d\ln z} = z\frac{dN'}{dz} = 0.56\omega\ e^{1.93(B-18.3)}\ , \tag{44}$$

where we've multiplied by the solid angle since the result in part a) was per solid angle. The observed ratio r_o is

$$r_o = \frac{n'(B_2)}{n'(B_{0.5})} = \exp[1.93(B_2 - B_{0.5})] \approx 5913\ , \tag{45}$$

where, we've plugged in the apparent magnitudes from above. Thus, the steady-state theory predicts a ratio of 8 and we observe a ratio of almost 6000. The steady-state theory must be wrong.

7. 20% Extra Credit

We want to find the space density of quasars in our actual universe. We start with

$$\Phi(M_B)dM_B\frac{dV_c}{dz}dz \equiv n(B,z)dBdz = 0.56e^{1.93(B-18.3)}dB\frac{dz}{z}\ , \tag{46}$$

where V_c is the comoving volume. Rearranging the differentials and we have

$$\Phi(M_B) = \frac{0.56}{z}\exp[1.93(B(M_B,z) - 18.3)]\frac{dB}{dM_B}\frac{dz}{dV_c}\ . \tag{47}$$

We first need to figure how the apparent magnitude depends on the absolute magnitude and redshift, just as in problem 5 but in the actual universe and for quasars with $\alpha = 1$. Following that derivation and using $\alpha = 1$ as in problem 6b yields,

$$M_B - B = -5\log_{10}\frac{D_L}{10\text{pc}} \;, \tag{48}$$

where D_L is the luminosity distance. As in problem 5, $D_L = (1+z)r$, where r is the comoving distance. Using $dr = cdt$ and converting to redshift in the usual manner it is given by

$$r = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')} = \frac{c}{H_0} I(z) \;, \tag{49}$$

where for our universe $E(z) = \sqrt{0.73 + 0.27(1+z)^3}$. For ease we've defined the integral as $I(z)$. Putting these into 48

$$B = M_B + 5\log_{10}\frac{c}{H_0 * 10\text{pc}} + 5\log_{10}[(1+z)I(z)] = M_B + 43.13 + 5\log_{10}[(1+z)I(z)] \;. \tag{50}$$

This equation gives the apparent magnitude as a function of the independent variables M_B and z. We are evaluating equation 47 at a particular redshift and given the magnitude relation, $dB/dM_B = 1$. We consider quasars at redshifts 0.5 and 2. Using Mathematica, $I(z = 0.5) = 0.445668$ and $I(z = 2) = 1.24223$. So for redshift 0.5 quasars $B = 16.256$. For redshift 2 quasars $B = 19.987$.

The next part we need is to know how the comoving volume changes with redshift. The comoving volume differential is

$$dV_c = 4\pi r^2 dr \;. \tag{51}$$

where r is the comoving coordinate of equation 49. Therefore

$$dV_c = 4\pi\frac{c^3}{H_0^3} I(z)^2 \frac{dz}{E(z)} \tag{52}$$

Plugging into equation 47

$$\Phi = \frac{0.56}{z}\frac{E(z)}{4\pi I(z)^2}\frac{H_0^3}{c^3}\exp[1.93(B - 18.3)] \tag{53}$$

$$\Phi = \frac{0.56}{z}\frac{\sqrt{0.73 + 0.27(1+z)^3}}{4\pi I(z)^2}\frac{H_0^3}{c^3}\exp[1.93(B - 18.3)] \tag{54}$$

Using the values for a redshift, $z = 0.5$, we get $\Phi = 1.47478 \times 10^{-13}$ Mpc^{-3} mag^{-1}. For $z = 2$, we get $\Phi = 1.40626 \times 10^{-11}$ Mpc^{-3} mag^{-1}. Quasars more common earlier in the universe.