Problem Set 4 - Solution

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1. **Age of the Universes**
As done in class, the present-day age of a Universe with density parameters $(\Omega_\Lambda, \Omega_k, \Omega_M, \Omega_r)$ is,

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{1}{E(z)} \frac{dz}{1+z} \tag{1}$$

where $E(z) = \sqrt{\Omega_{\Lambda 0} + \Omega_k(1+z)^2 + \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4}$, '$z$' denotes the redshift, and the subscript '0' refers to present-day values. Remember that $\sum_X \Omega_X = 1$, by definition. We also take $H_0 = 70$ km s^{-1} Mpc^{-1}. We plug in the appropriate Ω values (current density relative to the critical density) into $E(z)$ and evaluate the integral in equation 1 for each universe.

   a) For the current concordance model.
   $(\Omega_{\Lambda 0}, \Omega_{r0}, \Omega_{m0}, \Omega_k) \approx (0.73, 0, 0.27, 0) \Rightarrow t_0 = 13.9$ Gyr.

   b) Spatially flat (critical density) universe made entirely of pressureless matter.
   $(\Omega_{\Lambda 0}, \Omega_{r0}, \Omega_{m0}, \Omega_k) = (0, 0, 1, 0) \Rightarrow t_0 = 9.3$ Gyr.

   c) Spatially open universe with a pressureless matter density 0.27 of the critical density. No vacuum energy density (i.e. no cosmological constant)
   $(\Omega_{\Lambda 0}, \Omega_{r0}, \Omega_{m0}, \Omega_k) \approx (0, 0, 0.27, 0.73) \Rightarrow t_0 = 11.4$ Gyr.

   d) Spatially closed universe with pressureless matter density twice the critical density.
   $(\Omega_{\Lambda 0}, \Omega_{r0}, \Omega_{m0}, \Omega_k) \approx (0, 0, 2, -1) \Rightarrow t_0 = 8.0$ Gyr.

   I used Mathematica to evaluate the integral in 1.

2. **Ages in Our Universe**
Starting with the Friedman equation,

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_\Lambda + \rho_m + \rho_r) - \frac{kc^2}{a^2} \tag{2}$$

we can write,

$$\begin{aligned} H^2(a) &= \frac{8\pi G}{3}\left[\rho_{\Lambda 0} + \rho_{m0}\left(\frac{a_0}{a}\right)^3 + \rho_{r0}\left(\frac{a_0}{a}\right)^4\right] - \frac{kc^2}{a^2} \\ &= H_0^2\left[\Omega_{\Lambda 0} + \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{r0}\left(\frac{a_0}{a}\right)^4 - \frac{kc^2}{H_0^2 a_0^2}\left(\frac{a_0}{a}\right)^2\right] \\ \therefore H^2(z) &= H_0^2\ E^2(z) \end{aligned} \tag{3}$$

since $a/a_0 = 1/(1+z)$ and where $E(z)$ is the function defined above. H(z) comes in the integrand when calculating the age of the Universe at a particular redshift (say z_1, when the age of the Universe is t_1 and scale factor is a_1).

$$t_1 = \int_0^{t_1} dt = \int_0^{a_1} \frac{da}{\dot{a}} \times \frac{a}{a} = \int_0^{a_1} \frac{da}{a \cdot H(a)} = \int_{z_1}^{\infty} \frac{dz}{(1+z) \cdot H(z)} = \frac{1}{H_0} \int_{z_1}^{\infty} \frac{1}{E(z)} \frac{dz}{1+z} \tag{4}$$

Thus we can calculate the age of the current concordance model, $(\Omega_{\Lambda 0}, \Omega_{r0}, \Omega_{m0}, \Omega_k) \approx (0.73, 0, 0.27, 0)$, as,

a) $T_{CMB,1}/T_{CMB,0} = a_0/a_1 = 1 + z_1$. We know that $T_{CMB,1} = 3000$ K and $T_{CMB,0} \approx 2.73$ K. Therefore, $z_1 \approx 1100$. Integrating equation 4 using this value we get $t_1 = 0.49$ Myr

b) $z_1 = 8 \Rightarrow t_1 = 0.66$ Gyr

c) $z_1 = 3 \Rightarrow t_1 = 2.2$ Gyr

d) $z_1 = 1 \Rightarrow t_1 = 6.0$ Gyr

e) $z_1 = 0 \Rightarrow t_1 = 13.9$ Gyr