1. **Age of the Universes**

As done in class, the present-day age of a Universe with density parameters \((\Omega_\Lambda, \Omega_k, \Omega_M, \Omega_r)\) is,

\[
t_0 = \frac{1}{H_0} \int_0^\infty \frac{1}{E(z)} \frac{dz}{1+z} \tag{1}
\]

where \(E(z) = \sqrt{\Omega_\Lambda + \Omega_k(1+z)^2 + \Omega_m(1+z)^3 + \Omega_r(1+z)^4}\), \(z\) denotes the redshift, and the subscript '0' refers to present-day values. Remember that \(\sum \Omega_x = 1\), by definition. We also take \(H_0 = 70\) km s\(^{-1}\) Mpc\(^{-1}\). We plug in the appropriate \(\Omega\) values (current density relative to the critical density) into \(E(z)\) and evaluate the integral in equation 1 for each universe.

a) For the current concordance model.

\((\Omega_\Lambda, \Omega_r, \Omega_m, \Omega_k) \approx (0.73, 0, 0.27, 0) \Rightarrow t_0 = 13.9\) Gyr.

b) Spatially flat (critical density) universe made entirely of pressureless matter.

\((\Omega_\Lambda, \Omega_r, \Omega_m, \Omega_k) = (0, 0, 1, 0) \Rightarrow t_0 = 9.3\) Gyr.

c) Spatially open universe with a pressureless matter density 0.27 of the critical density.

No vacuum energy density (i.e. no cosmological constant)

\((\Omega_\Lambda, \Omega_r, \Omega_m, \Omega_k) \approx (0, 0, 0.27, 0.73) \Rightarrow t_0 = 11.4\) Gyr.

d) Spatially closed universe with pressureless matter density twice the critical density.

\((\Omega_\Lambda, \Omega_r, \Omega_m, \Omega_k) \approx (0, 0, 2, -1) \Rightarrow t_0 = 8.0\) Gyr.

I used Mathematica to evaluate the integral in 1.

2. **Ages in Our Universe**

Starting with the Friedman equation,

\[
H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} (\rho_\Lambda + \rho_m + \rho_r) - \frac{kc^2}{a^2} \tag{2}
\]

we can write,

\[
H^2(a) = \frac{8\pi G}{3} \left[ \rho_{\Lambda 0} + \rho_{m0} \left(\frac{a_0}{a}\right)^3 + \rho_{r0} \left(\frac{a_0}{a}\right)^4 \right] - \frac{kc^2}{a^2}
= H_0^2 \left[ \Omega_{\Lambda 0} + \Omega_{m0} \left(\frac{a_0}{a}\right)^3 + \Omega_{r0} \left(\frac{a_0}{a}\right)^4 - \frac{kc^2}{H_0^2 a_0^2} \left(\frac{a_0}{a}\right)^2 \right]
\]

\[
\therefore H^2(z) = H_0^2 E^2(z) \tag{3}
\]
since $a/a_0 = 1/(1 + z)$ and where $E(z)$ is the function defined above. $H(z)$ comes in the integrand when calculating the age of the Universe at a particular redshift (say $z_1$, when the age of the Universe is $t_1$ and scale factor is $a_1$).

$$t_1 = \int_{0}^{t_1} dt = \int_{0}^{a_1} \frac{da}{a} \times \frac{a}{a} = \int_{0}^{a_1} \frac{da}{a \cdot H(a)} = \int_{z_1}^{\infty} \frac{dz}{(1 + z) \cdot H(z)} = \frac{1}{H_0} \int_{z_1}^{\infty} \frac{dz}{E(z) \cdot (1 + z)} \quad (4)$$

Thus we can calculate the age of the current concordance model, $(\Omega_{\Lambda 0}, \Omega_{r 0}, \Omega_{m 0}, \Omega_{k}) \approx (0.73, 0, 0.27, 0)$, as,

a) $T_{CMB,1}/T_{CMB,0} = a_0/a_1 = 1 + z_1$. We know that $T_{CMB,1} = 3000$ K and $T_{CMB,0} \approx 2.73$ K. Therefore, $z_1 \approx 1100$. Integrating equation 4 using this value we get $t_1 = 0.49$ Myr

b) $z_1 = 8 \Rightarrow t_1 = 0.66$ Gyr

c) $z_1 = 3 \Rightarrow t_1 = 2.2$ Gyr

d) $z_1 = 1 \Rightarrow t_1 = 6.0$ Gyr

e) $z_1 = 0 \Rightarrow t_1 = 13.9$ Gyr