Problem Set 2 - Solution

E.S. Phinney, Kunal Mooley and J. Sebastian Pineda

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1. **The Virial Theorem:**

   a) Consider a collection of N particles interacting via attractive 'harmonic oscillator gravity.' The total force on mass, m_i is

$$\mathbf{F}_i = -\sum_{j\neq i} Gm_i m_j(\mathbf{r}_i - \mathbf{r}_j) \ . \tag{1}$$

So the force on particle i due to particle j is $\mathbf{F}_{ij} = -Gm_i m_j(\mathbf{r}_i - \mathbf{r}_j)$. After close examination of the derivation in C&O, you'll see that the beginning of the derivation is exactly the same despite the new force, so equations 2.40 and 2.41 still hold.

$$\frac{1}{2}\frac{d^2 I}{dt^2} - 2K = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \ . \tag{2}$$

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = \frac{1}{2}\sum_i \sum_{j\neq i} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j) \ . \tag{3}$$

To evaluate the RHS we use $\mathbf{F}_{ij}$,

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = -\frac{1}{2}\sum_i \sum_{j\neq i} Gm_i m_j |\mathbf{r}_i - \mathbf{r}_j|^2 \ , \tag{4}$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the scalar distance between the particles. So we can recognize the potential energy in the expression as $U_{ij} = \frac{1}{2}Gm_i m_j r_{ij}^2$.

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = -\sum_i \sum_{j\neq i} U_{ij} \ . \tag{5}$$

From the double sum the potential energy is counted twice, so the RHS evaluates to $-2U$, twice the total potential energy. Plugging back into 3 and taking time averages yields

$$\left\langle \frac{1}{2}\frac{d^2 I}{dt^2} \right\rangle - 2\langle K\rangle = -2\langle U\rangle \tag{6}$$

$$-2\langle K\rangle = -2\langle U\rangle \ , \tag{7}$$

where the average of the moment of inertia derivative vanishes over the periodic motion of our particle orbits. Equation 7 is thus our expression for the virial theorem. Since $\langle E\rangle = \langle K\rangle + \langle U\rangle$, we also have

$$\langle E\rangle = 2\langle K\rangle = 2\langle U\rangle \ . \tag{8}$$

b) We can ascribe a temperature T to the particles by defining kT as the average energy per particle, where k is the Boltzmann constant.

$$\frac{3}{2}kT = \frac{\langle K\rangle}{N} = \frac{\langle E\rangle}{2N} \ . \tag{9}$$

Thus, there is a direct relationship between the energy of the system and the temperature of the system. Since the number of particles remains fixed, when the energy drops the temperature drops and the system cools, so there is a positive specific heat capacity.

c) Consider two systems, one at temperature T_1 and the other at T_2 such that $T_1 > T_2$. A Heat, Q , will flow from hot to the cold. Heat is energy so, when the temperature of system 1 drops by $3\Delta T/2 = Q/2kN_1$, system 2 increases by the same amount. This is only true as energy and temperature are positively correlated. This will continue as long as $T_1 \neq T_2$ with the hotter system always giving energy to the colder system to warm it until they are equal and thus at equilibrium.

d) From C&O 2.46 and 2.47

$$\langle E\rangle = -\langle K\rangle \ . \tag{10}$$

So using our formalism from part b) we arrive at $3kT/2 = -\langle E\rangle/N$, meaning that changes in energy are negatively correlated to changes in temperature.

$$\Delta T \propto -\Delta E \ . \tag{11}$$

So if the energy drops, the temperature increases and vice versa. This corresponds to a negative specific heat capacity for self-gravitating systems.

e) If we consider two self gravitating systems in thermal contact. The hotter one will transfer heat to the colder one according to thermodynamics. The loss of heat means a drop in energy and according to equation 11 an increase in the temperature. The colder body will gain energy and get colder accordingly. The bodies will continue this and drive well out of equilibrium.

2. **II. Discover Dark Matter Yourself** a) From Figure 2, clustering is evident in the region $cz > 4000$ km/s, $cz < 9500$ km/s. Mean radial velocity of the selected galaxies (183 in number) is then cz = 6811 km/s. Then, $cz/H_0 = 6811/70 = 97$ Mpc (officialy, mean cz= 6900 km/s)

   b) Non-members of the cluster clearly have $cz < 4000$ km/s, $cz > 9500$ km/s (from Figure 2). Notice the second cluster behind the central region at cz= $50,000$ km/s. For the central region, $\sigma_r = 1016$ km/s (officialy, $\sigma_r = 1100$ km/s). For the outlying NGC 4839 region, $\sigma_r = 1088$ km/s (officialy, $\sigma_r = 1100$ km/s).

c) $2R = D\theta = (97 \text{ Mpc}) \cdot (1.3°) \cdot (\pi^c/180°) = 2.2$ Mpc.

d) $\sigma_r/H_0 = (1016 \text{ km/s})/(70 \text{ km/s/Mpc}) = 16$ Mpc.

e) 5 σ_r^2R / G = 2×10^{15} $M_\odot$. The 5 is dubious and depends on the density profile (Carrol & Ostlie take a flat, really centrally concentrated profile, so 2 is a better coefficient.)

f) M = (0.73 $M_\odot/L_{\odot K}$) $\cdot$ (2.3 $\times 10^9$ $L_{\odot K}$) $\cdot$ (97 Mpc/Mpc)^2 = 1.6×10^{13} $M_\odot$. This is 0.8 % of the total cluster mass estimated in part (e). See also Rines et al. 2001.

g) From C&O page 1068, $M_{gas} = 1.05 \times 10^{14}$, which is 5.2% of the total cluster mass estimated in part (e).

h) For the central region, kurtosis is -0.095 (implies that the distribution is near Gaussian), while that for the outlying region is -1.06 (implies that the distribution is rather flat-top). See also Figure 3.

i) As stated in the hint, Gaussian distributions in radial velocity result when the galaxies have had many momentum-exchanging encounters with each other. $t_{cross} = D/\sigma_r = (2.2\, Mpc)/(1016\, km/s) = 2.2 \times 10^{16}\, s = 0.7$ Gyr, which is about 5% of the age of the universe, 13.7 Gyr.

$$\text{No. of encounters} = (\text{covering fraction}) \times \left(\frac{Age}{t_{cross}}\right) \approx (\text{covering factor}) \times 0.05$$

From Figures 2 and 3 given in the Problem Set, we can see that the covering fraction (= fraction of the region occupied by galaxies in a 2-dimensional projection) of the central region is much larger than that of the outlying region.

$$\begin{aligned} \text{No. of encounters} &\gtrsim 1 \quad \text{for the central region} \\ &\ll 1 \quad \text{for the outlying region} \end{aligned}$$

Thus, the galaxies in the central region have had enough time to achieve an approximate Maxwellian distribution of radial velocities, but not so for the galaxies in the outlying region.

3. **Dynamical Friction**

This question follows the discussion on dynamical friction of section 26.1 in C&O (pg 1001 of 2nd Edition). Equation 26.3 of C& O expresses the maximum distance within which clusters would have spiraled into the galaxy due to dynamical friction.

$$r_{max} = \sqrt{\frac{tCGM}{2\pi v_M}}\ . \tag{12}$$

Using $t = 1.25 \times 10^{10}$ years as our age estimate for the galaxy, $C = 76$ for globular clusters, a cluster mass of $M = 2 \times 10^6 M_\odot$ and a rotation curve velocity for the Milky Way of $v_M = 220$ km s^{-1}, we get maximum distance of $r_{max} = 2.46$ kpc. The scale length of the galactic bulge

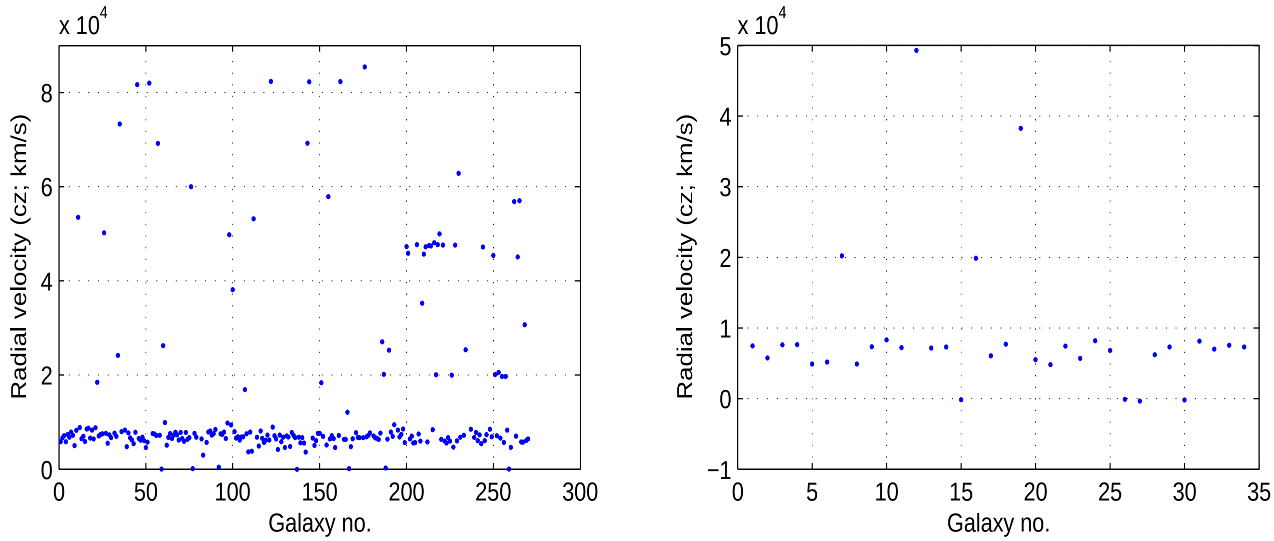

Figure 1: The clustering in radial velocity can be clearly seen in the region $4000 < cz < 9500$ km/s. Galaxies lying outside this region are not members the Coma cluster. Figure on the left is for the central region (253 points plotted) and that on the right is for the outlying region (34 points plotted).

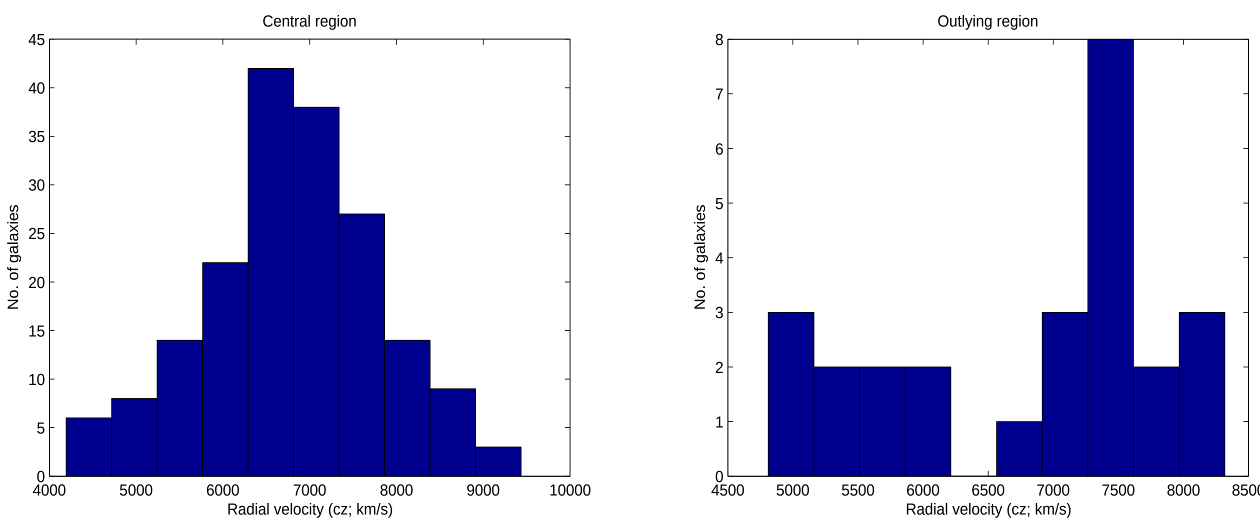

Figure 2: Histogram of the cz values of selected galaxies in the central and outlying regions of the Coma cluster. It shows the Gaussian nature of the distribution of radial velocities of galaxies in the central region, and the top-hat nature for those in the outlying region. Plots made using Matlab.

is usually fitted as about 1.9-2.7kpc (see e.g. Table 1, Table 3 and equations 1 and 2 of http://adsabs.harvard.edu/abs/2009A&A...498...95V. So, the dynamical friction result is comparable to the radius of the bulge as the outer edge is ill-defined.