Problem Set 1 - Solution

E.S. Phinney, Kunal Mooley and J. Sebastian Pineda

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1. **Finding and interpreting data on a galaxy:**

   a) Perusing the database you can find several images of NGC2639. The most useful is the HST image reproduced below. From the image you can guess that the galaxy is probably an early non-barred spiral galaxy of Hubble Type Sa.

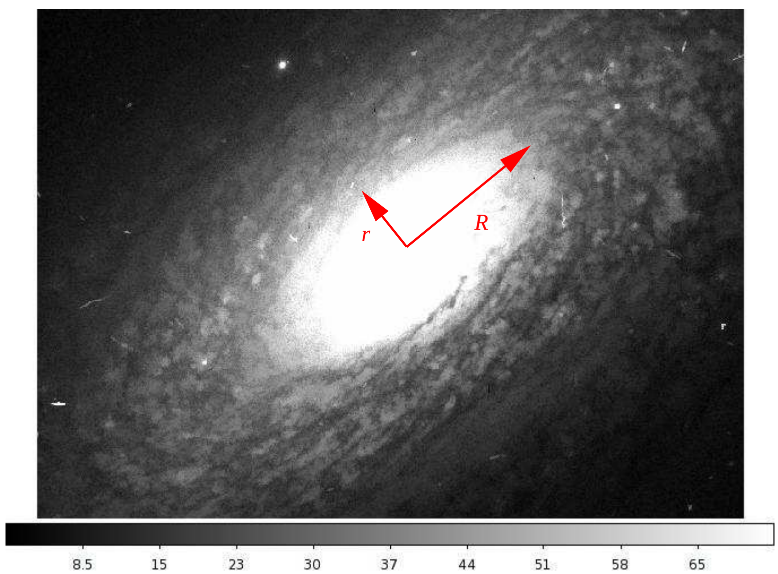

Figure 1: HST imge of NGC2639 with arrows overlaid showing the principal axes of the projected ellipse to an isophote.

   b) In order to estimate the inclination angle, ι, we assume the isophotes of the spiral galaxy would be very nearly circular if viewed face-on. When inclined, the projected isophotes will then appear elliptical, with an axis ratio independent of the isophote chosen to measure it. We can always rotate the galaxy in the plane of the sky, so we'll assume we're looking at the galaxy at the paralactic angle of 135°, to place the long axis horizontally. Along the long axis the distance from the center to the edge of the galaxy corresponds to the actual radius of the disk, R, and along the short axis the observed projection is the length r (see HST image, Figure 1). In Figure 2 we demonstrate the associated geometry for observing the galaxy.

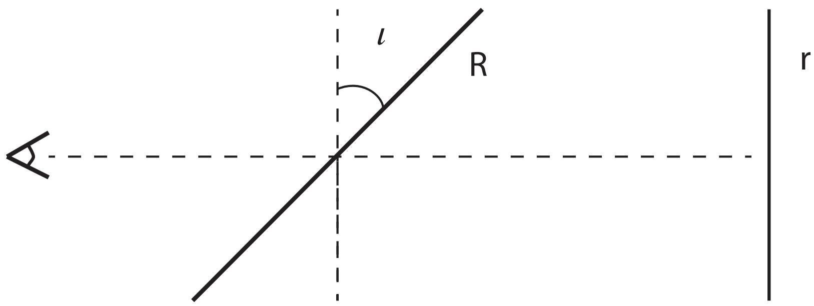

Figure 2: Diagram of geometry for projected ellipse of galaxy with observer on the left. R is the actual radius of the galaxy with inclincation angle ι and r is the observed distance of the short axis.

From the geometry you can see that the lengths are related by the inclination angle according to

$$r = R\cos\iota\ .$$

Measuring r and R to a given isophote (level of surface brightness) on a common length scale (you can just use a ruler placed over the screen) and taking the ratio gives a value of $\cos\iota \approx 0.42$ and an inclincation $\iota \approx 65°$.

c) We assume that the stars and gas of the spiral galaxy are orbiting in circles about the center and look at the radial velocity curve of NGC 2639 to figure out the recessional velocity. We then use that to get the circular velocity about the center while accounting for the factor due to the inclinatio (see Figure 3).

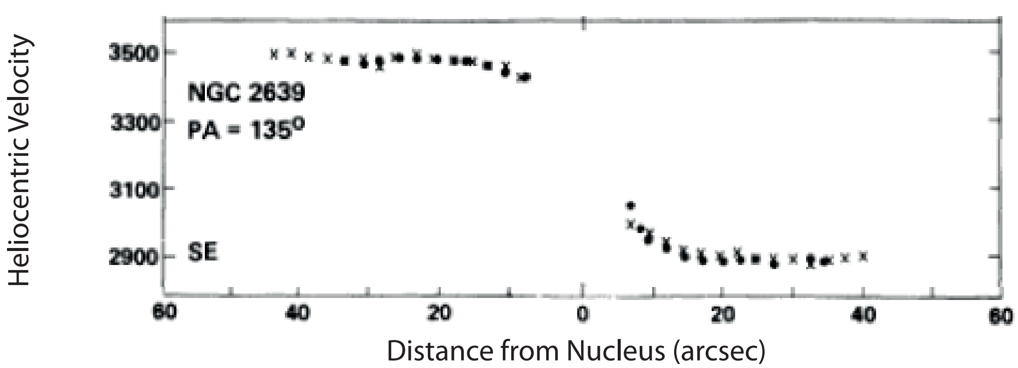

Figure 3: Radial velocity curve for NGC 2639

i. We can get the recessional velocity by noting that the measured radial velocities on each side of the nucleus are symmetric about the mean motion of the galaxy. Taking the max velocities in each wing as approximately 3500 km s^{-1} and 2900 km s^{-1} we can compute a recessional velocity of $v_r = (3500 + 2900)/2 = 3200$ km s^{-1}

ii. The observed radial velocities are a combination of the mean motion and the circular motion as

$$v_{obs} = v_r + v_c \sin \iota \ .$$

The maximum circular velocity is about 300 km s^{-1} relative the mean motion of the galaxy so we compute a corrected velocity of $v \approx 300/\sin 65° = 331$ km s^{-1}.

iii. If we take NGC 2639 and the Sun to be cosmic observers and we note that the galaxy is in the local universe we can compute the distance using

$$d = \frac{v_r}{H_0} \ .$$

Plugging in our value and using the current value of the Hubble constant, $H_0 =$ 73 km s^{-1}Mpc^{-1} gives $d \approx 44$ Mpc.

d) From the SDSS data we have apparent magnitudes of $ugriz = 14.05, 12.2, 11.29, 10.92, 10.55$. We can relate the magnitudes to the luminosity and distance,

$$m - M_\odot = -2.5 \log \frac{L \ (10 \text{ pc})^2}{L_\odot \ d^2}$$

Rearranging and plugging in with the g-band magnitudes and using our distance value from earlier we get a luminosity of $L_g = 2.85 \times 10^{10} \ L_{\odot,g}$

e) The Tully-Fisher diagram was populated with nearby galaxies in which the distance could be calculated using $d = v_r/H_0$. Using this distance value and the measured apparent magnitudes the absolute magnitudes of the vertical axis in figure 25.10 of C& O was computed. By changing the value of H_0 used in the calculation you shift the vertical axis of the figure. Using the same magnitude-distance relation we can compute the dependence the absolute magnitude has on the Hubble constant.

$$M = m + 5 \log \frac{10\text{pc} \ H_0}{v_r}$$

If we look at the difference we would get by using an updated value for the Hubble constant we get the required shift for the vertical axis.

$$M' = M + 5 \log \frac{H_0'}{H_0} \approx M + 0.872 \ ,$$

where M' corresponds to the updated vertical axis and M corresponds to the old axis with $H_0 = 50$ km s^{-1}Mpc^{-1} and $H_0' = 73$ km s^{-1}Mpc^{-1}.

f) Our circular velocity is 331 km s^{-1} and we can compute the absolute B-band magnitude with the distance, so $M_B = m + 5\log(10\text{pc}/d) = -21.0$. Figure 4 reproduces the figure from C&O with the points shifted according to part e) . The red square is our new point plotted in the diagram and the blue circle is centered on what is likely the point on the original plot corresponding to NGC 2639. We note that they coincide rather well.

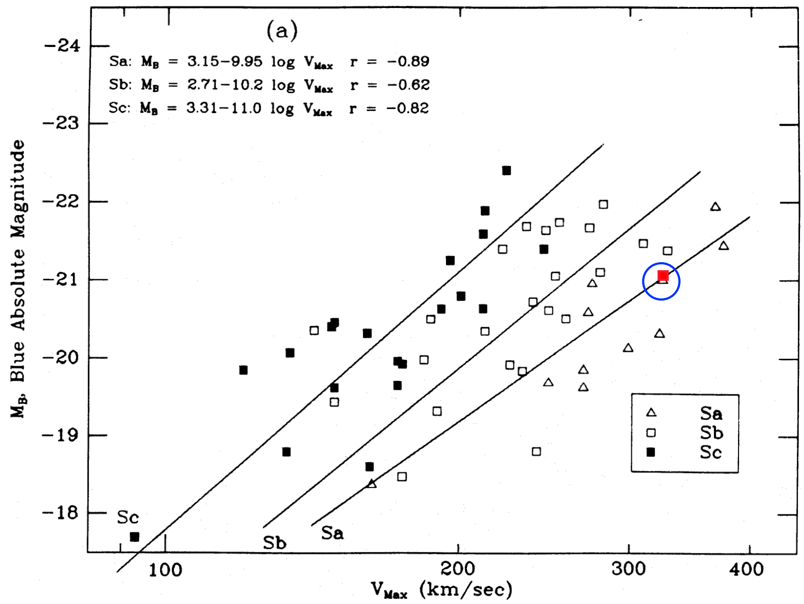

Figure 4: Tully-Fisher relation for spirals. Figure 25.10 of C& O with points shifted to account for updated Hubble constant

g) We assume that the circular velocity remains flat out to the edge of the galaxy corresponding to 60 arcsec. We relate the acceleration experience by stars out at this radius to the mass enclosed at this distance.

$$\frac{v_c^2}{R} = \frac{GM}{R^2} ,$$

where in this case M is the mass and R is the radius of the galaxy. Using our distance from earlier and the seperation of 60 arcsec we get $R = 60 * 44 \times 10^6/206265 = 12.8$ kpc. Plugging in and solving for mass we get $M = 3.26 \times 10^{11}\ M_\odot$.

h) To calculate the mass-to-light ratio you take the answer in part g) and divide by the answer in part d). So $3.26 \times 10^{11}/2.85 \times 10^{10} = 11.4$ in solar masses to solar luminosities. That this number is rather greater than unity indicates that the mass responsible for the dynamics off the galaxy is unaccounted for in the observed light.

2. **Measure a redshift yourself:**

Use the defintion of redshift: $\lambda_o/\lambda_e = 1 + z$. Examples of lines which can be used to calculate the redshift are given in Table 1 and shown in Figure 1. You can form a matrix or try and identify individual absorption lines. Consideration of either mean, median or mode of the (1+z) values should give you an answer close to the actual value, $z = 0.1428$.

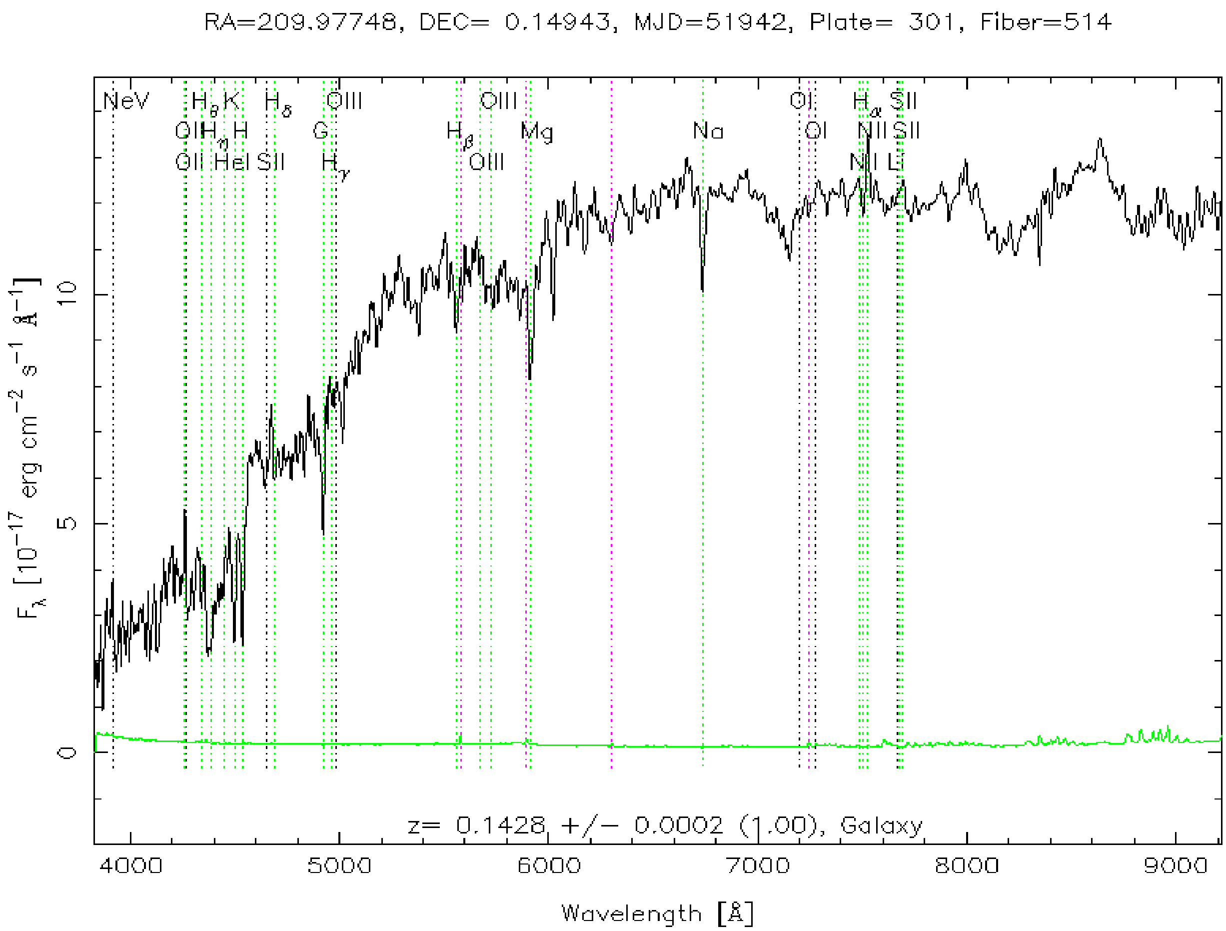

Figure 5: The given spectrum is that of galaxy SDSS J135954.59+000857.9. Some spectroscopic lines which might be considered for the calculation of redshift are marked. Additional exercises based on this data can be found on the SDSS webpage.

Table 1: Expected and observed wavelengths of some absorption lines.

ion	λ_e (Å)	λ_o (Å)
Ca II, K	3934	4496
Ca II , H	3968	4535
Fe I, G	4308	4923
Hβ	4861	5555
Hα	6563	7500

3. **Build-a-galaxy:**

a) We are building a galaxy that is infinite and plane-parallel, from that symmetry to be true our distribution function can only be a function of z and v_z. We want to get a mass density from our distribution function $f(z, v_z)$. We know that the distribution function describes the particle number density per velocity range. Since we assume that the particles are all of the same mass we can get a mass density per velocity range by multiplying by the mass of each particle. In order to get the mass density you must then integrate over all possible velocities hence,

$$\rho(z) = \int_{-\infty}^{\infty} m f(z, v_z) dv_z \ . \tag{1}$$

b) We use a gravitational potential that does not change with time, so that the *energy* $E = \phi + v_z^2/2$ is a constant for each particle. Thus, $\partial E/\partial t = 0$. If we choose $f(z, v_z) = f(E)$ then we can show that the distribution function does not change with time and the galaxy will be unchanging.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}E}\frac{\partial E}{\partial t} = 0 \ . \tag{2}$$

f does not change with time and thus the galaxy remains the same over time.

c) In Newtonian gravity there is an expression analogous to Gauss' Law in electrodynamics relating the gravitational field to the density of sources i.e. mass.

$$\vec{\nabla} \cdot \vec{\mathrm{g}} = -4\pi G\rho \ , \tag{3}$$

where $\vec{\mathrm{g}}$ is the gravitational field and ρ is the mass density. Since the field is the gradient of the potential, $\vec{\mathrm{g}} = -\vec{\nabla}\phi$, we can substitute in equation 3 and use equation 1 to get,

$$\frac{d^2\phi}{dz^2} = 4\pi G m \int_{-\infty}^{\infty} f(\phi + v_z^2/2)\, dv_z \ , \tag{4}$$

where the only non-zero derivatives remaining are with respect to z since we are making a infinite plane-parallel galaxy.

d) We take the stars to be distributed within H of the mid-plane with surface density Σ.

i. The surface density of the disk is give by

$$\Sigma = \int_{-\infty}^{\infty} \rho \, \mathrm{d}z \; . \tag{5}$$

If we assume that the stars are distributed roughly uniformly with density ρ_0 near the mid-plane and that most of the stars are confined to $|z| < H$ then

$$\Sigma = \rho_0 \int_{-H}^{H} \mathrm{d}z = 2\rho_0 H \; , \tag{6}$$

which can be solved for ρ_0.

ii. Because of the symmetry of our infinite plane-parallel galaxy we know that the field lines are parallel to the z axis, similar to the case of an infinite slate of charge.

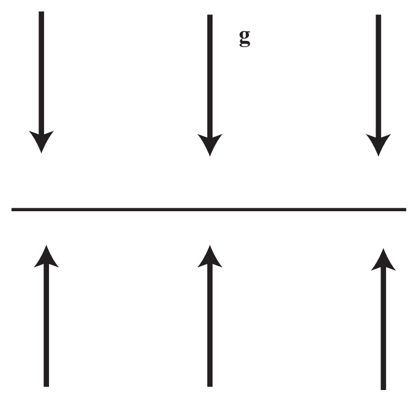

Figure 6: Gravitational field for infinite plane-parallel galaxy. Test mass would fall toward the plane.

If we take a pillbox of surface area A and height $2z$ centered on the mid-plane and integrate equation 3 with the volume enclosed we get, after applying Gauss' theorem,

$$\int \vec{\mathrm{g}} \cdot \mathrm{d}\vec{A} = -4\pi G \int \rho \, \mathrm{dV} \; , \tag{7}$$

For $z \ll H$, we evaluate the integral using ρ_0 as the value of the density near the mid-plane. The surface area is oriented outward anitparallel to the field, so

$$\begin{aligned} -|\vec{\mathrm{g}}|A &= -8\pi G\rho_0 A z & (8)\\ |\vec{\mathrm{g}}| &= 8\pi G \rho_0 z \; . & (9) \end{aligned}$$

Next, $\vec{\mathrm{g}} = -\vec{\nabla}$, so $\mathrm{d}\phi/\mathrm{d}z = 8\pi G\rho_0 z$. Integrating the last equation we get

$$\phi = 4\pi G \rho_0 z^2 \; , \tag{10}$$

where we don't care about the constant offset of the potential.

For the case $z \gg H$, we follow the same procedure but now the bounds of the integral go out past H. Since the stars are contained within $|z| < H$, H replaces z in the limits. So eq. 9 becomes,

$$|\vec{\mathrm{g}}| = 4\pi G\rho_0 H = 2\pi G\Sigma \tag{11}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = 2\pi G\Sigma \; . \tag{12}$$

Integrating the last equation thus gives

$$\phi = 2\pi G\Sigma z \; , \tag{13}$$

where again we don't care about the constant offset.

iii. We consider the particles to turn around at around H since beyond this height there are no more stars. The energy (per mass), $E = \phi + v_z^2/2$ is a constant of the motion. At turnaround the particle isn't moving and so the total energy is given by $E = 4\pi G\rho_0 H^2$. At the mid-plane $z = 0$ the potential goes to zero so the energy is all in the kinetic motion of the particle, so setting $E = v_z^2/2$ and solving for the velocity gives

$$v_z \approx \sqrt{8\pi G\rho_0} H \; . \tag{14}$$

e) Using an equation for $f(E)$ based on thermal physics, we can set up equation 4 as

$$\frac{d^2\phi}{dz^2} = 4\pi Gm \int_{-\infty}^{\infty} \frac{\rho_o}{m\sqrt{2\pi\sigma^2}} e^{-E/\sigma^2}\, dv_z \; . \tag{15}$$

Plugging in for E and taking the factors out that don't depend on v_z we get a gaussian integral that can be evaluated readily. We divide both sides by σ^2 to get the form,

$$\frac{d^2(\phi/\sigma^2)}{dz^2} = \frac{4\pi G\rho_0}{\sigma^2} e^{-\phi/\sigma^2} \; . \tag{16}$$

To examine we velocity distribution we need to look at the integral of the distribution function over the volume.

$$f_v = \int f(E)\mathrm{d}V \tag{17}$$

$$= \frac{\rho_0}{m\sqrt{2\pi\sigma^2}} e^{-v_z^2/2\sigma^2} \int e^{-\phi/\sigma^2}\mathrm{d}V \; . \tag{18}$$

The expression for the velocity distribution is gaussian in the velocity and has a factor that depends on the integral of the potential. The factor however will not change the form of the distribution. Thus, the velocity distribution is Maxwellian.

f) From our description of the field (Figure 6) we see that the field is symmetric about the mid-plane. At the mid-plane the gravitational field cannot point in any direction without breaking this symmetry. Additionally, at the mid-plane the gravitational force due to all the other masses must add to zero. Therefore since the field is the gradient of the potential, $\mathrm{d}\phi/\mathrm{d}z(0) = 0$. Because we have an infinite plane-parallel galaxy it does not make sense to set the potential at infinity to zero instead we set the zero point of the potential to be at the mid-plane. The constant doesn't really matter but setting it to zero at the origin avoids any extra normalization factor being introduced into the distribution function, so $\phi(0) = 0$.

g) We wish to make the equations dimensionless by introducing $\zeta = z/H$, $H = \sigma/\sqrt{8\pi G\rho_0}$ and $\psi = \phi/\sigma^2$.

$$\begin{aligned}
\frac{\mathrm{d}^2(\phi/\sigma^2)}{\mathrm{d}z^2} &= \frac{4\pi G\rho_0}{\sigma^2} e^{-\phi/\sigma^2} \\
\frac{1}{H^2}\frac{\mathrm{d}^2(\psi)}{\mathrm{d}\zeta^2} &= \frac{4\pi G\rho_0}{\sigma^2} e^{-\psi} \\
\frac{8\pi G\rho_0}{\sigma^2}\frac{\mathrm{d}^2(\psi)}{\mathrm{d}\zeta^2} &= \frac{4\pi G\rho_0}{\sigma^2} e^{-\psi} \\
2\frac{\mathrm{d}^2\psi}{\mathrm{d}\zeta^2} &= e^{-\psi} \; .
\end{aligned} \tag{19}$$

From the definition of ψ it is thus evident that $\phi(0) = 0$ and $\mathrm{d}\phi/\mathrm{d}z(0) = 0$ implies $\psi(0) = 0$ and $\mathrm{d}\psi/\mathrm{d}\zeta(0) = 0$.

h) Making the substitution $u(\psi) = d\psi/d\zeta$ into equation 19 and using the chain rule,

$$\begin{aligned}
2\frac{\mathrm{d}^2\psi}{\mathrm{d}\zeta^2} &= e^{-\psi} \\
2\frac{\mathrm{d}u}{\mathrm{d}\zeta} &= e^{-\psi} \qquad ; \frac{\mathrm{d}u}{\mathrm{d}\zeta} = \frac{\mathrm{d}u}{\mathrm{d}\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} \\
2u\frac{\mathrm{d}u}{\mathrm{d}\psi} &= e^{-\psi} \\
\frac{\mathrm{d}u^2}{\mathrm{d}\psi} &= e^{-\psi} \; .
\end{aligned} \tag{20}$$

Since $\phi(0) = 0$ and $\mathrm{d}\phi/\mathrm{d}z(0) = 0$ and $u(\psi) = d\psi/d\zeta$ then $u(\phi = 0) = 0$

i) Directly integrating the equation gives $u^2 = C - \exp(-\psi)$, applying the boundary condition of $u(\psi = 0) = 0$ sets the value of $C = 1$.

$$u^2 = 1 - \exp(-\psi) \; . \tag{21}$$

j) Now taking the square root and using the definition of u leads to $u = \mathrm{d}\psi/\mathrm{d}\zeta = \sqrt{1 - \exp(-\psi)}$. This integral can be done by using hyperbolic trigonometric identities and using the substitution method. Use the substitution $\operatorname{sech}^2 x = e^{-\psi}$ and the identity $\tanh^2 x + \operatorname{sech}^2 x = 1$ to simplify the integral,

$$
\begin{aligned}
\int \mathrm{d}\zeta &= \int (1-e^{-\psi})^{-\frac{1}{2}} \mathrm{d}\psi \\
\int \mathrm{d}\zeta &= 2\int \mathrm{d}x \\
\zeta + C &= 2x\ .
\end{aligned} \tag{22}
$$

From the hyperbolic identity you can show that $\tanh x = \sqrt{1-e^{-\psi}}$. Thus, you use the inverse function to get x and use the boundary condition of $\psi(0) = 0$ to set $C = 0$ since $\tanh 0 = 0$ and you get,

$$
\zeta = 2\tanh^{-1}\sqrt{1-\exp(-\psi)}\ . \tag{23}
$$

k) We have the density ρ defined in equation 1 where we plug our distribution function from part e),

$$
\rho = m\int \frac{\rho_o}{m\sqrt{2\pi\sigma^2}} e^{-E/\sigma^2} \mathrm{d}v_z\ . \tag{24}
$$

Putting in $E = \phi + v_z^2/2$ and evaluating the gaussian integral then simplifying gives

$$
\rho = \rho_0 e^{-\phi/\sigma^2} = \rho_0 e^{-\psi}\ . \tag{25}
$$

From rearranging the solution of 23 we get

$$
\rho(\zeta)\ /\rho_0 = \mathrm{sech}^2(\zeta/2)\ , \tag{26}
$$

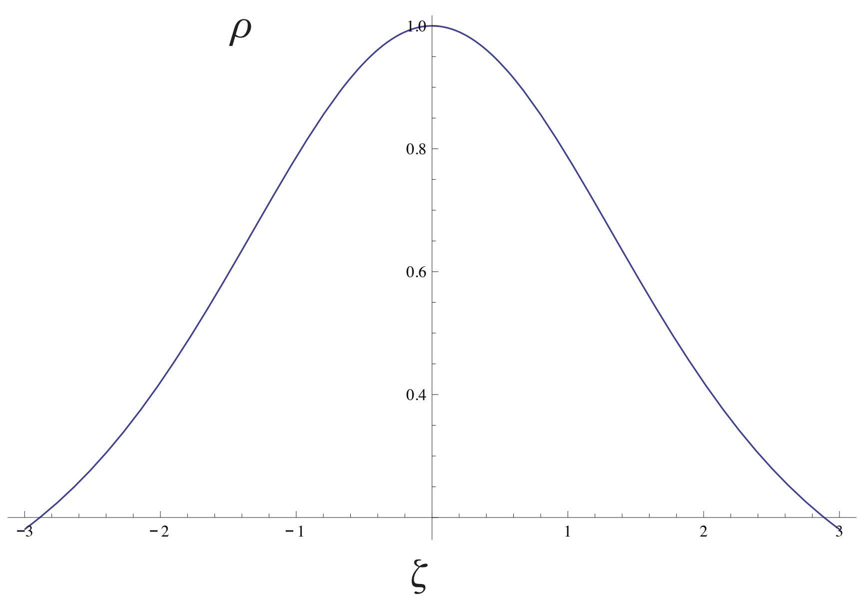

Figure 7: Density profile of unchanging galaxy