Ay 20 Basic Astronomy and the Galaxy
Problem Set 5

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1 Photons in a blackbody spectrum

a) The Planck intensity is

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}. \tag{1}$$

First, we need to integrate over solid angle to remove the dependence on direction. Since B_λ does not depend on direction, this amounts to multiplying by 4π. To convert from energy flux to (specific) energy density, imagine a surface of area dA. In a time dt, the photons move a distance cdt and are contained in a volume $cdAdt$. The energy density is then $dE_\lambda/(cdAdt) = F_\lambda/c = 4\pi B_\lambda/c$. Finally, to convert from specific energy density to number density, we divide by the photon energy hc/λ:

$$n_\lambda = \frac{4\pi\lambda}{hc^2}B_\lambda = \frac{8\pi}{\lambda^4}\frac{1}{e^{hc/\lambda kT}-1} \tag{2}$$

To get the total number density over all wavelengths, we integrate over λ:

$$\int_0^\infty n_\lambda d\lambda = 8\pi\int_0^\infty \frac{d\lambda}{\lambda^4(e^{hc/\lambda kT}-1)} = 8\pi\left(\frac{kT}{hc}\right)^3\int_0^\infty \frac{x^2dx}{e^x-1}, \tag{3}$$

where $x = hc/(\lambda kT)$. The value of the integral can be performed numerically and is approximately 2.404. Inserting the physical constants, we get

$$n = 20.29T^3 \tag{4}$$

with n in cm^{-3} and T in K.

b) Plugging in $T = 2.726$ K gives $n = 411$ cm^{-3}.

c) The total energy density is

$$\frac{8\pi^5}{15}\left(\frac{kT}{hc}\right)^3 kT, \tag{5}$$

from previous homework, while the number density is

$$n = 8\pi(2.404)\left(\frac{kT}{hc}\right)^3 \tag{6}$$

from (a). Dividing gives the average energy per photon:

$$\langle E\rangle = \frac{8\pi^5/15}{8\pi(2.404)}kT = 2.70kT \tag{7}$$

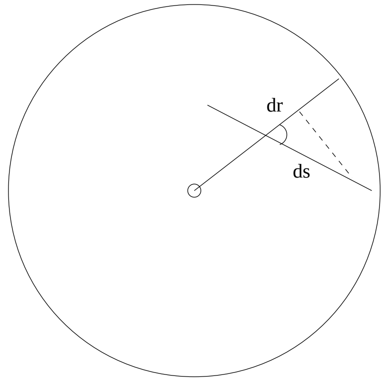

Figure 1: Geometry of problem 2

d) Inserting the given temperature into part (d) gives 3.65 keV at the center of the sun and 1.34 eV in the photosphere.

2 Carroll and Ostlie 9.16

a) The radiative transfer equation, integrated over wavelength, says

$$\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda + S_\lambda, \tag{8}$$

where τ_λ is measured along the light path. Figure 1 shows that ds is related to dr by $ds = dr/\cos\theta'$, where θ' is the angle marked. (Note that is relation is true differentially, but a statement like $s = r/\cos\theta'$, which many of you wrote, does not make sense. As one moves along the light path, the local radial vector and hence θ' are changing.) Finally, the optical depth is defined by $d\tau_\lambda = \kappa_\lambda \rho ds$. Putting the pieces together yields

$$-\frac{\cos\theta'}{\kappa_\lambda \rho}\frac{dI_\lambda}{dr} = I_\lambda - S_\lambda. \tag{9}$$

b) To obtain the pressure gradient we would like to take the first moment. We might as well pick a point to study, place the origin there, and orient the polar axis along the outward radial vector so that there is no distinction between θ and θ'. Multiplying the result from (a) by $\cos\theta'$ and integrating over the sphere gives

$$-\int \frac{\cos^2\theta'}{\overline{\kappa}\rho}\frac{dI}{dr}d\Omega = \int I\cos\theta' d\Omega - \int S\cos\theta' d\Omega, \tag{10}$$

where the λ dependence has been removed by integration over wavelength. (We can take κ to be the Rosseland mean opacity $\overline{\kappa}$.) Note that the solid angle element is $d\Omega = \sin\theta' d\theta' d\phi$. The first term on the RHS is just the flux F_{rad}. Since S is assumed isotropic, it has no angular dependence and can be taken outside the integral, which then vanishes since

$$\int \cos\theta' d\Omega = \int_{\phi=0}^{\phi=2\pi}\int_{\theta'=-\pi/2}^{\theta'=\pi/2} \cos\theta' \sin\theta' d\theta' d\phi = 0. \tag{11}$$

Finally, recalling that

$$\int I\cos^2\theta' d\Omega = cP_{rad}, \tag{12}$$

we get

$$\frac{d}{dr}\int I\cos^2\theta' d\Omega = \frac{dP_{rad}}{dr} = -\frac{\overline{\kappa}\rho}{c}F_{rad}. \tag{13}$$

3 Carroll and Ostlie 9.20

For a plane grey atmosphere in LTE under the Eddington approximation, Carroll and Ostlie derive

$$\frac{4\pi}{3}\langle I\rangle = F_{rad}\left(\tau_v + \frac{2}{3}\right) \tag{14}$$

in Eqn 9.50. Using $F_{rad} = \sigma T_e^4$ (by definition) and $\langle I\rangle = S = B = \sigma T^4/\pi$ (in LTE), this becomes

$$T^4 = \frac{3}{4}T_e^4\left(\tau_v + \frac{2}{3}\right). \tag{15}$$

From this we see that $T = T_e$ at $\tau_v = \frac{2}{3}$. (Note that along any line of sight, you are seeing photons from an average $\tau = 1$; however, when averaged over angles, this corresponds to an average vertical depth of $\tau_v = 2/3$.) This can also be derived by assuming a linear source function $S = a + b\tau$, which is another form of the Eddington approximation.

4 Carroll and Ostlie 9.26

a) From the given λ and W we can calculate $\log W/\lambda$ then read the corresponding $\log fN_a/(\lambda/500 \text{ nm})$ off of the curve of growth (obviously there is some error in reading the plot):

λ (nm)	W (nm)	log W/λ	$\log fN_a/(\lambda/500$ nm)
330.239	0.0067	-4.69	16.7
589.594	0.0560	-4.02	18.6

Now we can find N_a using

$$\log N_a = \log\left(\frac{fN_a}{\lambda/500 \text{ nm}}\right) - \log\left(\frac{f}{\lambda/500 \text{ nm}}\right). \tag{16}$$

From this,

λ (nm)	f	$\log f/(\lambda/500$ nm)	$\log N_a$
330.239	0.0049	–2.1	18.9
589.594	0.325	–0.6	19.2

The average value is about $\log N_a = 19.0$, with N_a in m^{-2}.

b) The values of $\log N_a$ found in Example 9.5.5 are 19.0 and 18.92. Combining these with our values from part (a), the mean is 19.0. Using this, we can plot the positions of the four transitions given in the homework problem plus the example:

λ (nm)	$\log W/\lambda$	$\log fN_a/(\lambda/500$ nm)
330.298	-4.69	16.51
589.594	-4.02	18.58
330.238	-4.57	17.15
588.997	-3.91	18.88

These lie in good agreement with the curve in Fig. 9.22.

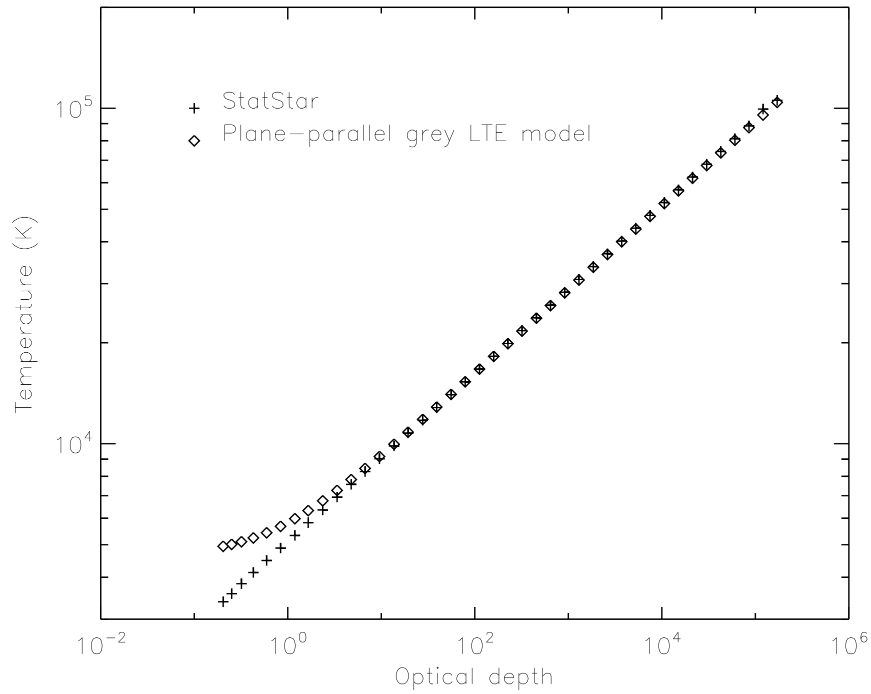

Figure 2: $T(\tau)$ for a StatStar model atmosphere and for the LTE calculation in a gray plane atmosphere under the Eddington approximation.

5 Carroll and Ostlie 9.28

a) – c) The final two pages show an IDL code performing the required calculations and its output. Note that for (c), T is computed as

$$T^4 = \frac{3}{4} T_e^4 \left(\tau + \frac{2}{3} \right), \tag{17}$$

where $T_e = 5504$ K. The results are plotted in Fig 2.

d) The calculations of both models are suspect in the outermost regions of the atmosphere. StatStar fixes its initial conditions here (note *not* at $\tau = 2/3$) to $P = \tau = 0$, which is approximately true, but still could imply a substantial fractional error on the next several points, where P and τ are quite small. (Both methods agree well once τ is well above 1, since the atmospheric contribution is then minute and its details unimportant.) LTE is also a poor assumption in the outer atmosphere, since the mean free path of photons becomes very long.

6 Isothermal atmosphere

a) Hydrostatic equilibrium tells us that

$$\frac{dP}{dr} = -\frac{GM\rho}{R^2} = -g\rho, \tag{18}$$

where $g = GM/R^2$. (Note that since the atmosphere is thin, we can consider $r = R$ to be constant.) Introducing the definition of optical depth, $d\tau = -\rho\kappa dr$, we get

$$\frac{dP}{d\tau} = \frac{g}{\kappa} = \frac{g}{\rho f(T)}. \tag{19}$$

Using the ideal gas law $P = \rho kT/\mu$, this becomes

$$\frac{dP}{d\tau} = \frac{gkT}{\mu f(T)P}. \tag{20}$$

This is a separable differential equation:

$$\int_0^P P'dP' = \int_0^\tau \frac{gkT}{\mu f(T)} d\tau' \tag{21}$$

Note that the integrand on the RHS consists of constants only since the temperature is fixed at $T = T_0$. We are integrating here from $r = \infty$, where $P = \tau = 0$, inward. The result of performing the integration is

$$P = \sqrt{\frac{2gkT}{\mu f(T)}\tau}. \tag{22}$$

If you explore the behavior of ρ, you will find it to be exponential in r, i.e. $\rho \propto e^{-r/r_H}$, where the scale height $r_H = kT/\mu g$. This is a classic result for an isothermal atmosphere.

b) The equations developed in the chapter for the plane-parallel gray atmosphere in LTE under the Eddington approximation (e.g., 9.53) are not useful for this case. (For instance, note that since there is no temperature gradient, the net radiation flux is 0.) Rather, consider the cause of limb darkening in that atmosphere. The photons you see along a given line of sign emerge on average from $\tau = 1$. This is the location at which they last interacted with the atmosphere and were influenced by its properties, including temperature. Looking along different lines of sight, $\tau = 1$ is reached at various depths measured perpendicular to the atmosphere; in particular, $\tau_v = \tau/\cos\theta$. These various depths correspond to different temperatures, which in LTE dictates the source function $S = B(T)$ and thus the intensity. As an example: Looking toward the edge of a star, the line of sight is more grazing and pierces less deeply into the atmosphere to reach $\tau = 1$. The gas here is cooler, and we therefore see less intensity.

For this problem we are considering an isothermal atmosphere. No matter the depth to which a line of sight penetrates, the local temperature is always the same. Therefore, there is no limb darkening effect, provided the atmosphere is optically thick. (If the medium is optically thin instead, the intensity along a line of sight is proportional to the distance that line travels through the medium, and geometry will determine the limb darkening.) You can write this as $I_\nu(\mu, \tau = 0)/I_\nu(\mu = 1, \tau = 0) = 1$.

r (m)	T StatStar	tau	T LTE
7.10080E+08	0.00000E+00	0.00000E+00	4.62829E+03
7.09320E+08	3.38000E+03	2.03935E-01	4.94765E+03
7.09250E+08	3.57300E+03	2.51057E-01	5.01328E+03
7.09180E+08	3.82600E+03	3.19653E-01	5.10444E+03
7.09100E+08	4.13300E+03	4.28739E-01	5.24008E+03
7.09010E+08	4.48800E+03	5.94428E-01	5.42789E+03
7.08910E+08	4.88700E+03	8.38400E-01	5.67326E+03
7.08800E+08	5.32900E+03	1.19131E+00	5.98003E+03
7.08690E+08	5.81500E+03	1.65323E+00	6.32137E+03
7.08560E+08	6.34800E+03	2.36607E+00	6.75931E+03
7.08420E+08	6.93000E+03	3.36776E+00	7.25921E+03
7.08270E+08	7.56700E+03	4.76596E+00	7.81981E+03
7.08110E+08	8.26200E+03	6.70989E+00	8.44126E+03
7.07930E+08	9.02200E+03	9.56006E+00	9.15963E+03
7.07730E+08	9.85100E+03	1.36851E+01	9.96942E+03
7.07520E+08	1.07600E+04	1.93285E+01	1.08312E+04
7.07280E+08	1.17500E+04	2.77313E+01	1.18240E+04
7.07030E+08	1.28200E+04	3.91322E+01	1.28650E+04
7.06750E+08	1.40000E+04	5.57657E+01	1.40387E+04
7.06450E+08	1.52900E+04	7.89830E+01	1.53017E+04
7.06120E+08	1.67000E+04	1.12249E+02	1.66968E+04
7.05760E+08	1.82300E+04	1.59511E+02	1.82219E+04
7.05370E+08	1.99100E+04	2.26189E+02	1.98784E+04
7.04950E+08	2.17400E+04	3.19706E+02	2.16699E+04
7.04480E+08	2.37300E+04	4.56044E+02	2.36785E+04
7.03980E+08	2.59100E+04	6.44932E+02	2.58187E+04
7.03430E+08	2.83000E+04	9.15543E+02	2.81801E+04
7.02830E+08	3.08900E+04	1.29997E+03	3.07597E+04
7.02170E+08	3.37300E+04	1.85053E+03	3.35975E+04
7.01460E+08	3.68300E+04	2.62179E+03	3.66540E+04
7.00680E+08	4.02100E+04	3.72507E+03	4.00172E+04
6.99840E+08	4.39100E+04	5.27190E+03	4.36465E+04
6.98920E+08	4.79400E+04	7.47781E+03	4.76318E+04
6.97910E+08	5.23400E+04	1.06307E+04	5.20106E+04
6.96820E+08	5.71400E+04	1.50606E+04	5.67425E+04
6.95640E+08	6.23800E+04	2.13038E+04	6.18816E+04
6.94350E+08	6.81000E+04	3.01878E+04	6.75156E+04
6.92950E+08	7.43500E+04	4.27335E+04	7.36440E+04
6.91430E+08	8.11600E+04	6.04638E+04	8.03191E+04
6.89780E+08	8.86000E+04	8.55139E+04	8.75898E+04
6.87990E+08	9.94200E+04	1.20885E+05	9.55076E+04
6.86040E+08	1.05600E+05	1.71019E+05	1.04161E+05

```
pro CO928

readcol,"Table95",r,T,rho,kappa    ; load data
n = n_elements(r)
tau = fltarr(n)
!fancy = 3                         ; make plot look better

; Integrate by trapezoidal rule.
tau[0] = 0.
for i=0,n-2 do $
  tau[i+1] = tau[i] + 0.5*(kappa[i]*rho[i]+kappa[i+1]*rho[i+1]) $
                        *(r[i]-r[i+1])
; Plot
plot,tau[1:n-1],T[1:n-1],xtitle="Optical depth",ytitle="Temperature (K)",$
psym=1,/xlog,/ylog,xrange=[1e-2,1e6],/xstyle,yrange=[3e3,2e5],/ystyle

; Compute the plane-parallel grey LTE model and overplot.
Te = 5504.
T_LTE = ((3./4)*Te^4*(tau+2./3))^(0.25)
oplot,tau[1:n-1],T_LTE[1:n-1],psym=4

; Add a legend
oplot,[1e-1],[1e5],psym=1
oplot,[1e-1],[8e4],psym=4
xyouts,2e-1,1e5,"StatStar"
xyouts,2e-1,8e4,"Plane-parallel grey LTE model"

; Output results
openw,lun,"CO928.out",/get_lun
for i=0,n-1 do printf,lun,r[i],T[i],tau[i],T_LTE[i],$
format='(E11.5,x,E11.5,x,E11.5,x,E11.5)'
free_lun,lun

end
```