1 The 51 Pegasi system

a) Mass  Refering to Figure 7.7 in C&O, we can see that the points on the log–log plot of L
versus M is a straight line - thus mass and luminosity must be related by a power law. We can fit a
straight line to it by reading off any two points. We see that (log M, log L) = (-0.75, -3) and (1.25,
5) are points on the line. Also, the the line passes through (0,0). This tells us that if we measure
L in units of solar luminosity (L⊙) and mass in units of solar masses M⊙ then we have:

\[ \frac{L}{L_\odot} = \left( \frac{M}{M_\odot} \right)^4 \]  

Substituting L=1.3 L⊙ gives M=1.1 M⊙.

b) Radius  To calculate the radius, we use the luminosity and effective temperature. We have,

\[ L = 4\pi R^2 T^4 \]  

Substituting L⊙=3.8×10^{33} \text{erg s}^{-1}, T_*=5660 \text{K}, we get:

\[ R_* = 8.3 \times 10^{10} \text{cm} = 1.2 R_\odot \]  

c) Variation of Hβ wavelength  The wavelength of the Hβ line measured in air is \( \lambda_0 = 4861.34 \text{Å} \). We see from Figure 1 in the problem set that \( v_r \) varies from -60 m/s to +55 m/s. We can then calculate the variation in range of variation of \( \lambda \) by using the formula,

\[ \Delta \lambda = \frac{v}{c} \lambda \]  

Hence \( \lambda \) varies from \( \lambda_0 - 9.7 \times 10^{-4} \) Å to \( \lambda_0 + 8.9 \times 10^{-4} \) Å, corresponding to a total variation of 0.19 pm.

d) Selecting a good line  The full width at half maximum (FWHM) of the Hβ line in the solar
spectrum is about 0.1 Å. To measure velocities of \( \sim 5 \text{m/s} \), we have to measure \( \Delta \lambda \sim 10^{-4} \) Å. Hence
we have to centroid the line to 10^{-4} times its width, which will be prone to a lot of errors. A better
choice will be to use one of the narrower lines in the spectrum, which are much sharper. Note
that the Hβ line is broad and hence is the main contributor to the signal - resolving narrow lines
requires high resolution spectroscopy and a high signal–to–noise ratio.
e) The planet

i. Semi major axis  We use Kepler’s law to calculate the semimajor axis of the orbit:

$$\frac{P^2}{a^3} = \frac{1}{M_s + M_p}$$  \hspace{1cm} (5)

where $P$ is the period in years, $a$ is the semimajor axis in AU, $M_s$ is mass of the star, $M_p$ is the mass of the planet, both measured in solar masses. Neglecting the mass of the planet, we get:

$$a = \sqrt[3]{1.1 \times \left(\frac{4.231}{365.25}\right)^2} = 0.053 \text{ AU} = 7.9 \times 10^{11} \text{ cm}$$  \hspace{1cm} (6)

ii. Mass of the planet  To estimate the mass of the planet, we use the modulations in the star’s velocity. We know that:

$$M_s v_s = M_p v_p$$  \hspace{1cm} (7)

$$v_p = \frac{2\pi a}{P} = 1.4 \times 10^7 \text{ cm s}^{-1}$$  \hspace{1cm} (8)

Substituting this value in Equation 7 we get,

$$M_p = 1.1 M_\odot \times \frac{(55 - (-60)) \times 100}{2} \text{ cm s}^{-1} \times \frac{1}{1.4 \times 10^7 \text{ cm s}^{-1}} = 4.7 \times 10^{-4} M_\odot = 9.3 \times 10^{29} \text{ gm}$$  \hspace{1cm} (9)

f) Temperature of the planet  We assume that the planet rotates rapidly and the surface is at a uniform temperature. The flux from the star at the planet’s orbit is given by,

$$F = \frac{L_s}{4\pi a^2}$$  \hspace{1cm} (10)

The energy absorbed by the planet is then:

$$E_{in} = 0.5 \times \pi R_p^2 F = \frac{\pi R_p^2 L_s}{8\pi a^2}$$  \hspace{1cm} (11)

This energy is re–radiated as a blackbody:

$$E_{out} = 4\pi R_p^2 \sigma T_p^4$$  \hspace{1cm} (12)

Solving Equations 11 and 12 together we get:

$$T_p = \sqrt[4]{\frac{L_s}{4\pi a^2} \times \frac{\pi R_p^2}{2} \times \frac{1}{4\pi R_p^2 \sigma}} = 1100 \text{ K}$$  \hspace{1cm} (13)

In reality, the planet is orbiting so close to the star, that it is “tidally locked” to the star, just as moon is locked to the earth. So, the same surface of the planet faces the star all the time. At a place on the planet which has high noon, the starlight is incident normally on the “ground”. The temperature can then be calculated by equating the energy absorbed and re–radiated by a unit area:

$$0.5 \times \frac{L_s}{4\pi a^2} = \sigma T^4$$  \hspace{1cm} (14)
This gives a temperature $\sqrt{2}$ times higher than the previous case: $T_p \sim 1550$ K

2 Finding Exoplanets

![Image of Venus transiting the sun, 8 June 2004. (Webcam on 6 inch Newtonian telescope with solar filter)](image)

Figure 1: Image of Venus transiting the sun, 8 June 2004. (Webcam on 6 inch Newtonian telescope with solar filter)

a) Detecting Jupiter Since we are searching for a planet around a distant star, we can consider parallel rays coming from the star. If we could resolve the disc of the star, we would see the transit simply as a dark dot on the stars disc. (For example see Figure 1 for an image of Venus transiting the disc of the sun.) For a distant star, we cannot resolve the disc, so we will just note a drop in the flux of the star. The fractional decrease is simply given by the ratio of the areas:

$$ f = \frac{\pi R_p^2}{\pi R_J^2} = 0.01 \quad (15) $$

Where we have used $R_J = 7.14 \times 10^8$ cm. A useful website to get such constants is [http://www.astro.wisc.edu/~dolan/constants.html](http://www.astro.wisc.edu/~dolan/constants.html). Here, we have also neglected any emission from Jupiter itself. If Jupiter was also radiating significantly, there would be another correction to this, given by:

$$ \Delta f = \frac{T_J^4}{T_\odot^4} \quad (16) $$

Let us again assume an albedo of 0.5. We can either estimate $T_J$ or we can directly use Equation 13, substituting $L_* = 4\pi R_\odot^2 \sigma T_\odot^4$, to get:

$$ \frac{T_J^4}{T_\odot^4} = \frac{R_J^2}{8a^2} \approx 10^{-7} \quad (17) $$
Hence, we can ignore this correction in calculating the fractional decrease in brightness.

![Diagram showing range of angles for transit/occultation](image)

**Figure 2:** Range of angles in which an observer will see a transit/occultation

**b) Probability of finding Jupiter** Refer to Figure 2. The planet will occult the star only if the angular momentum vector is in the range L1 to L2, which means $\theta' < \sin^{-1}\left(\frac{R_\odot}{a}\right)$ where $R_\odot$ is the radius of the star (in this case the sun), and $a$ is the star–planet separation. Thus we can calculate the probability of the occultation:

\[
P = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta'=\frac{\pi}{2}}^{\frac{\pi}{2} - \theta'} \sin(\theta) d\theta d\phi
\]

(18)

\[
P = -\frac{1}{2} \left[ \cos\left(\frac{\pi}{2} + \theta'\right) - \cos\left(\frac{\pi}{2} + \theta'\right) \right]
\]

(19)

\[
P = -\frac{1}{2} \left[ -\sin(\theta') - \sin(\theta') \right] = \sin(\theta') = \frac{R_\odot}{a}
\]

(20)

Substituting $a = 1$ AU we get $P = 8.9 \times 10^{-4}$.

By a similar argument, the probability of a grazing transit will be $2R_p/a$, or $\sim 1.8 \times 10^{-4}$.

c) Detecting 51 Pegasi

First let us calculate the radius of 51 Peg:

\[
R_* = \sqrt{\frac{L}{4\pi\sigma T_4^4}} = 8.3 \times 10^{10} \text{cm} = 1.2 R_\odot
\]

(21)

Then we assume that the orbiting planet has the same radius as Jupiter, and proceed like part (a) to get,

\[
\Delta f = 7.4 \times 10^{-3} = 0.74\%
\]

(22)

Again, we have to verify that the correction due to the planets emission is small. Indeed we see that $T_j^4/T_*^4 = 1.4 \times 10^{-3}$, which is negligible.

d) Probability of finding 51 Pegasi–like systems

Again proceeding like part (b), we use Equation 20 to get,

\[
P_{51 \text{ Peg}} = \frac{R_*}{a_*} = 0.1
\]

(23)
3 Hydrogen gas

This is a straightforward application of the Boltzmann Equation. For having only 1% atoms in the first excited state,

\[
10^{-2} = \frac{2 \times 2^2}{2 \times 1^2}e^{-\left[\frac{-13.6 \text{eV}}{2^2} - \frac{-13.6 \text{eV}}{1^2}\right]/kT}
\]

Hence we get \( T = 2.0 \times 10^4 \text{K} \). For having 10% atoms in the excited state, we get \( T = 3.2 \times 10^4 \text{K} \).

4 Solving the Saha Equation

See separate pdf of hand-written solution below:

5 Giant stars and spectral type

The key to this problem is understanding the meaning of spectral type as a set of particular line ratios. Then, using the Saha equation, the argument is straightforward: two parameters are free (free electron density and temperature), and thus a giant star with a low density will be cooler to have the same line ratios.
\[ n_e V = N_\Pi \]
\[ N_\pi V = \frac{\varrho V}{m_\rho} \]

\[ \varrho = 10^{-6} \text{ kg m}^{-3} = 10^{-9} \text{ g cm}^{-3} \]

\[ \frac{N_\pi}{N_\pi N_\pi} = \frac{2 \times \frac{1}{\eta_c \times 2}}{h^2} \left( \frac{2 \pi m_e kT}{h^2} \right)^{3/2} e^{-X_\pi/kT} \]

But \[ \eta_c = \frac{N_\pi}{V} = \frac{N_\pi \varrho}{m_\rho N_\pi} \]

:. we get

\[ 1 = \left( \frac{N_\pi}{N_\pi} \right) \left( \frac{N_\pi}{N_\pi} \right) \left( \frac{m_\rho}{\varrho} \right) \left( \frac{2 \pi m_e kT}{h^2} \right)^{3/2} e^{-X_\pi/kT} \]

\[ \left( \frac{N_\pi}{N_\pi} \right)^2 = \left( \frac{m_\rho}{\varrho} \right) \left( \frac{2 \pi m_e kT}{h^2} \right)^{3/2} e^{-X_\pi/kT} \]

\[ - \frac{N_\pi}{N_\pi} \left( \frac{m_\rho}{\varrho} \right) \left( \frac{2 \pi m_e kT}{h^2} \right)^{3/2} e^{-X_\pi/kT} \]
b) We have a quadratic in \( x = \frac{N\pi}{N_A} \) expressing as \( ax^2 + bx + c = 0 \), we get

\[
  a = 1 \\
  b = \frac{m_p}{\frac{g}{\hbar^2}} \left( \frac{2\pi m e k T}{h^2} \right)^{3/2} e^{-\frac{\xi_1}{kT}} \\
  c = -b \\
\]

Since \( x = \frac{N\pi}{N_A} \) is > 0,

\[
  x = \frac{-b + \sqrt{b^2 + 4b}}{2} 
\]

See graph above. It matches well with 8.8.