Ay 20 Basic Astronomy and the Galaxy
Problem Set 2

October 19, 2008

1 Angular resolutions of radio and other telescopes

Angular resolution for a circular aperture is given by the formula,

$$\theta_{min} = \frac{1.22\lambda}{d} \tag{1}$$

Where θ_{min} is the least resolvable angle (in radians), d is the diameter of the aperture, and λ is the wavelength of observation. The factor 1.22 is for circular apertures. In case of synthesized apertures like interferometry, the angular resolution is ill-defined as the least resolvable angle varies as a function of diameter. The resolution is still proportional to λ/d but the numerical coefficient can be different from 1.22. For the purpose of this problem, we use 1.22 as the factor even for interferometric observations. The resolutions of the various telescopes are:

Telescope	Wavelength	Resolution (radians)	Resolution (other units)
a) Owens Valley	21 cm	6.5×10^{-3}	22'
b) CARMA	2.6 mm	1.6×10^{-6}	0.33"
c) Arecibo	21 cm	8.4×10^{-4}	2.9'
d) VLA	21 cm	7.1×10^{-6}	1.5"
e) VLBA[a]	21 cm	2.6×10^{-8}	5.3 mas[b]
f) Hubble	380 nm	1.9×10^{-7}	40 mas
g) Spitzer	24 μm	3.4×10^{-5}	7"
h) TMT	1μm	4.1×10^{-8}	8.4 mas
i) JWST	1.2μm	2.4×10^{-7}	50 mas

[a] The longest baseline for the VLBA is from Virgin Islands to Hawaii, roughly 9500 km.
[b] milli-arc seconds

2 What we'd like to see

The angular size of an object of size l (note that l should be diameter, not a radius, to be consistent with the definitions of problem 1) at a distance d is given by:

$$\theta = \frac{l}{d} \tag{2}$$

Object	Size (cm)	Distance (cm)	θ (rad)	θ (other units)
a) Sun-like star	1.4×10^{11}	3.1×10^{20}	4.5×10^{-10}	93 μas
b) Red giant star	1.4×10^{14}	3.1×10^{20}	4.5×10^{-7}	93 mas
c) Binary star orbit, 5 day period	2.2×10^{12}	3.1×10^{20}	7.2×10^{-9}	1.5 mas
d) Binary star orbit, 100 yr period	8×10^{14}	3.1×10^{20}	2.6×10^{-6}	0.56"
e) Supernova shell after 1 month	5.2×10^{15}	3.1×10^{25}	1.7×10^{-10}	34 μas
f) Region around central BH	5.3×10^{19}	3.1×10^{26}	1.7×10^{-7}	35 mas
g) Cluster in interacting galaxies	6.2×10^{18}	3.1×10^{27}	2×10^{-9}	0.4 mas

To find the maximum separation a of the binary systems, use Kepler's Third Law:

$$a = \left(\left(\frac{P}{2\pi} \right)^2 G(M_1 + M_2) \right)^{1/3} \tag{3}$$

where P is the orbital period and M_i are the masses of the stars. The diameters entered in the table are the relative major axis $2a$, though of course projection effects could reduce this somewhat.

To find the radius of the region around the central black hole where stars have orbital velocities >500 km/s, use the expression for circular velocity:

$$v_c = \sqrt{\frac{GM}{R}} \tag{4}$$

using $M = 10^8 M_\odot$.

3 Fading into the distance

(a) P60

The Palomar 60-inch can detect up to $m_V = 22$.

(i) Since we know the absolute magnitude of the sun in V-band (4.83), we can use the following relation between apparent (m) and absolute (M) magnitudes to find the distance at which it our sun would have $m_V = 22$:

$$m - M = 5 \log \left(\frac{d}{10 \text{ pc}} \right) \tag{5}$$

giving $\boxed{d = 27 \text{ kpc}}$. To get this so-called **distance modulus**, recall that the definition of absolute magnitude is the apparent magnitude an object would have at a distance of 10 pc, and that d^{-2} shows up in flux.

(ii) A white dwarf with radius 0.01 $R_\odot$ with the same surface temperature as the sun will have a luminosity of 10^{-4} $L_\odot$, since $L \sim R^2$. And since

$$M_{\text{WD}} - M_\odot = -2.5 \log \left(\frac{F_{\text{WD}}}{F_\odot} \right) \tag{6}$$

and $F \sim L$, then $M_{\text{WD}} = M_\odot + 10 = 14.83$, using the distance modulus above gives a distance of $\boxed{270 \text{ pc}}$ for the white dwarf to have $m_V = 22$.

Another way to see this is to notice that in order to have the same magnitude, two objects must have the same flux, and so:

$$F_\odot = F_{\rm WD} \rightarrow \frac{L_\odot}{d_\odot^2} = \frac{L_{\rm WD}}{d_{\rm WD}^2} = \frac{10^{-4} L_\odot}{d_{\rm WD}^2} \rightarrow d_{\rm WD} = 10^{-2} d_\odot \tag{7}$$

(iii) If $\nu L_\nu = 10^{46}$, then $L_\nu = 1.83 \times 10^{31}$ erg s^{-1} Hz^{-1} at $\nu = 5.45 \times 10^{14}$ Hz (or equivalently, $L_\lambda = 1.82 \times 10^{43}$ erg s^{-1} nm^{-1} at 550 nm). Within a bandpass of 100 nm, the luminosity is then roughly $L_V = 1.82 \times 10^{45}$ erg s^{-1}.

To find the absolute magnitude M_V of this quasar then:

$$M_V - M_\odot = -2.5 \log \left(\frac{F_V}{F_\odot} \right) = -2.5 \log \left(\frac{L_V}{L_{\odot,V}} \right). \tag{8}$$

(Note that the flat νL_ν throughout optical and UV is important here; it allows us to not worry about how redshift affects these calculations.) Now, in order to find the luminosity of the sun in the V-band, approximate the sun as a black body at 5800 K, and find the fraction of the sun's radiation that falls within this band:

$$\frac{\int_{600\ \text{nm}=5\times10^{14}\ \text{Hz}}^{500\ \text{nm}=6\times10^{14}\ \text{Hz}} B_\nu d\nu}{\int_0^\infty B_\nu d\nu} = 0.13 \tag{9}$$

Thus, $L_{\odot,V} = 0.13\ L_{\odot,\rm tot} = 0.13 \times 4 \times 10^{33}$ erg/s $= 5.2 \times 10^{32}$ erg/s.

Using this in (7), we obtain $M_V = -26.5$ for this quasar, giving us a distance of $\boxed{50.5 \text{ Gpc}}$. Again, using the equivalent of (7) gives the same result quicker. This is interesting scientifically, as it implies that even the most distant possible luminous quasars are easily detectable with modest instruments, indicating that the observed rarity of very high-z quasars is in fact intrinsic rather than a selection effect. This is also a good place to point out that the "distance" talked about here is what's known as the "luminosity distance" in cosmology, which is significantly greater than geometric distance due to the redshifting of photon energies.

(b) The Heroic Hubble Deep Field

The AB magnitude system is defined such that:

$$m_{AB} = -2.5 \log F_\nu - 48.6 \tag{10}$$

We assume throughout that this filter goes from $\lambda = 556$-656 nm, or $\nu = 4.57$-5.40×10^{14} Hz.

(i) We begin by finding out m_{AB} for the sun, for which we need F_ν of the sun at earth at 606 nm:

$$F_\nu = \int B_\nu \cos\theta d\Omega = B_\nu \Delta\Omega \text{ (for small } \theta) \ = B_\nu \cdot \pi \left(\frac{R_\odot}{1 \text{ AU}} \right)^2 \tag{11}$$

Plugging in

$$R_\odot = 6.96 \times 10^{10} \text{ cm},\ 1 \text{ AU} = 1.5 \times 10^{13} \text{ cm},\ B_{\nu=4.96\times10^{14}\ \text{Hz}}(5800 \text{ K}) = 3.01 \times 10^{-5} \text{ cgs}$$

gives $F_\nu = 2.04 \times 10^{-9}$ erg s^{-1} cm^{-2} Hz^{-1}, which from (9) gives $m_{AB,\odot} = -26.9$. From this we can find the distance at which the sun would have an AB magnitude of 27.5:

$$27.5 + 26.9 = 5 \log \left(\frac{d}{1 \text{ AU}} \right) \rightarrow d = 1.14 \times 10^{24} \text{ cm}. \tag{12}$$

And so, our sun at a distance of $\boxed{369 \text{ kpc}}$ would have $m_{AB} = 27.5$.

(ii) By analogy to part (a), $d_{\text{WD}} = 10^{-2} d_\odot$, giving $\boxed{3.69 \text{ kpc}}$.

(iii) Here again the calculation is essentially the same as in part (a), except the slightly different band means that $L_{V,\odot} = 0.12 L_\odot$ instead of $0.13 L_\odot$. Using the same idea as (7), we get $d_{\text{quasar}} = 1.95 \times 10^6 d_\odot$, or $\boxed{719 \text{ Gpc}}$ for our quasar to have $m_{AB} = 27.5$. With current cosmological models that you will learn about next term, this corresponds to a redshift $z = 57$, so the observed V band is really 100 Angstrom soft X-ray radiation in the quasar rest frame!

4 Spectroscopy is a lot slower than imaging

(a) Compute the specific flux F_ν and F_λ from a $m_V = 22$ galaxy. Again, we can compare to the sun, since we know $M_{V,\odot} = 4.83$, and differences in magnitudes are related to logs of ratios of fluxes:

$$22 - 4.83 = -2.5 \log \left(\frac{F_{V,\text{galaxy}}}{F_{V,\odot,10\text{pc}}} \right) \tag{13}$$

Remember, we already found out that $L_{V,\odot} = 0.13 L_\odot$, and $F = L/(4\pi d^2)$, so

$$F_{V,\odot,10\text{pc}} = \frac{0.13 \times 4 \times 10^{33}}{4\pi (10\text{pc})^2} = 4.35 \times 10^{-8} \text{ erg s}^{-1} \text{ cm}^{-2} \tag{14}$$

giving

$$F_{V,\text{galaxy}} = (4.35 \times 10^{-8}) 10^{-(22-4.83)/2.5} = 5.89 \times 10^{-15} \text{ erg s}^{-1} \text{ cm}^{-2}. \tag{15}$$

But remember, we're looking for specific flux F_ν, and $F = F_\nu \Delta\nu$, where $\Delta\nu$ is the width of the V-band as defined above (from 500-600 nm: i.e. a width of 1×10^{14} Hz). So,

$$F_\nu = \frac{F_{V,\text{galaxy}}}{10^{14} \text{ Hz}}, \tag{16}$$

giving $\boxed{F_\nu = 5.89 \times 10^{-29} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1}}$. Similarly,

$$F_\lambda = \frac{F_{V,\text{galaxy}}}{1000 \text{ Å}}, \tag{17}$$

giving $\boxed{F_\lambda = 5.89 \times 10^{-18} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Å}^{-1}}$

The V-band photon number flux is given by:

$$F_\gamma = \frac{F_V}{h\nu} = \frac{5.89 \times 10^{-15}}{(6.626 \times 10^{-27})(5.45 \times 10^{14})} = 1.6 \times 10^{-3} \text{photons s}^{-1} \text{ cm}^{-2} \tag{18}$$

giving a total of $\boxed{330 \text{ photons s}^{-1}}$ striking over the area of the P200.

(b) The surface brightness of the sky in V-band is 20.4 mag/arcsec^2. The effective magnitude of a 2" × 2" patch is thus the equivalent of four mag 20.4 stars, so:

$$m_{V,\text{sky}} = 20.4 - 2.5 \log 4 = 18.9 \text{ mag} \tag{19}$$

(c) So the key here (and this is what makes spectroscopy so slow) is the narrowness of the band, which is 1/100 the width of the entire V-band (for one resolution element). Thus, the number of photons in each resolution element from the galaxy is 0.33 photons/s, (accounting for the 10% efficiency), giving $\boxed{\text{1188 photons}}$. The sky is 3.1 mag brighter, and thus is $10^{0.4 \cdot 3.1} = 17.4$ times brighter, or about $\boxed{\text{20700 photons}}$.

(d) Even background-subtracting the sky, you have to account for the poisson error in the sky as well as the galaxy. In this case, $\sqrt{N}$ for the sky counts is a noise level of 144 photons/s. Including a Poisson noise of 34 photons/s for the galaxy pushes this to ∼149 photons/s, giving a total signal-to-noise per resolution element of $\boxed{8}$.

5 Zodiacal light and the search for extrasolar earth–like planets

(a) Flux from zodiacal light We know that the area of a hemisphere is 2π steradians. Let us calculate this area in square degrees. Consider a 1 square degree patch of a spherical surface of radius r. The length l of each side of this square is given by, $l = \frac{\theta}{360} \cdot 2\pi r$ where $\theta = 1°$. The surface area of this square is l^2. Hence the solid angle in radians is:

$$\Omega = \frac{l^2}{r^2} = \theta^2 \cdot \left(\frac{2\pi}{360}\right)^2 = 3.04 \times 10^{-4} \text{ sr} \tag{20}$$

Notice that equation 20 gives us a conversion factor from square degrees to steradians. Thus the area of the hemisphere is 20,626 deg^2. The area of the entire sky is $4\pi/3.04 \times 10^{-4} = 41{,}253$ deg^2 – a useful number to know.

1 S_{10} means that the light coming from 1 square degree of the sky equals the flux from a star with m_V=10. Hence, the total flux from the zodiacal light is,

$$60\, S_{10} \cdot 20626 \,\text{deg}^2 = 1.2 \times 10^6 \text{ stars with } \mathrm{m_v} = 10 \tag{21}$$

The total V–mag is thus,

$$m_{dust} = 10 - 2.5 \log(1.2 \times 10^6) = -5.2 \tag{22}$$

This is the equivalent magnitude of all the zodiacal dust, at a distance of 1 AU from us. The sun has an apparent V–magnitude of -26.8 at the same distance from us. So, ratio of the fluxes can be calculated:

$$f_{dust} = 10^{(m_{dust} - m_\odot)/-2.5} = 2 \times 10^{-9} \tag{23}$$

(b) Sunlight scattered by earth As we learned when calculating the equilibrium temperature of the earth in class, the earth has some albedo A, which represents the fraction of sunlight that is reflected (a fraction $(1-A)$ is absorbed). Let's assume $A \sim 0.5$, half of the sunlight incident on earth is reflected, or 'scattered.' It is useful to think of a sphere of radius 1 AU centered on the sun. The light from the sun spreads on the entire surface, and the earth scatters light corresponding to it'd cross–sectional area (Radius of earth, $R_\oplus$=6376 km). Hence, fraction of sunlight scattered by the earth is,

$$f_{earth} = A\frac{\pi R_\oplus^2}{4\pi(1\ \mathrm{AU})^2} \approx 2.3 \times 10^{-10} \tag{24}$$

(c) Magnitude of earth–like planet At largest elongation, the earth-like planet will appear as half illuminated. Thus, half of the light scattered by the planet is in our direction. Thus, the ratio of the light from the star to light from the planet is $0.5 \times f_{earth}$ with f_{earth} as defined in Equation 24 above. We already know that the absolute magnitude of the sun is M_V=4.79, which is defined as the apparent magnitude of the sun if it was 10 pc away. We can then use the magnitude formula to find the apparent magnitude of the planet, m_p:

$$m_p = M_{V,\odot} - 2.5\log\left(\frac{F_{planet}}{F_\odot}\right) = 4.79 - 2.5\log(0.5 \times 2.3 \times 10^{-10}) = 29.7 \tag{25}$$

(d) Mass of zodiacal dust A thin circular sheet of paper with radius equal to radius of the earth and thickness t would scatter just as much light as the earth, but its mass would be only $\sim t/R_\oplus$ times the mass of the earth. Ripping the paper into tiny pieces does not change its total cross sectional area. Moving it to some distance a from the sun changes the fraction of sunlight it scatters proportional to $1/a^2$. Hence, we get a *lower* limit on mass by assuming the smallest sized particles possible. The smallest possible size is $a \sim 1\mu$m, as the particles have to be larger than the wavelength of visible light. So, for dust grains of thickness $t = 1\mu$m to scatter $N \sim 10$ times as much light as the earth scatters, when at a distance a (in AU) from the sun, requires them to have a total mass (in earth masses):

$$\frac{M_{dust}}{M_\oplus} = N\frac{t}{R_\oplus} \times \frac{\rho_{dust}}{\rho_\oplus} \times \left(\frac{a}{AU}\right)^2 \approx 2 \times 10^{-13} \tag{26}$$

Here we have assumed the dust to be slightly less dense than earth, corresponding to $\rho_{dust} \sim 3\,\mathrm{g\,cm^{-3}}$. We have also assumed that all the dust is **at 2 AU** from the earth, instead of considering a continuous distribution.

(e) Comparing fluxes from dust and the planet First, we calculate the physical size corresponding to a single resolution element of the telescope.

$$x = \frac{1.22\lambda}{d} \times (\text{Distance to planetary system}) = 2 \times 10^{12}\,\mathrm{cm} = 0.14\,\mathrm{AU} \tag{27}$$

Where the aperture d is 10 m, and the wavelength λ is 550 nm for green light. Hence a single resolution element of the telescope collects a fraction $\left(\frac{0.14\,\mathrm{AU}}{3\,\mathrm{AU}}\right)^2$ of the light from the zodiacal disc. Hence ratio of zodiacal flux in the resolution element to zodiacal flux from the planet is,

$$f_{ratio} = \frac{\text{Ratio of zodiacal flux to sun}}{\text{Ratio of planet flux to sun}} \times \left(\frac{0.14\,\text{AU}}{3\,\text{AU}}\right)^2 \approx 0.02 \tag{28}$$

(f) Infrared searches

i. Peak λB_λ: We obtain the peak of λB_λ by differentiating the equation and setting the derivative to 0:

$$\frac{d}{d\lambda}\left(\lambda \cdot \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right) = 0 \tag{29}$$

$$\frac{1}{e^{\frac{hc}{\lambda kT}}-1} \cdot \frac{(-4)2hc^2}{\lambda^5} + \frac{2hc^2}{\lambda^4}\frac{(-1)(-1)e^{\frac{hc}{\lambda kT}}}{\left(e^{\frac{hc}{\lambda kT}}-1\right)^2} \cdot \frac{hc}{\lambda^2 kT} = 0 \tag{30}$$

$$\frac{hc}{\lambda kT} \cdot \frac{e^{\frac{hc}{\lambda kT}}}{e^{\frac{hc}{\lambda kT}}-1} = 4 \tag{31}$$

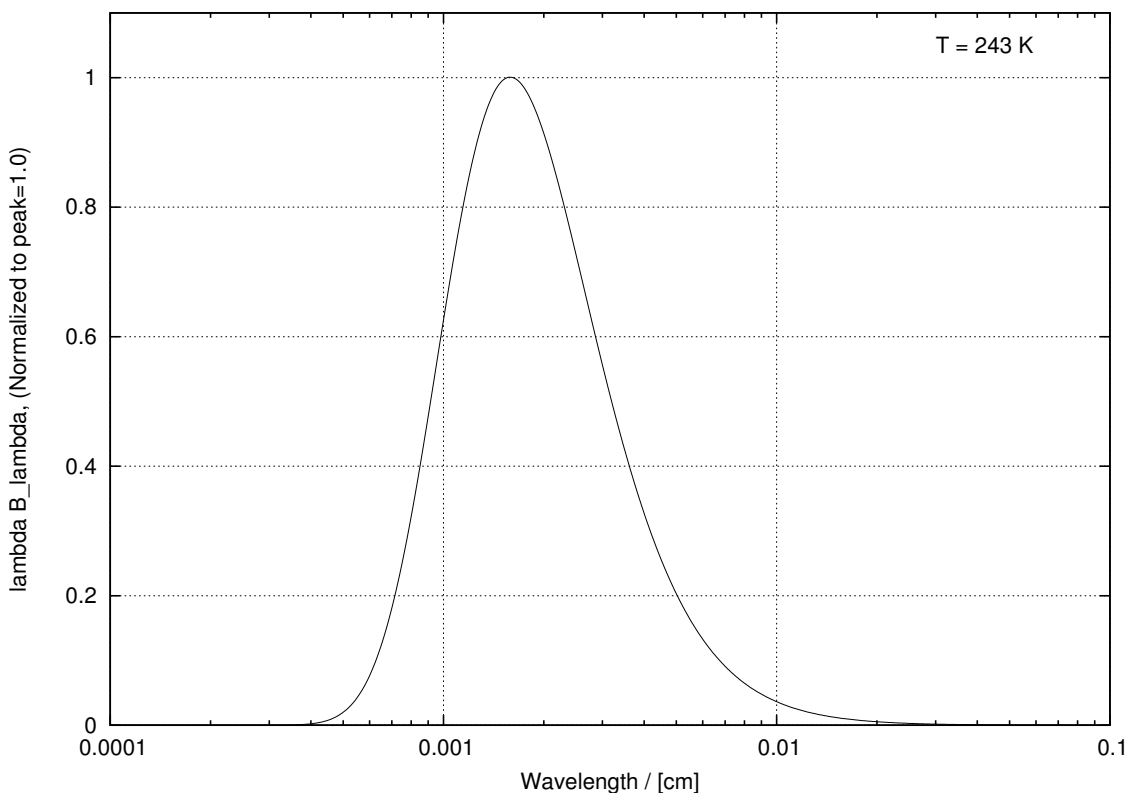

Figure 1: Plot of λB_λ as a function of wavelength for a blackbody with $T = 234\,K$.

Let $hc/\lambda kT = x$. This equation can be solved numerically, or by noting that for $e^x >> 1$, we simply get x=4, which in turn satisfies the condition that $e^4 = 54.6 >> 1$. Hence the peak wavelength is given by the condition $x_p \approx 4$. Iterating, we get $x_p = 3.92$, or:

$$\lambda_p = \frac{hc}{3.92kT} \tag{32}$$

Now, we are looking at light re–radiated by the dust, so we need to calculate the temperature of the dust first. For this we equate the total energy received by the dust grains to the total energy radiated by them. Note that the particles reflect 50% of the light and absorb only 50%.

$$\frac{L_*}{4\pi(1\,\mathrm{AU})^2} \cdot \pi(1\,\mu\mathrm{m})^2 \cdot 0.5 = 4\pi(1\,\mu\mathrm{m})^2 \times \sigma T_{dust}^4 \tag{33}$$

Using the solar luminosity $L_* = 3.84 \times 10^{33}\,\mathrm{ergs\,cm^{-2}\,s^{-1}}$, we get $T_{dust} = 234\,\mathrm{K}$. Thus the wavelength of peak emission is $\lambda_p = 1.6 \times 10^{-3}\,\mathrm{cm} = 16\,\mu\mathrm{m}$. We can verify this by plotting λB_λ – see Figure 1.

ii. Stellar and zodiacal flux: For $\lambda >> \lambda_p$, we can use the Rayleigh–Jeans law to find the ratio of fluxes emitted by both the blackbodies:

$$B_\lambda(T) \simeq \frac{2ckT}{\lambda^4} \tag{34}$$

Hence, at a given wavelength, we have,

$$\frac{B_*}{B_d} = \frac{T_*}{T_d} \tag{35}$$

$$\frac{F_*}{F_d} = \frac{B_* \cdot 4\pi r_*^2}{B_d \cdot N_{dust} 4\pi r_d^2} \tag{36}$$

Where r_* is the radius of the star and $r_d = 1\,\mu\mathrm{m}$ is the radius of the dust particles. Next we need the total area of the dust particles, for which we use their radius, density and mass (Equation 26):

$$N_{dust} = \frac{M_{dust}}{4\pi/3(1\,\mu\mathrm{m})^3 \rho_{dust}} = 10^{26} \tag{37}$$

Hence, the ratio of fluxes is:

$$\frac{F_*}{F_d} = \frac{T_* \cdot r_*^2}{T_d \cdot N_{dust} \cdot r_d^2} = 1.2 \times 10^5 \tag{38}$$

This is much better than the optical case (Equation 23) where this ratio was 4.3×10^8.

iii. Stellar and planet flux: We have to follow the same procedure as Part (ii) above. Note that in Equation 33, the radius of the body has canceled out - so the planet will have the same temperature as the dust. So, the ratio of fluxes is just the ratio of areas. We use the radius of the earth $R_p = 6376\,\mathrm{km}$ to get,

$$\frac{F_p}{F_d} = \frac{4\pi r_p^2}{N_{dust} 4\pi r_d^2} = 0.4 \tag{39}$$

The flux from the entire zodiacal dust is about 2.5 times the flux from the planet. We will see below that if we consider the flux within a single resolution element, the ratio shifts in favour of detecting the planet.

iv. Resolving the planet from the star: We are observing at $\lambda_p = 16\,\mu$m. Separation between the star and the planet is,

$$\theta_{*p} = \frac{1\,\mathrm{AU}}{10\,\mathrm{pc}} = 0.1'' = \frac{0.1}{206265}\mathrm{radians} = 4.8 \times 10^{-7} \tag{40}$$

Where we have used the fact that 1 AU subtends an angle of 1" at a distance of 1 pc, and that 206265" make a radian (This is a useful number to know. By definition, it is also the number of AU in a parsec). To match this resolution, the diameter of the telscope should be:

$$d_{tele} = \frac{1.22 \cdot 16 \times 10^{-4}}{4.8 \times 10^{-7}} \approx 40\,\mathrm{m} \tag{41}$$

In reality, the airy rings of the image of the star will still dominate over the flux of the planet at this angular resolution, hence a much bigger telescope diameter is needed. We now calculate the ratio of the flux from the zodiacal dust in one resolution element to the flux from the planet: it is $\left(\frac{1\,AU}{3\,AU}\right)^2$ times the ratio calculated in Equation 39. Thus, the flux from the planet is now 3.5 times higher than the flux from zodiacal dust within a single resolution element.