Astronomy 127

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Cosmology and Galaxy formation

27 Apr 2007

TA office hours for this problem set

Dan Grin: Weds 5-6pm. Adrienne Erickcek: Thurs 5-7pm. If many students turn up, the TAs may move to 124 Bridge ("Interaction Room").

Readings:

Review chapter 14, read chapters 15 and 18 of Coles.

Problem Set 5

Due in class *in class Fri, 4 May 2007*

Homework Problems:

1. **Angular momentum conservation** Using the linearized equations derived in class for perturbations to a homogeneous Newtonian expanding universe, show that perturbations with zero overdensity ($\delta = \delta\rho/\rho = 0$) of the form $\delta\mathbf{v}(\mathbf{r}, t) = \delta\mathbf{v_k}(t)\exp(i\mathbf{k}\cdot\mathbf{r})$ (where $\mathbf{r}$ is a comoving coordinate) must be transverse waves:

$$\delta\mathbf{v_k}\cdot\mathbf{k} = \mathbf{0}$$

and evolve according to

$$\dot{\delta}\mathbf{v_k} + \frac{\dot{a}}{a}\delta\mathbf{v_k} = 0\ ,$$

and therefore decay $\delta\mathbf{v_k} \propto a^{-1}$ as the universe expands. Can you connect this result to the conservation of specific angular momentum and/or the behavior of vorticity in a perfect fluid?

2. **Sound waves in an expanding universe** Again using the linearized equations derived in class, but now for density perturbations to a homogeneous Newtonian expanding universe, show that (longitudinal) acoustic waves of wavelength much shorter than the Jeans length ($\lambda \ll \lambda_J$ or comoving wave number $k \gg k_J = a\sqrt{4\pi G\rho}/c_s$) decline slowly in amplitude according to

$$\delta_k \propto \frac{1}{\sqrt{ac_s}}\exp\left(\pm ik\int\frac{c_s dt}{a}\right)\ .$$

Notice that when the universe is radiation dominated $c_s = c/\sqrt{3}$), the solution is oscillatory in *conformal* time $\eta = \int dt/a$, not in cosmic time t. [Hint: seek solutions of the WKBJ or 'two-timing' form $\delta_k(t) = A(t)\exp(i\int^t\omega(t')dt'$, where ω is the "fast" variable, and $A(t)$ and $\omega(t)$ vary on the much slower Hubble time, much longer than $1/\omega$ in the $k \gg k_J$ limit. Solve for $\omega(t)$ and $A(t)$, dropping terms of 'second order in slowness'.]

3. **Direct observation!**

a) Make use of the Fourier transform of Poisson's equation to show that the Fourier amplitudes ϕ_k of gravitational potential fluctuations are related to those of fractional density perturbations $\delta_k \equiv \delta\rho_k/\bar{\rho}$ by

$$\phi_k = -4\pi G\bar{\rho}a^2\delta_k k^{-2} ,$$

for comoving wavenumber k. Argue, using the results derived in class, that for linear growing modes, $\bar{\rho}a^2\delta$ is independent of time for perturbations outside the horizon.

b) Thus show for an $\Omega = 1$ universe that if the power spectrum of density perturbations has the Harrison-Zel'dovich form, $P(k) = \langle|\delta_k|^2\rangle = Ak$, the potential fluctuations contributed by scales k per $\ln k$ (i.e. between k and $2.718k$) that are outside the horizon are given by

$$(\delta\phi/c^2)^2 \sim 4\pi k^3\phi_k^2/c^4 = 9\pi(H_0/c)^4 A ,$$

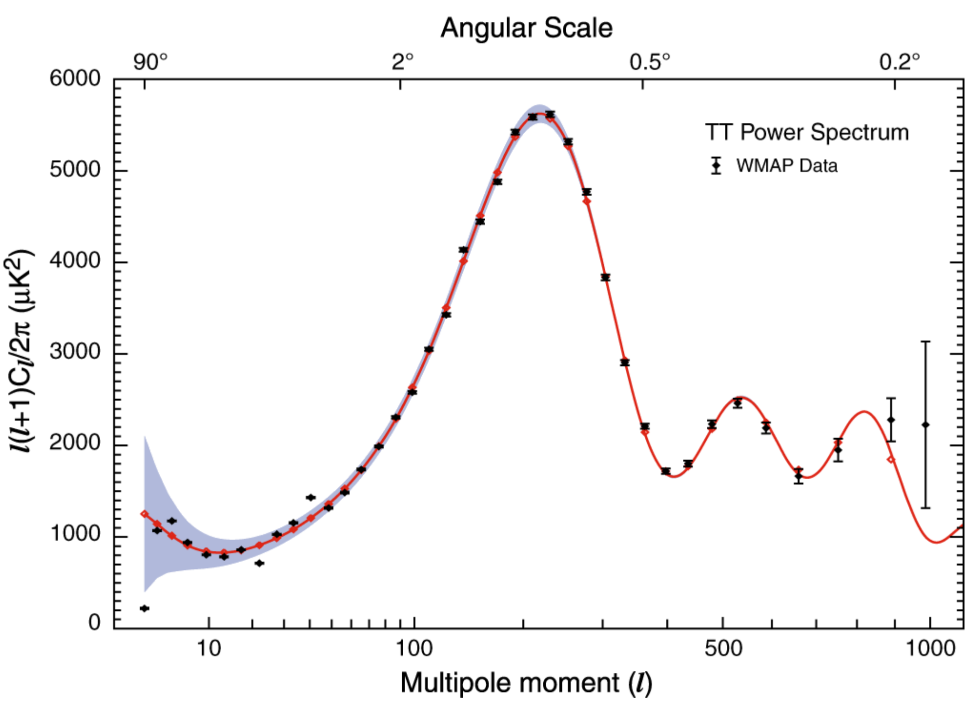
These are independent of time as long as they are outside the horizon (i.e the scale just entering the horizon now has about this same amplitude in $\delta\phi$ as it did at $z = 1100$).

c) On large scales, we will see next week that fluctuations in the gravitational potential like the ones you just found in part (b) lead to fluctuations in the observed temperature of the cosmic microwave background, which as Sachs & Wolfe (1967) showed have (for an $\Omega = 1$ universe) magnitude one third of the obvious gravitational redshift

$$\frac{\delta T}{T} = \frac{1}{3}\frac{\delta\phi}{c^2}.$$

Use this to show that the variance in temperature fluctuations produced by scales between k and $2.718k$ is

$$\left(\frac{\delta T}{T}\right)^2 = \frac{1}{9}\left(\frac{\delta\phi}{c^2}\right)^2 = \pi(H_0/c)^4 A .$$



d) The upper part of the figure on the previous page showing WMAP year 3 results plots a quantity $\ell(\ell+1)C_l/2\pi$ which is just the $(2.725\mathrm{K})^2\langle(\delta T/T)^2\rangle$ on the angular scale defined by spherical harmonic index l. Identify those fluctuations produced by waves which were outside the horizon at the last CMB scattering, and use the data, with your result in (c) to compute the dimensionless quantity $(H_0/c)^4 A$. You should get about 3.4×10^{-11}.

4. **Entering the horizon** For each of the perturbations whose wavelength is defined in parts (a-d) below find (i) the comoving wavelength λ in Mpc, (ii) the redshift at which one comoving wavelength just fitted into the horizon diameter (or equivalently $\lambda/2$ just fitted in the horizon radius).

   a) The Milky Way has a baryon mass of $2 \times 10^{10} M_\odot$. Suppose that this comprises all the baryons from the collapse of the positive half of a perturbation wave: i.e. from a sphere of *diameter* $\lambda/2$.

   b) The virialised portion of the Coma cluster has mass $\sim 2 \times 10^{15} M_\odot$, most of which is dark matter. Suppose that this comprises all the dark matter from the collapse of the positive half of a perturbation wave: i.e. from a sphere of *diameter* $\lambda/2$.

   c) On scales less than the Jean's scale at redshift z, the growth of fluctuations in the baryons are suppressed by pressure forces. The (physical) Jean's scale is defined by a physical (at z) wavelength $\lambda_J/(2\pi) = c_s/\sqrt{4\pi G\rho}$, where ρ is the total (dark plus baryon) mass density. Use the comoving wavelength corresponding to the physical λ_J evaluated immediately after recombination (show also that the Jean's comoving wavelength is independent of redshift provided the matter has the same temperature as the CMB - which is true until Compton heating and cooling 'freeze out'), and evaluate the baryon mass defined by this wavelength in the same way as was used in part (a).

   d) The baryons in the universe were reionized (photoionized) by the first generations of stars and black holes to a temperature of $T_{re} \sim 10^4$ K at redshift $1 + z_{re} \sim 10$. Use the comoving wavelength defined by the post-reionization Jean's scale (defined as in (c), but now with the T_{re} instead of T_{CMB}).