Ay127: Spring 2007 Homework #4 Solutions

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Problem #1:
a) i) The number density of a non-relativistic particle in chemical equilibrium is

$$n_i = g_i \left(\frac{m_i k_B T}{2\pi\hbar^2}\right)^{3/2} e^{\frac{\mu_i - m_i c^2}{k_B T}} \tag{1}$$

where i labels species, m_i is the mass of the ith species, g is a degeneracy factor, k_B is the Boltzmann constant, and μ_i is the chemical potential. Consider the reaction $e^- + p \rightarrow H + \gamma$. In chemical equilibrium $\mu_p + \mu_{e^-} = \mu_H$, since $\mu_\gamma = 0$. Note also that the binding energy of hydrogen is defined by $E_b = m_p c^2 + m_{e^-} c^2 - m_H c^2$. Here $g_{e^-} = 2$ and $g_p = 2$, while $g_H = 4$ due to the 4 possible spin states of neutral hydrogen. Forming the combination $\frac{n_{e^-} n_p}{n_H n_{\rm tot}}$ and using the definition of x_{e^-} provided, we see that

$$\frac{x_{e^-}^2}{1 - x_{e^-}} = \frac{1}{n_{\rm tot}} \left(\frac{m_e k_b T}{2\pi\hbar^2}\right)^{3/2} e^{-E_b/T}, \tag{2}$$

where we have made the approximation $m_p \simeq m_H$ outside the exponent, which is very reasonable. Plugging in numbers, it is easily shown that

$$\left(\frac{m_e k_b T}{2\pi\hbar^2}\right)^{3/2} = 1.5 \times 10^{23}\ \text{cm}^{-3} \left(\frac{T}{E_b}\right)^{3/2}. \tag{3}$$

Meanwhile,

$$\begin{aligned} n_{\rm tot} = X_p n_b = \\ X_p \eta n_\gamma = \\ X_p \eta \frac{2\zeta(3)}{\pi^2} T^3 = \\ 7.97 \times 10^{16}\ \text{cm}^{-3}\ X_p \eta \left(\frac{T}{E_b}\right)^3, \end{aligned} \tag{4}$$

where n_b is the baryon density at temperature T, η is the ratio of baryon-photon number densities, ζ is the Riemann zeta function, X_p is the mass fraction in hydrogen, and n_γ is the photon density. In the last step we have gone from natural units to physical units using the usual conversion factors at the back of Kolb and Turner. Combining Eqs. (3) and (4), we obtain the desired result:

$$\frac{x_{e^-}^2}{1 - x_{e^-}} = \frac{1.9 \times 10^6}{X_p \eta} \left(\frac{E_b}{T}\right)^{3/2} e^{-E_b/T}. \tag{5}$$

ii) Let's solve for the baryon density today: $n_{b,0} = \frac{\Omega_b \rho_{\rm crit}}{m_b} = 2.5 \times 10^{-7}\ \text{cm}^{-3}$, where best fit WMAP third year values were used in the last step. Meanwhile, $n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3$, so plugging in the CMB temperature today, $T = 2.35 \times 10^{-4}$ eV, we see that the photon number density today is $n_{\gamma,0} \simeq 413\ \text{cm}^{-3}$, so $\eta = n_{b,0}/n_{\gamma,0} \simeq 6.1 \times 10^{-10}$.

iii) For the canonical values $X_p = 0.75$ and $\eta = 6 \times 10^{-10}$, Eq. (5) yields

$$\frac{x_{e^-}^2}{1 - x_{e^-}} = \exp\left[36.0 + \frac{3}{2} \ln\left(E_b/T\right) - E_b/T\right], \tag{6}$$

as stated in the prompt. Defining $f(T) \equiv \exp\left[36.0 + \frac{3}{2} \ln\left(E_b/T\right) - E_b/T\right]$, we can solve for x_{e^-} as a function of temperature:

$$x_{e^-} = \frac{1}{2}\left(-f(T) + \sqrt{4f(T) + f^2(T)}\right). \tag{7}$$

The free electron fraction as a function of temperature is shown in Fig. (1), and either by inspection or iterative solution of Eq. (6), we see that $x_{e^-} = 0.5$ around $T = 3700$ K.

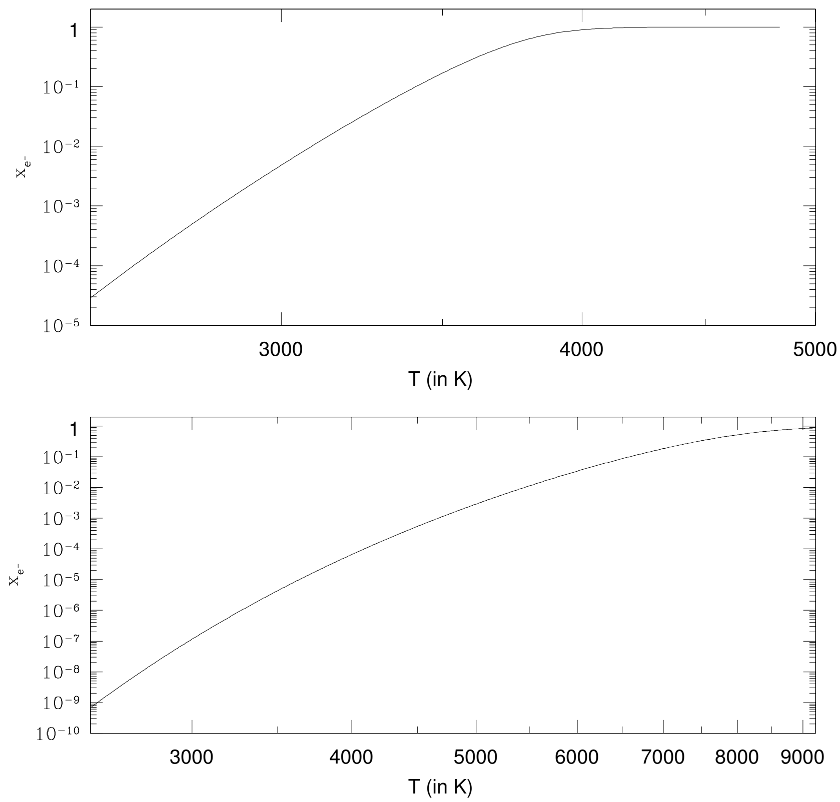

FIG. 1: Free electron fraction x_{e^-} as a function of temperature T for our universe ($\eta = 6 \times 10^{-10}$, top panel), and an alternative one ($\eta = 1$, bottom panel).

iv) In the alternate cosmology considered ($\eta = 1$), the Saha equation is modified to

$$\frac{x_{e^-}^2}{1 - x_{e^-}} = \exp\left[14.7 + \frac{3}{2}\ln\left(E_b/T\right) - E_b/T\right], \tag{8}$$

and either by inspecting Fig. (1) or iterative solution of Eq. (8), we see that $x_{e^-} = 0.5$ around $T = 8000$ K.

v) In thermal equilibrium, the ratio of numbers of atoms in excited states to those in the ground state is

$$\frac{n_i}{n_1} = e^{-(E_i - E_1)/T} = e^{-\frac{E_b}{T}\left(1 - \frac{1}{n_i^2}\right)}, \tag{9}$$

where we have applied the standard Rydberg formula in the last step. During recombination $T/E_b < 0.03$, so for the $n = 2$ state, $n_2/n_1 \sim 10^{-16}$. Higher excited states have even stronger Boltzmann suppression, and thus the population of all excited states is highly suppressed. If we wanted to include excited states in our treatment of recombination, we would have to sum over all excited states of neutral hydrogen, including the appropriate Boltzmann suppression factors for each excited state. For a lengthier explanation that uses equations to say what was just said in words, see Chapter 14 in Kippenhahn and Weigert's excellent text on stellar structure and evolution.

b) To calculate the freeze-out temperature of this process, we must set $n_e \left<\sigma v\right>_{\rm rec} = H(T)$.

i) In natural units, for today's preferred values of the cosmological parameters, it is easily shown that

$$H = 1.1 \times 10^{-26}\ \text{eV}\left(\frac{T}{E_b}\right)^{3/2} \tag{10}$$

if we assume complete matter domination. Setting this equal to the rate as stated above, and plugging in some of the numbers, we see that

$$1.1\times10^{-26}\ \mathrm{eV}\left(\frac{T}{E_b}\right)^{3/2} =$$
$$2.0\times10^{-15}\ \mathrm{eV}^{-2}\left\{2.8\times10^{-7}x_{e^-}(T)\ \mathrm{eV}^3\left(\frac{T}{E_b}\right)^3\right\}\times$$
$$(E_b/T)^{1/2}\ln\left(E_b/T\right). \tag{11}$$

ii) To figure out when freeze-out occurs, we may use the equilibrium expressions for $x_{e^-}(T)$ derived before, as they very nearly hold until freeze-out. We feed this to Mathematica, and after a little haggling see that $T_f \simeq 0.0171\times 13.6$ eV $= 0.23$ eV.

c) i) This is a numerical solution. Inserting this back into the expression for the equilibrium abundance, Eq. (7), we estimate that the freeze-out value of the free-electron fraction is $x_{e^-} = 1.2\times10^{-3}$. However, this will be a severe over-estimate, as the free-electron fraction does evolve (albeit slowly) after freeze-out, and this additional fall-off can-not be neglected here.

ii) We drop the ionization term and verify that this is self consistent at the end of the problem; presumably we are deep into the recombined regime and this assumption should be good. The freeze-out free electron fraction is ultimately due to inefficient recombination (due to the Hubble expansion) and not due to additional ionization reactions. Then

$$\frac{dx_{e^-}}{dt}\simeq -5.6\times10^{-22}\ \mathrm{eV}\ \left(\frac{T}{E_b}\right)^{5/2}x_{e^-}^2(T)\ln\left(E_b/T\right); \tag{12}$$

since we seek to evolve this equation past freeze-out, this $x_{e^-}(T)$ does not denote the equilibrium value. Now we define $y=\frac{E_b}{T}$. Then $\frac{dy}{dt}=-\frac{1}{T^2}\frac{dT}{dt}=yH(y)$, where the last step follows from setting $T=1/a$. In terms of this new variable, we then have

$$\frac{dx_{e^-}}{dy}\simeq -\frac{5.6\times10^{-22}\ \mathrm{eV}\ y^{-7/2}x_{e^-}^2(y)\ln(y)}{H(y)}. \tag{13}$$

Now if we plug in our expression for the Hubble parameter from Eq. (10) to obtain

$$\frac{dx_{e^-}}{dy}\simeq -5.1\times10^4 x_{e^-}^2 y^{-2}\ln(y). \tag{14}$$

Integrating Eq. (14) from $y=y_f$, it's value at freeze-out, to $y=\infty$ (low temperature/late time), we see that

$$\frac{1}{x_{e^-}(\infty)}-\frac{1}{x_{e^-}(y_f)}\simeq -5.1\times10^4\left|\frac{1}{y}+\frac{\ln(y)}{y}\right|_{y_f}^{\infty}. \tag{15}$$

Thus $x_{e^-}(\infty)\simeq\frac{y_f}{y_f/x_{e^-}(y_f)+5.1\times10^4\times[1+\ln(y_f)]}\simeq 1.9\times10^{-4}$.

Now let's make sure that it was self-consistent to drop the ionization term. We form the ratio of ionization to recombination rates:

$$\mathcal{R}=4.3\times10^{15}e^{-E_b/T}\left(\frac{E_b}{T}\right)^{3/2}\frac{1-x_{e^-}(T)}{x_{e^-}^2(T)} \tag{16}$$

Generalizing the result of Eq. (15) to allow earlier final y, we see that

$$x_{e^-}(y)\simeq\frac{y_f}{y_f/x_f+5.1\times10^4\left[1+\ln(y_f)-\frac{y_f}{y}-\frac{y_f}{y}\ln(y)\right]}. \tag{17}$$

Evaluating $\mathcal{R}$ shortly after freeze-out ($T\approx 0.14$ eV), we see that $\mathcal{R}<10^{-19}$ for *most* of the period of interest. Note that right near the decoupling temperature, this approximation breaks down, but this is a good first pass

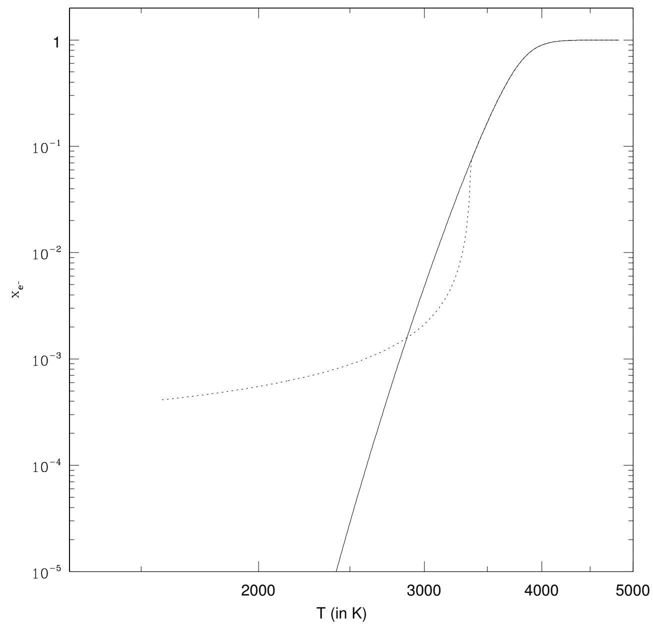

FIG. 2: Free electron fraction x_{e^-} as a function of temperature T obtained using the Saha equation (solid line) and our rough treatment of recombination (dashed line).

at the right answer! In Fig. 2, we can see the evolution of the free-electron fraction using the approximations we have used. For $T > T_f$ (where T_f is the freeze-out temperature derived above), we use equilibrium abundances as obtained for the Saha equation, and for $T < T_f$ we use the result in Eq. (17). For low enough temperatures, this curve looks like the usual freeze-out curves one has for dark matter, baryons, and other species. However, near the freeze-out temperature, our treatment breaks down, under-predicting the free-electron fraction at early times (high temperatures). This makes sense, since we neglected ionization at early times and noted that this approximation was imperfect very near freeze-out. By neglecting the effect of additional ionization at that epoch, we under-estimate the free-electron fraction until the universe cools enough for ionization to be negligible. The important thing to take away from this plot is that due to the freeze-out of the processes that determine the ionization fraction, the asymptotic free-electron fraction of the universe is much higher than it would be if the Saha equation were true at all times.

2) In this problem, we solve for the behavior of the scale-factor in a closed, positively curved universe filled with pressureless dust.

a) It is easiest to do this problem in terms of the conformal time, defined by

$$d\eta = \frac{dt}{a}. \tag{18}$$

Then $\frac{da}{dt} = \frac{d\eta}{dt}\frac{da}{d\eta} = \frac{1}{a}\frac{da}{d\eta}$. In terms of coordinate time, the Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}, \tag{19}$$

or in terms of conformal time,

$$\frac{1}{a^4}\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G\rho_{m,0}}{3}\left(\frac{a_0}{a}\right)^3 - \frac{1}{a^2}, \tag{20}$$

where $\rho_{m,0}$ is the matter density today and a_0 is the scale factor today. Defining $K \equiv \frac{8\pi G\rho_{m,0}a_0^2}{3}$, Eq. (20) can be simplified to obtain

$$\left(\frac{da}{d\eta}\right) = a_0\left[K\frac{a}{a_0} - \frac{a^2}{a_0^2}\right]^{1/2}. \tag{21}$$

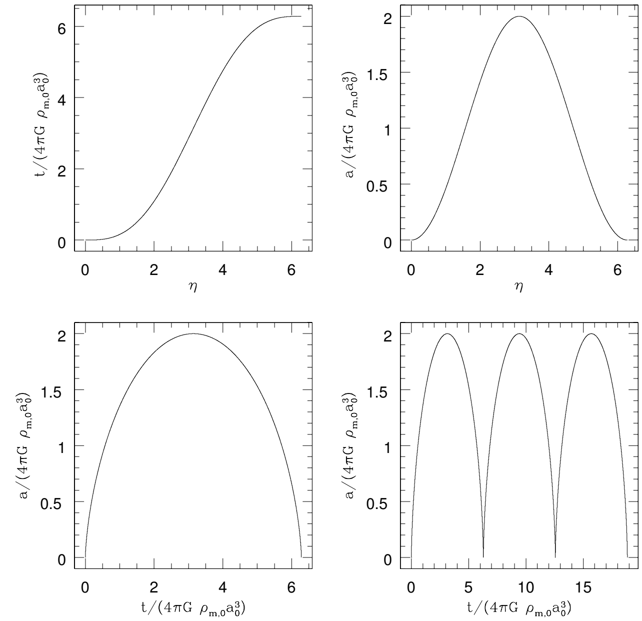

FIG. 3: Cycloid solution to the Friedmann equations for k=1. The bottom right panel shows the solution when we expand the range of η to $0 \leq \eta \leq 4\pi$.

Now define $u^2 = \frac{a}{a_0 K}$ and use $2u\; du = \frac{da}{a_0 K}$ and apply it to Eq. (21) to obtain (after a bit of algebra)

$$\frac{du}{d\eta} = 2\left[1 - u^2\right]^{1/2}. \tag{22}$$

This is easily integrated to obtain

$$\sin^{-1} u = \frac{\eta}{2}, \tag{23}$$

so applying the familiar half-angle identity $\sin(x/2) = \sqrt{\frac{1-\cos x}{2}}$, we see that

$$u = \sqrt{\frac{1-\cos\eta}{2}}, \tag{24}$$

and simplifying further by plugging in for the definition of u,

$$a = Ka_0\left(\frac{1-\cos\eta}{2}\right). \tag{25}$$

Since $dt = ad\eta$,

$$t = \int Ka_0\left(\frac{1-\cos\eta}{2}\right) d\eta, \tag{26}$$

we see that

$$t = \int Ka_0\left(\frac{\eta-\sin\eta}{2}\right). \tag{27}$$

Finally, plugging in for the definition of K, we obtain the desired result

$$a = \frac{4\pi G\rho_{m,0}a_0^3}{3}\left(1-\cos\eta\right) \tag{28}$$

$$t = \frac{4\pi G\rho_{m,0}a_0^3}{3}\left(\eta-\sin\eta\right). \tag{29}$$

b) Now we plot this behavior. In the top panel of Fig. 3, we see that this leads to a big-bang/big-crunch cosmology, in which the universe re-collapses to have zero size in finite time. If you read popular discussions of cosmology from old textbooks/newspapers, etc, you'll see uncertainty as to whether or not our own universe awaits such a fate. Today, we more or less know that the presence of a cosmological constant, yielding a flat universe, prevents such a fate from befalling us. Some authors (Weinberg and Kamionkowski 2002) have suggested that the uglier fate of a big-rip may await us if the dark energy is sufficiently mis-behaved. In the bottom panel of this plot, we see that if we expand the conformal time coordinate to twice its range in the top panel, we get a repeated series of big bangs/crunches, although it is not really kosher to extrapolate the Einstein equation (and thus the Friedmann equation) past the singularity at $\eta = 2\pi$.

c) In the $\eta \ll 1$ limit, we wish to obtain the more usual functional form $a(t)$. Taylor expanding to second/third order respectively for $a(\eta)$ and $t(\eta)$, we see that

$$a(\eta) \simeq \frac{2\pi G\rho_{m,0}a_0^3}{3}\eta^2 \tag{30}$$

$$t(\eta) \simeq \frac{2\pi G\rho_{m,0}a_0^3}{9}\eta^3. \tag{31}$$

If you're worried that we're expanding to different orders in η, you shouldn't be, because the power series for a cosine contains only even terms, while that for a sine contains only odd terms, so we're actually expanding to third order in η in both cases; the third order term vanishes for $\sin\eta$. Solving for $\eta(t)$ and plugging back into the expression for a, we see that $a \propto t^{2/3}$, as one would expect for a flat matter-dominated universe. This has to be so, because for $\eta \ll 1$, a/a_0 becomes small and $\Omega_a \gg \Omega_k$. In this limit, we had better recover a matter-dominated cosmology.

d) Because χ is the polar coordinate on a 3-sphere, it ranges from $0 \rightarrow \pi$, analogous to θ for a 2-sphere; however $\chi = 0$ and $\chi = \pi$ are the same point in space, unlike $\theta = 0$ and $\theta = \pi$, which are the opposing poles on a 2-sphere. Because the scale factor returns to *zero* at $\eta = 2\pi$, the interval $\eta = 0 \rightarrow 2\pi$ covers the history of the universe. θ and ϕ have their usual ranges on a 3-sphere. In these coordinates, the metric is

$$ds^2 = a^2(\eta)\left\{-d\eta^2 + d\chi^2 + \sin^2(\chi)\left[d\theta^2 + \sin^2\theta d\phi^2\right]\right\}. \tag{32}$$

For a radial null geodesic, $ds^2 = 0$ and $d\phi = d\theta = 0$ and photon rays most obey

$$\frac{d\chi}{d\eta} = \pm 1. \tag{33}$$

In other words, in $c = 1$ units, photons travel on 45-degree lines in spacetime diagrams, just like in special relativity. In other words, this spacetime is conformally (angles and photon trajectories are the same) equivalent to a flat, Minkowski space-time, justifying the name 'conformal time' for the new coordinate η. Placing an observer on a worldline at $\chi = 0$ and a galaxy on the world line at χ_g, we can see, using the attached sketch (Fig 4), that we will see two images of every source in the collapsing phase.

Problem #3:
a) Given that $y \equiv a/a_{eq}$,

$$\frac{\partial\delta}{\partial t} = \frac{\dot{a}}{a_{eq}}\frac{\partial\delta}{\partial y},$$

$$\frac{\partial^2\delta}{\partial t^2} = \frac{\ddot{a}}{a_{eq}}\frac{\partial\delta}{\partial y} + \frac{\dot{a}^2}{a_{eq}^2}\frac{\partial^2\delta}{\partial y^2},$$

where $\cdot$ denotes differentiation with respect to t. Using these expressions, the equation for the growth of density perturbations $\delta \equiv \delta\rho/\rho$ becomes

$$\frac{\dot{a}^2}{a_{eq}^2}\frac{\partial^2\delta}{\partial y^2} + \left(\frac{\ddot{a}}{a_{eq}} + 2\frac{\dot{a}^2}{a_{eq}a}\right)\frac{\partial\delta}{\partial y} = 4\pi G\rho\delta, \tag{34}$$

where we have taken $c_s = 0$ for cold dark matter. Finally, we use $y \equiv a/a_{eq}$ to eliminate a_{eq} from this expression, obtaining

$$\frac{\dot{a}^2}{a^2}y^2\frac{\partial^2\delta}{\partial y^2} + \left(\frac{\ddot{a}}{a}y + 2\frac{\dot{a}^2}{a^2}y\right)\frac{\partial\delta}{\partial y} = 4\pi G\rho\delta. \tag{35}$$

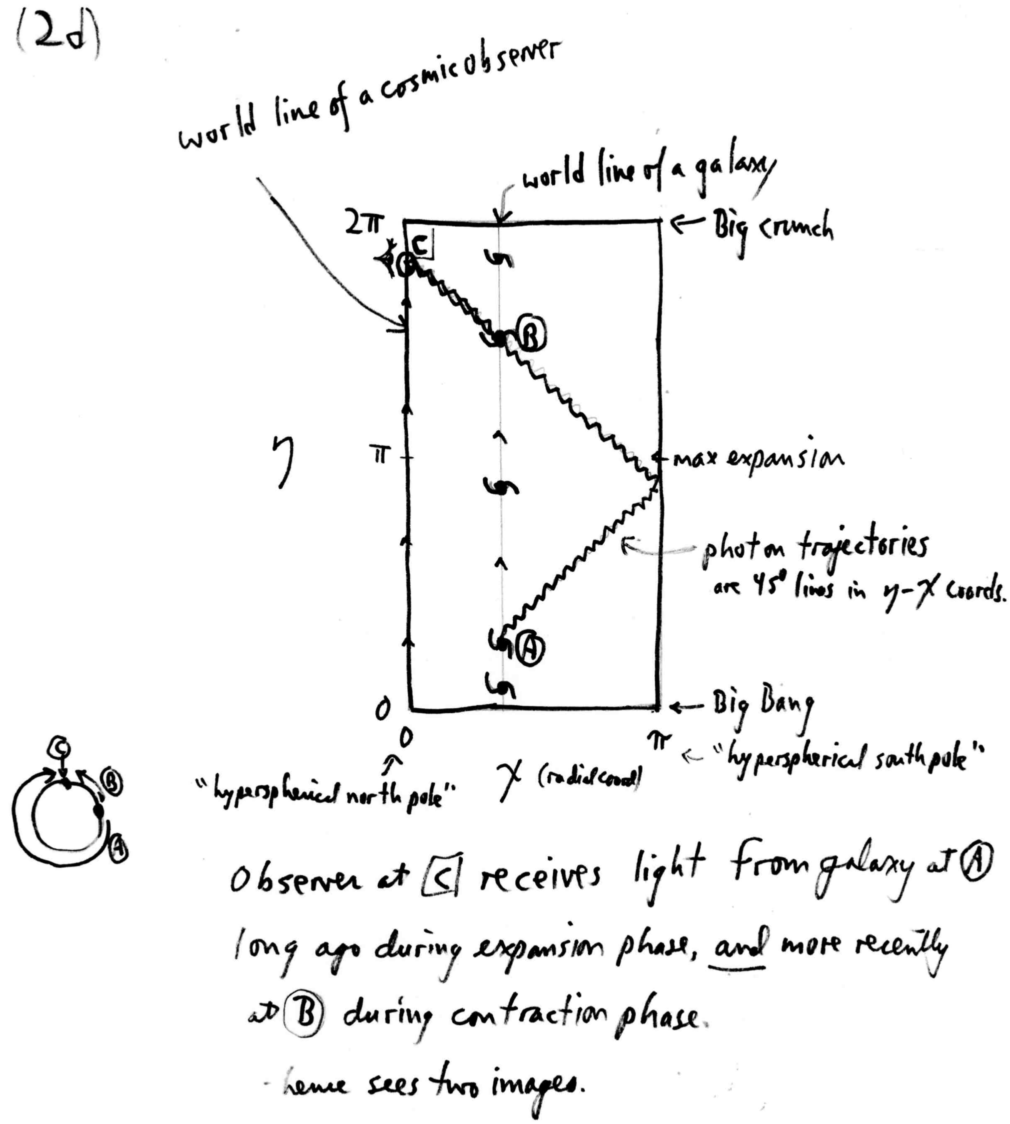

FIG. 4: Space-time diagram for the cycloid solution to Friedmann equations for a closed (hyperspherical) universe with $k = 1$. During the collapsing phase every observer sees two images of every (suitably long-lived) source.

b) Given that $\ddot{a}/a = -(4\pi/3)G(\rho + 3p)$ in units where $c = 1$,

$$\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho_m + 2\rho_r), \\
&= -\frac{4\pi G}{3}\left[\frac{\rho_{eq}}{2}\left(\frac{a_{eq}}{a}\right)^3 + 2\frac{\rho_{eq}}{2}\left(\frac{a_{eq}}{a}\right)^4\right], \\
&= -\frac{2\pi G}{3}\rho_{eq}\left[\frac{1}{y^3} + \frac{2}{y^4}\right],
\end{aligned}$$

where we have used $p_r = \rho_r/3$ for the radiation pressure and we have defined $\rho_{eq} \equiv \rho_{tot}(a_{eq}) = \rho_m(a_{eq})/2 = \rho_r(a_{eq})/2$.

c) Applying the Friedmann equation with $\rho = \rho_m + \rho_r$, we have

$$\begin{aligned}
\frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 \frac{y}{1+y} &= \frac{3}{2}\frac{8\pi}{3}G\left[\rho_m + \frac{\rho_{eq}}{2}\left(\frac{a_{eq}}{a}\right)^4\right]\frac{y}{1+y}, \\
&= 4\pi G\left[\rho_m + \rho_m\frac{1}{y}\right]\frac{y}{1+y}, \\
&= 4\pi G\rho_m.
\end{aligned}$$

d) In terms of y and ρ_{eq} the Friedmann equation may be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi}{3}G\rho_{eq}\left[\frac{1}{y^3} + \frac{1}{y^4}\right]. \tag{36}$$

Using the expressions above to eliminate $\dot{a}^2/a^2$, $\ddot{a}/a$, and $4\pi G\rho$, Eq. (35) becomes

$$y(1+y)\frac{\partial^2\delta}{\partial y^2} + (1+\frac{3}{2}y)\frac{\partial\delta}{\partial y} = \frac{3}{2}\delta. \tag{37}$$

e) We define $\delta_1(y) = 1 + (3/2)y$. Since $\partial^2\delta_1/\partial y^2 = 0$ and $\partial\delta_1/\partial y = 3/2$, it is clear that δ_1 is a solution to Eq. (37).

f) We set $\delta_2(y) = [1 + (3/2)y]v(y)$. Plugging this ansatz into Eq. (37) gives us a differential equation for $v(y)$:

$$y(1+y)\left[3\frac{dv}{dy} + \left(1+\frac{3}{2}y\right)\frac{d^2v}{dy^2}\right] + \left(1+\frac{3}{2}y\right)^2\frac{dv}{dy} = 0. \tag{38}$$

This is a second-order differential equation for v, but it is first-order differential equation for $w(y) \equiv dv/dy$:

$$\frac{dw}{w} = -\left[\frac{\left(1+\frac{3}{2}y\right)^2 + 3y(1+y)}{\left(1+\frac{3}{2}y\right)y(1+y)}\right]dy.$$

Integrating both sides of this equation (using the technique of partial fractions) gives

$$-\ln w = \ln y + \frac{1}{2}\ln(1+y) + 2\ln\left(\frac{2}{3}+y\right) + C$$

where C is an undetermined constant of integration. Defining $c_1 = 4C/9$, we have

$$w \equiv \frac{dv}{dy} = \frac{c_1}{\left(1+\frac{3}{2}y\right)^2 y\sqrt{1+y}}.$$

Clearly w, and by extension δ_2, may be multiplied by any constant. This freedom is also apparent in Eq. (37) since each term is linear in δ. Setting $c_1 = 1$ and integrating gives

$$\begin{aligned}
\delta_2 &= \left(1+\frac{3}{2}y\right)\int^y \frac{dy'}{\left(1+\frac{3}{2}y'\right)^2 y'\sqrt{1+y'}} \qquad (39) \\
&= \left(1+\frac{3}{2}y\right)\left[\frac{3\sqrt{1+y}}{\left(1+\frac{3}{2}y\right)} + \ln\left(\frac{\sqrt{1+y}-1}{\sqrt{1+y}+1}\right)\right] \\
&= 3\sqrt{1+y} + \left(1+\frac{3}{2}y\right)\ln\left(\frac{\sqrt{1+y}-1}{\sqrt{1+y}+1}\right) \qquad (40)
\end{aligned}$$

g) When the universe is radiation dominated, $a \ll a_{eq}$, which implies that $y \ll 1$. In this limit,

$$\begin{aligned}
\delta_1 &\simeq 1, \\
\delta_2 &\simeq 3 + \ln\left(\frac{\sqrt{1+y}-1}{\sqrt{1+y}+1}\right) \simeq 3 - \ln[4] + \ln[y] \simeq \ln[y]
\end{aligned}$$

When the universe is matter dominated, $a \gg a_{eq}$, which implies that $y \gg 1$. In this limit,

$$\delta_1 \simeq \frac{3}{2} y.$$

Extracting the behavior of δ_2 in the $y \gg 1$ limit is trickier. It is useful to define $x \equiv 1/y$ and work in the limit that $x \ll 1$. In this notation, we may rewrite Eq. (40) as

$$\begin{aligned} \delta_2 &= \frac{3}{\sqrt{x}}\sqrt{1+x} + \left(1 + \frac{3}{2x}\right) \ln\left(1 - \frac{2}{\sqrt{1+\frac{1}{x}}+1}\right) \\ &\simeq \frac{3}{\sqrt{x}}\left[1 + \frac{x}{2} - \frac{x^2}{8} + \mathcal{O}(x^3)\right] + \left(1 + \frac{3}{2x}\right)\left[-2\sqrt{x} + \frac{x^{3/2}}{3} - \frac{3x^{5/2}}{20} + \mathcal{O}(x^{7/2})\right] \\ &\simeq \frac{-4}{15} x^{3/2} + \mathcal{O}(x^{5/2}), \end{aligned}$$

where *Mathematica* was used to obtain the series expansion of the logarithmic term (go ahead and try to do it by hand – I dare you!). Thus we see that $\delta_2 \propto y^{-3/2}$ during matter domination.

Problem #4:
a) As in problem 3, we are considering perturbations in the density of cold dark matter, so $c_s = 0$. Defining $y \equiv a/a_c$, we see that Eq. (35) in the solution to problem 3 is still applicable:

$$\frac{\dot{a}^2}{a^2} y^2 \frac{\partial^2 \delta}{\partial y^2} + \left(\frac{\ddot{a}}{a} y + 2\frac{\dot{a}^2}{a^2} y\right) \frac{\partial \delta}{\partial y} = 4\pi G \rho \delta. \tag{41}$$

Now the Friedmann equation takes the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi}{3} G \rho_c \left[\frac{1}{y^3} + 1\right], \tag{42}$$

where $\rho_c \equiv \rho_{tot}(a_c) = \rho_m(a_c)/2 = \rho_\Lambda(a_c)/2$. We also know that $\ddot{a}/a = -(4\pi/3)G(\rho + 3p) = -(4\pi/3)G(\rho_m - 2\rho_\Lambda)$, since $p_\Lambda = -\rho_\Lambda$ and the cold dark matter is pressureless. Rewriting this expression in terms of y gives

$$\frac{\ddot{a}}{a} = -\frac{2\pi G}{3} \rho_c \left[\frac{1}{y^3} - 2\right] \tag{43}$$

Finally, we can see that

$$\begin{aligned} 4\pi G \rho_m &= 4\pi G \left(\frac{\rho_c}{2} y^{-3}\right) \left(\frac{\dot{a}}{a}\right)^2 \frac{1}{\frac{4\pi}{3} G \rho_c \left[\frac{1}{y^3} + 1\right]} \\ &= \frac{3}{2} y^{-3} \left(\frac{\dot{a}}{a}\right)^2 \frac{y^3}{1+y^3} \\ &= \frac{3}{2} \left(\frac{\dot{a}}{a}\right)^2 \frac{1}{1+y^3}. \end{aligned}$$

Using these expressions to eliminate $\dot{a}^2/a^2$, $\ddot{a}/a$, and $4\pi G \rho$, Eq. (41) becomes

$$(1+y^3) y^2 \frac{\partial^2 \delta}{\partial y^2} + \frac{3}{2} y (1 + 2y^3) \frac{\partial \delta}{\partial y} = \frac{3}{2} \delta \tag{44}$$

b) Consider the solution $\delta_1 = \sqrt{1 + y^{-3}}$. Differentiating with respect to y gives

$$\begin{aligned} \frac{\partial \delta_1}{\partial y} &= \frac{-3y^{-4}}{2\sqrt{1+y^{-3}}} \\ \frac{\partial^2 \delta_1}{\partial y^2} &= \frac{6y^{-5}}{\sqrt{1+y^{-3}}} - \frac{9y^{-8}}{4\left(1+y^{-3}\right)^{3/2}}. \end{aligned}$$

Substitution into Eq. (44) verifies that δ_1 is a solution. When the Universe is matter-dominated, $a \ll a_c$ so that $y \ll 1$. In this limit $\delta_1 \simeq y^{-3/2}$, so this is a decaying mode. When the Universe is Λ-dominated, $a \gg a_c$ so that $y \gg 1$. In this limit, $\delta_1 \simeq 1$, so this mode is frozen.

c) We define $\delta_2(y) = v(y)\sqrt{1+y^{-3}}$, where $v(y)$ is an unknown function. Plugging this ansatz into Eq. (44) gives us a differential equation for $v(y)$:

$$(1+y^3)y^2\sqrt{1+y^{-3}}\frac{d^2v}{dy^2} + \left[\frac{3}{2}y(1+2y^3)\sqrt{1+y^{-3}} - (1+y^3)\frac{3y^{-2}}{\sqrt{1+y^{-3}}}\right]\frac{dv}{dy} = 0.$$

This is a second-order differential equation for v, but it is first-order differential equation for $w(y) \equiv dv/dy$,

$$\frac{dw}{w} = \left[\frac{3/2}{y(1+y^3)} - \frac{3}{y(1+y^{-3})}\right]dy,$$

which may be integrated to give

$$\ln w = \frac{3}{2}\left[\ln y - \frac{1}{3}\ln\left(1+y^3\right)\right] - \ln\left(1+y^3\right) + C,$$

where C is an undetermined constant. This constant reflects the fact that if δ_2 is a solution to Eq. (44), then $C\delta_2$ is also a solution for any constant C. Setting this arbitrary constant to 1, we have

$$\begin{aligned} w \equiv \frac{dv}{dy} &= \left(\frac{y}{1+y^3}\right)^{3/2}, \\ \delta_2 &= \sqrt{1+y^{-3}}\int^y \left(\frac{y'}{1+y'^3}\right)^{3/2} dy'. \end{aligned} \tag{45}$$

When the Universe is matter-dominated so that $y \ll 1$, we may approximate the integrand as $y^{3/2}$. Therefore, in the matter-dominated Universe,

$$\delta_2 \simeq \sqrt{y^{-3}}\left(\frac{2}{5}y^{5/2}\right) \simeq \frac{2}{5}y. \tag{46}$$

d) Using *Mathematica* to numerically integrate Eq. (45) from 0 to ∞, we find that

$$\int_0^\infty \left(\frac{y'}{1+y'^3}\right)^{3/2} dy' = 0.575. \tag{47}$$

When the universe is Λ-dominated, $a \gg a_c$ so that $y \gg 1$. In that case, $\sqrt{1+y^{-3}} \simeq 1$, so we see that $\delta_2(y \to \infty) = 0.575$. Since δ_2 approaches a finite value as $a \to \infty$, δ_2 cannot grow indefinitely. The growth-cap is even more apparent when we consider

$$\begin{aligned} \frac{d\delta_2}{dy} &= \sqrt{1+y^{-3}}\left(\frac{y}{1+y^3}\right)^{3/2} - \frac{3y^{-4}}{2\sqrt{1+y^{-3}}}\int^y \left(\frac{y'}{1+y'^3}\right)^{3/2} dy' \\ &\simeq \sqrt{1}y^{-3} - \frac{3}{2}y^{-4}(\text{finite constant}) \simeq 0 \text{ for } y \gg 1. \end{aligned}$$

Thus, we see that δ_2 effectively stops growing when the Universe is Λ-dominated. We already showed in part (b) that δ_1 does not grow during Λ-domination either. Since the Universe is currently Λ-dominated and will remain so (unless something truly bizarre happens to the dark energy at some future time), density contrasts such as voids and super-clusters will not evolve significantly from their current state. The era of structure formation is at its end.