Problem Set 3

Due **in class**, Thursday, 1 February 2018

Reading: See the on-line syllabus for lecture-by-lecture readings.

Collaboration policy: See the on-line collaboration policy.

Homework Problems:

1. **The Cosmic Neutrino background:** At temperatures high enough that all species are ultrarelativistic, and if all species have 0 chemical potential, by integrating over an appropriate thermal distribution,

   a) show that the energy densities of photons, neutrinos and electrons are

   $$\rho_\gamma = aT^4, \quad \rho_{\nu_i} = \rho_{\bar{\nu}_i} = \frac{7}{16}aT^4, \quad \rho_{e^+} = \rho_{e^-} = \frac{7}{8}aT^4\,, \tag{1}$$

   where $a = \frac{\pi^2}{15}\frac{k^4}{c^3\hbar^3}$, and $i = e, \mu, \tau$ (and no more, according to accelerator measurements of the Z^0 width!). Also

   b) show that in physical volume V, the entropies are

   $$S_i = \frac{4}{3}\rho_i \frac{V}{T}\,. \tag{2}$$

   c) As was shown in class, neutrinos decoupled from the rest of matter well before positrons, electrons and photons fell out of thermal equilibrium. Use these results to show that neutrinos today form a thermal background like the cosmic microwave background, but with $T_\nu = (4/11)^{1/3}T_{CMB}$. When many classic cosmology texts were written, neutrinos were believed to be massless. Why does the fact that neutrinos are now known to have masses of order 10^{-2}eV not affect this result?

   d) Assuming neutrinos are massless, estimate the number of cosmic neutrinos passing through your body each second. How would this be changed if $m_\nu = 0.1$eV or 10^{-2}eV?

   e) Has a cosmic background neutrino ever interacted with a nucleus in your body? In anyone's?

2. **Optical depth after reionization.** In the standard cosmological scenario, all the electrons and protons in the Universe combine to form hydrogen at a redshift $z \sim 1100$, —the epoch of recombination. However, at some redshift $z_{\rm reion}$, stars begin to form and emit radiation that ionizes all the hydrogen in the Universe. If so, then cosmic microwave background (CMB) photons may Thomson scatter from the free electrons en route from the surface of last scattering. Calculate the optical depth $\tau_{\rm reion}$ for Thomson scattering of CMB photons as a function of the reionization redshift $z_{\rm reion}$ for a flat universe $\Omega_m = 0.3$ (sum of cold dark plus baryonic matter), $\Omega_b = 0.045$ and $\Omega_\Lambda = 0.7$. Assume a helium mass fraction $Y \simeq 0.23$. Derive an analytic approximation for redshifts $z_{\rm reion} \gg \Omega_m$. At what $z_{\rm reion}$ would $\tau_{\rm reion} = 1$?

3. **Cosmic Microwave Background Radiation.** In this problem, you will reproduce roughly the reasoning that led Alpher and Gamow to predict the existence and temperature of the cosmic microwave background in 1948, 17 years before its discovery. Assume, as discussed in class, that deuterium begins to form when the universe's temperature drops to $T = 10^9$K (at higher temperatures it is photodissociated).

   a) Calculate the age of the universe when $T = 10^9$K. Show that if we waited much longer than this, most of the neutrons from which deuterium is formed will have decayed.

   b) Require that neutron capture be efficient enough to form light elements but not so efficient as to leave no deuterium, i.e. $\langle \sigma v \rangle n_b t \sim 1$, where n_b is the baryon density. Using $\langle \sigma v \rangle = 5 \times 10^{-20}\text{cm}^3\text{s}^{-1}$ at 10^9K, and the age of the universe you found in (a), estimate the required baryon density n_b when $T = 10^9$K.

   c) Assuming the present-day baryon number density gives $\Omega_{b0} = 0.045$, what is the scale factor $(1/(1+z))$ at $T = 10^9$K?

   d) What is (roughly) the resulting predicted temperature of the background radiation today?

4. **Freeze-out and WIMP dark matter:**

   a) Show that the number density today of stable relics of a particle species X of mass m_X which falls out of equilibrium when it is nonrelativistic (this is called "cold dark matter") is proportional to $m_x^{-1}\sigma_a^{-1}$, where σ_a is the annihilation cross-section (at the energies characteristic of the freeze-out time).

   b) Thus show that their contribution to Ω_0 depends only on σ_a, and show that (for chemical potential $\mu_X = 0$)

$$\Omega_X \simeq \frac{7 \times 10^{-27}\,\text{cm}^3\text{s}^{-1}}{\langle \sigma_a v \rangle} \times (\text{slowly varying logarithms})\,. \tag{3}$$

   c) Before precision electroweak experiments showed that there is room only for 3 neutrino types, it used to be popular to let X be a fourth neutrino species. Show that this would contribute significantly to Ω only if $m_x \sim 1$ GeV. (Hints for those with a weak particle-physics background: with $\hbar = c = 1$, the weak coupling constant is $G_F = 1.2 \times 10^{-5}\,\text{GeV}^{-2}$, $1\,\text{GeV}^{-2} = 0.4 \times 10^{-27}\,\text{cm}^2$, and $\sigma_a \sim G_F^2 m_X^2$; those able to do so may justify and perhaps improve on the expression for σ_a). [Note for information only: A decade ago it was popular to let X be the lightest supersymmetric particle, perhaps a neutralino (a linear combination of the supersymmetric partners to the photon, Z^0 and/or Higgs boson), which can annihilate into all the usual particles including Higgs bosons. Unfortunately, the cross-sections for these and scattering cross-sections which might also make them visible in the lab or astrophysically, depend on many unknown parameters. Current experimental limits show that these are many orders of magnitude less than the required annihilation cross-sections. This does not significantly constrain model-builders, but does mean that their models cannot be the simplest natural WIMP models anymore.]