Ay127: Spring 2007 Homework #2 Solutions

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Problem #1: Recall that the luminosity distance is given by $d_L(z) = a_0(1+z)S(z)$, where a_0 is a dimensionful factor and $S(z)$ is a dimensionless distance given by

$$S(z) = \left.\begin{array}{ll} \sinh\left[\frac{c}{a_0 H_0}\int_0^z \frac{dz'}{E(z')}\right] & \text{if } k=-1 \\ \left[\frac{c}{a_0 H_0}\int_0^z \frac{dz'}{E(z')}\right] & \text{if } k=0 \\ \sin\left[\frac{c}{a_0 H_0}\int_0^z \frac{dz'}{E(z')}\right] & \text{if } k=1 \end{array}\right\}, \tag{1}$$

where generally, $E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda + \Omega_k(1+z)^2}$. For an Einstein-deSitter (EdS) universe ($\Omega_M = 1.0$, $\Omega_\Lambda = 0$), we see that $E(z) = \sqrt{(1+z)^3}$, and evaluating Eqn. 1 yields

$$d_L = 6000h^{-1}\ \text{Mpc}(1+z-\sqrt{1+z}). \tag{2}$$

For the curved cases, we must carefully evaluate a_0. To deal with this, we apply the Hubble equation today:

$$H_0^2 = \frac{8\pi G\rho_{m,0}}{3} - \frac{k}{a_0^2} + \frac{\Lambda}{3} \tag{3}$$

Defining $\Omega_m \equiv \frac{\rho_{m,0}}{\rho_{crit}}$, $\Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2}$, and $\Omega_k = -\frac{k}{a_0^2 H_0^2}$, we obtain

$$a_0 = \frac{c}{\sqrt{(|1-\Omega_m-\Omega_\Lambda|)}H_0}. \tag{4}$$

Here $\rho_{m,0}$ is the matter density today. For the open CDM case, we have $\Omega_m = 0.26$, $\Omega_\Lambda = 0$, yielding

$$d_L(z) = \frac{c(1+z)}{H_0|\sqrt{1-\Omega_m-\Omega_\Lambda}|}\sinh\left(\sqrt{|1-\Omega_m-\Omega_\Lambda|}\int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z)^3+\Omega_\Lambda+\Omega_k(1+z)^2}}\right) = \tag{5}$$

$$3490h^{-1}\ \text{Mpc}(1+z)\sinh\left(\sqrt{0.74}\int_0^z \frac{dz'}{\sqrt{0.26(1+z)^3+0.74(1+z)^2}}\right). \tag{6}$$

For the flat ΛCDM case, we have $\Omega_m = 0.26, \Omega_\Lambda = 0.74$, so

$$d_L(z) = \frac{c(1+z)}{H_0}\times\int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z)^3+\Omega_\Lambda}} = \tag{7}$$

$$3000h^{-1}\text{Mpc}\ (1+z)\int_0^z \frac{dz'}{\sqrt{0.26(1+z)^3+0.74}}. \tag{8}$$

For a flat universe with a non-trivial equation of state parameter $w \neq 1$, we need to use the fluid continuity equation in an expanding universe to figure out how $E(z)$ changes in the presence of dark energy as opposed to a simple cosmological constant:

$$\frac{d\rho}{dt} + 3\frac{\dot a}{a}(\rho+P) = 0. \tag{9}$$

Recall that $dt = \frac{da}{aH}$ (from Hubble's equation) and that $w \equiv \frac{P}{\rho}$. This yields

$$\frac{d\rho}{da} = -3(1+w)\rho/a \rightarrow \tag{10}$$

$$\rho \propto a^{-3(1+w)}, \tag{11}$$

TABLE I: Luminosity distance $d_L(z)$ calculated for four different 'true' cosmologies, at three different redshifts.

	$z = 0.1$	$z = 0.5$	$z = 1.0$
$\Omega_m = 1, \Omega_\Lambda = 0, k = 0$	$307h^{-1}$Mpc	$1650h^{-1}$Mpc	$3510h^{-1}$Mpc
$\Omega_m = 0.26, \Omega_\Lambda = 0, k = -1$	$313h^{-1}$Mpc	$1810h^{-1}$Mpc	$4160h^{-1}$Mpc
$\Omega_m = 0.26, \Omega_\Lambda = 0.74, k = 0$	$323h^{-1}$Mpc	$2010h^{-1}$Mpc	$4740h^{-1}$Mpc
$\Omega_m = 0.26, \Omega_\Lambda = 0.74, k = 0, w = -0.9$	$322h^{-1}$Mpc	$1980h^{-1}$Mpc	$4640h^{-1}$Mpc

TABLE II: Distance Modulus with h dependence removed: $m - M + 5\log h$

Values Today			
$\Omega_m = 1, \Omega_\Lambda = 0, k = 0$	37.44	41.09	42.73
$\Omega_m = 0.26, \Omega_\Lambda = 0, k = -1$	37.48	41.29	43.10
$\Omega_m = 0.26, \Omega_\Lambda = 0.74, k = 0$	37.55	41.52	43.38
$\Omega_m = 0.26, \Omega_\Lambda = 0, k = 0, w = -0.9$	37.54	41.48	43.33

where the second equation follows from integrating the first, and we see that in the case $w = -1$ (simple cosmological constant), we recover the fact that the energy density due to a cosmological constant does not depend on redshift. Modifying the ΛCDM case accordingly, we see that

$$d_L(z) = \frac{c(1+z)}{H_0} \times \int_0^z \frac{dz^{'}}{\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda(1+z)^{3(1+w)}}} = \tag{12}$$

$$3000h^{-1}\text{Mpc}(1+z)\int_0^z \frac{dz^{'}}{\sqrt{0.26(1+z)^3 + 0.74(1+z)^{0.3}}}. \tag{13}$$

We evaluate all the cases of interest in Mathematica to generate a table (Table I) of distances ($w = -1$ unless explicitly otherwise stated). We also calculate the table of distance moduli with the h dependence removed, that is,($m - M + 5\log h$) (See Table II). By looking at this table, we see that 0.15 mag errors can distinguish between flat and open standard cold dark matter universes for $z = 0.5$ and $z = 1.0$. In this case, we can also distinguish the two ΛCDM cosmologies from either standard CDM variant. At $z = 0.1$, these magnitude errors have no constraining power between the different cosmological scenarios. However, with these redshift errors, even with the presence of high-z supernovae in our sample, the table clearly shows us that we will not be able to distinguish between the two variants of ΛCDM with different equation of state parameters for the dark energy.

If magnitude errors were beaten down to 0.01, supernovae at all these redshifts could be used to distinguish between dark energy models with $w = -1$ and $w = -0.9$. It should be noted though, that this analysis assumes that w is a constant, which may or not be well motivated from a particle physics standpoint (see papers by Steinhardt and others on quintessence, k-essence, and other alternatives to the cosmological constant). It has been pointed out that there are numerous degeneracies between different dark-energy models that cannot be broken by supernovae data alone. Also, note that the $w = -0.9$ model has lower distances than the pure cosmological constant model. This is because vacuum energy with this 'softened' equation of state parameter comes to dominate at later times than $w = -1$ vacuum energy, and thus has a slightly lessened effect on the expansion history of the universe.

Problem #2:
Now we are asked to repeat Problem #1 for little green Martians (LGMs). There are two ways to read the prompt, **A. Assume each 'true' cosmology succession, calculate z_{LGM}, the redshift of the LGMs, and then calculate, H, Ω_m, Ω_Λ, Ω_k self-consistently at the time when the LGMs are observing. Then once we have the cosmological parameters evaluated t = −4.5 billion years ago, evaluate all the distances as before., B. Assume that ΛCDM with w = −1 is the 'true cosmology' today. Calculate z_{LGM} *just once* in this cosmology and figure out the Martian's observed value of h in terms of ours, and then use z_{LGM} to scale back all the parameters in the alternative cosmologies to z_{LGM}. In other words, figure out their version of our 'wrong' cosmologies. Then repeat as in #1, calculating all the distances and distance moduli.** After making one of these two choices, this exercise becomes identical to #1, but with different numbers to feed into our analytic expressions and numerical integrations. Since even your hard-

working TAs were unclear on this point, we will give full credit for correct calculations in either interpretation of the problem, as long as you present a self-consistent understanding of what you were asked to do and what you actually did.

Solution A:

First we must evaluate the age of the universe as a function of redshift for the different cosmologies. The Friedmann equation can be re-arranged to yield

$$t = \frac{1}{H_0}\int_0^z \frac{dx}{(1+x)\,E(x)}, \tag{14}$$

where $E(x) = \sqrt{\Omega_m (1+x)^3 + \Omega_\Lambda a^{-3(1+w)} + \Omega_k (1+x)^2}$. For the CDM case ($\Omega_m = 1$, $\Omega_k = 0$, $\Omega_\Lambda = 0$, $h = 0.71$), this integral is easy, yielding

$$t = \frac{2}{3H_0}\left[1 - \frac{1}{(1+z)^{3/2}}\right] \rightarrow$$
$$1 + z = \left(1 - \frac{3H_0 t}{2}\right)^{-2/3}. \tag{15}$$

Evaluated at $t = 4.5$ billion years for the given value of the Hubble constant, this yields $1 + z = 1.57$.

For the $k = -1$, $\Omega_m = 0.26$ case, we can evaluate the integral numerically making the $x = (1+z)^{-1}$ substitution:

$$t = \frac{1}{H_0}\int_{\frac{1}{1+z}}^1 \frac{x dx}{\sqrt{0.26x + 0.74x^2}} \tag{16}$$

We solve this numerically for $t = 4.5$ billion years to obtain $1 + z = 1.53$.

Last week we showed that for a pure cosmological constant ($w = -1$)

$$a = a_0\left[\sqrt{\frac{\Omega_m}{1-\Omega_m}}\sinh\left(\frac{3H_0 t\sqrt{1-\Omega_m}}{2}\right)\right]^{2/3}. \tag{17}$$

We have to be careful here though, because t is measured from the beginning of time and not from today, so we must solve this equation for $t = 13.8\text{Gyr} - 4.5\text{Gyr} = 9.3\text{Gyr}$. Evaluating this for the given values of the cosmological parameters (We have to plug in $h = 0.71$), we see that in this case, $1 + z = a^{-1} = 1.42$, where the flat universe allows us to make the choice $a_0 = 1$.

For the Λ CDM, $w = -0.9$ case we can make the variable change $x = (1+z)^{-1}$ and the substitutions $w = -0.9$, $\Omega_m = 0.26$, $\Omega_\Lambda = 1 - \Omega_m$ to obtain

$$t = \frac{1}{H_0}\int_a^1 \frac{x^{1/2} dx}{\sqrt{\Omega_m + (1-\Omega_m)\,x^{2.7}}}. \tag{18}$$

We use Mathematica to solve this numerically for $t = 4.5$ billion years, obtaining $1 + z = a^{-1} = 1.46$. Now we have to be careful. For the i^{th} species, $\Omega_i(a) = \frac{\rho(a)}{\rho_{crit}(a)}$, so

$$\Omega_m(a) = \frac{\Omega_{m,0}}{\Omega_{m,0} + \Omega_{\Lambda,0}a^{-3w} + \Omega_{k,0}a}$$
$$\Omega_\Lambda(a) = \frac{\Omega_{\Lambda,0}a^{-3w}}{\Omega_{m,0} + \Omega_{\Lambda,0}a^{-3w} + \Omega_{k,0}a}$$
$$\Omega_k(a) = \frac{\Omega_{k,0}a}{\Omega_{m,0} + \Omega_{\Lambda,0}a^{-3w} + \Omega_{k,0}a} \tag{19}$$

TABLE III: Luminosity distance $d_L(z)$ calculated for the four cosmologies under consideration, at three different redshifts, in the world of little Green martians, 4.5 billion years ago, in units of h^{-1} Mpc, where h is the value we observe today. l superscripts indicate values evaluated at the redshift of the LGMs.

$\Omega_{m,0}, \Omega_{\Lambda,0}, \Omega_{k,0}, w, h$	z_l	$\Omega_m, \Omega_\Lambda, \Omega_k, h$ evaluated at z_l	$z = 0.1$	$z = 0.5$	$z = 1.0$
$\Omega_m = 1, \Omega_\Lambda = 0, k = 0$	0.57	$\Omega^l_m = 1.0$, $\Omega^l_\Lambda = 0.0$,$\Omega^l_k = 0.0$, $h^l = 1.40$	156	838	1780
$\Omega_m = 0.26, \Omega_\Lambda = 0, k = -1$	0.53	$\Omega^l_m = 0.35, \Omega^l_\Lambda = 0, \Omega^l_k = 0.65, h^l = 0.87$	255	1460	3310
$\Omega_m = 0.26, \Omega_\Lambda = 0.74, k = 0$	0.42	$\Omega^l_m = 0.50, \Omega^l_\Lambda = 0.50, \Omega^l_k = 0, h^l = 0.87$	259	1520	3410
$\Omega_m = 0.26, \Omega_\Lambda = 0.74, k = 0, w = -0.9$	0.46	$\Omega^l_m = 0.52, \Omega^l_\Lambda = 0.48, \Omega^l_k = 0, h^l = 0.88$	255	1490	3330

TABLE IV: Distance Modulus with h dependence removed: $m - M + 5 \log h$ for the LGM

	$z = 0.1$	$z = 0.5$	$z = 1.0$
$\Omega_m = 1, \Omega_\Lambda = 0, k = 0$	35.97	39.62	41.25
$\Omega_m = 0.26, \Omega_\Lambda = 0, k = -1$	37.03	40.82	42.60
$\Omega_m = 0.26, \Omega_\Lambda = 0.74, k = 0$	37.07	40.91	42.66
$\Omega_m = 0.26, \Omega_\Lambda = 0, k = 0, w = -0.9$	37.03	40.87	42.61

where the 0 subscripts indicate the 'true' values today. We have to be careful, because the LGMs will have a different Hubble constant too. The value is easily scaled back:

$$H_{LGM} = H_0 \left(\frac{\Omega_{m,0}}{a^3} + \Omega_\Lambda a^{-3(1+w)} + \frac{\Omega_k}{a^2} \right)^{1/2} \tag{20}$$

Now we can go away and calculate things. Note that w does not change as a function of time in the models under consideration. I tabulate the results in Tables III and IV. Superscripts l denote values of parameters evaluated **at** z_l. The re-calculated distance moduli have some implicit h-dependence because we had to evaluate the redshift of the LGM to evaluate the cosmological parameters when they were doing cosmology. Note that wherever an h appears in a numerical answer, it is h measured today. As discussed in the problem session, the little green martians will observe a different Hubble parameter and this has been incorporated into our calculations using Eq. (20), but we've left everything in terms of the h we observe today to avoid drowning in a notational swamp. Looking at the entries in Table IV, we see that while the flat CDM cosmology can be easily distinguished from any of the others with $\delta m = 0.15$ photometry at all redshifts, it is inadequate in telling any of the other cosmologies apart. In particular, unlike today, we've lost the ability to tell an open universe apart from a cosmological constant dominated universe. This makes plenty of sense. Since $\Omega_\Lambda = 0.50$ for the little green martians, they should have to work harder to discover dark energy. Little martian supernovae astronomers seeking tenure just need to work a little harder though, because with $\delta m = 0.01$ photometry, little martians will be able to tell all these cosmologies apart.

Part g, ΛCDM, common to both interpretations

Now we need to evaluate the redshifts of objects at some chosen redshifts as seen by the little Martians. The easiest way to think about this is by remembering that the comoving distance between us and an object stays fixed. Then we define z_0 and z_1 as the redshifts of the object today and 4.5 billion years ago. In a ΛCDM cosmology with $w = -1$, the comoving distance is then given by $d_c = \frac{c}{a_0 H_0} \int_0^{z_0} \frac{dx}{\sqrt{\Omega_m(1+x)^3 + (1-\Omega_\Lambda)}}$ in terms of numbers today. Another expression can be obtained by substituting values 4.5 billion years ago. Equating the two, since comoving distance does not change, yields

$$\begin{aligned} \frac{a_{\rm LGM} H_{LGM}}{a_0 H_0} \int_0^{z_0} \frac{dx}{\sqrt{\Omega_m(1+x)^3 + (1-\Omega_m)}} &= \int_0^{z_1} \frac{dx}{\sqrt{\Omega_{m,\rm LGM}(1+x)^3 + (1-\Omega_{m,\rm LGM})}} \\ \frac{\sqrt{\Omega_m(1+z_{\rm LGM})^3 + (1-\Omega_m)}}{1+z_{\rm LGM}} \int_0^{z_0} \frac{dx}{\sqrt{\Omega_m(1+x)^3 + (1-\Omega_m)}} &= \int_0^{z_1} \frac{dx}{\sqrt{\Omega_{m,\rm LGM}(1+x)^3 + (1-\Omega_{m,\rm LGM})}} \end{aligned} \tag{21}$$

Plugging in the numbers we've derived, we see that for $z_0 = 0.5$ in this cosmology, $z_1 = 0.456$, and for $z_0 = 1.0$, $z_1 = 0.96$.

TABLE V: Luminosity distance $d_L(z)$ calculated for the four cosmologies under consideration, at three different redshifts, in the world of little Green martians, 4.5 billion years ago, in units of h^{-1} Mpc, where h is the h we observe today. l superscripts indicate values evaluated at $z = z_l$.

$\Omega_{m,0}, \Omega_{\Lambda,0}, \Omega_{k,0}, w, h$	z_l	$\Omega_m, \Omega_\Lambda, \Omega_k, h$ evaluated at z_l	$z = 0.1$	$z = 0.5$	$z = 1.0$
$\Omega_m = 1, \Omega_\Lambda = 0, k = 0$	0.42	$\Omega_m^l = 1.0$, $\Omega_\Lambda^l = 0.0, \Omega_k^l = 0.0$, $h^l = 0.87$	251	1350	2870
$\Omega_m = 0.26, \Omega_\Lambda = 0, k = -1$	0.42	$\Omega_m^l = 0.33, \Omega_\Lambda^l = 0, \Omega_k^l = 0.67, h^l = 0.87$	255	1460	3330
$\Omega_m = 0.26, \Omega_\Lambda = 0.74, k = 0$	0.42	$\Omega_m^l = 0.50, \Omega_\Lambda^l = 0.50, \Omega_k^l = 0, h^l = 0.87$	259	1520	3410
$\Omega_m = 0.26, \Omega_\Lambda = 0.74, k = 0, w = -0.9$	0.42	$\Omega_m^l = 0.48, \Omega_\Lambda^l = 0.52, \Omega_k^l = 0, h^l = 0.87$	259	1510	3400

TABLE VI: Distance Modulus with h dependence removed: $m - M + 5\log h$ for the LGM

	$z = 0.1$	$z = 0.5$	$z = 1.0$
$\Omega_m = 1, \Omega_\Lambda = 0, k = 0$	37.00	40.65	42.29
$\Omega_m = 0.26, \Omega_\Lambda = 0, k = -1$	37.03	40.82	42.61
$\Omega_m = 0.26, \Omega_\Lambda = 0.74, k = 0$	37.07	40.91	42.66
$\Omega_m = 0.26, \Omega_\Lambda = 0, k = 0, w = -0.9$	37.07	40.89	42.66

Note from your humble and apologetic TAs: In office hours, a different (and flawed) approach to part (g) was proposed: Given that $1 + z_0 = a_0/a_e$ and $1 + z_1 = a_{lgm}/a_e$, where a_e is the scale factor at the time the light was emitted, there appears to be a simple relationship between z_0 and z_1. The problem is that the LGM observe light that was emitted from a given galaxy at an earlier time than the light that we observe from the same galaxy. Therefore, a_e for the LGM is smaller that a_e for us for the same object. This approach gives the redshifts for the same time of emission–not the same object. This is why this approach leads to negative redshifts for the LGM; if $a_e > a_{lgm}$ then the light that they are supposed to be observing hasn't been emitted yet! Given our confusion on this point, we did not deduct points if this approach was used.

Solution B:
First we must evaluate the age of the universe as a function of redshift for the true cosmology. Last week we showed that for a pure cosmological constant ($w = -1$)

$$a = a_0 \left[\sqrt{\frac{\Omega_m}{1 - \Omega_m}} \sinh\left(\frac{3H_0 t\sqrt{1 - \Omega_m}}{2} \right) \right]^{2/3}. \tag{22}$$

We have to be careful here though, because t is measured from the beginning of time and not from today, so we must solve this equation for $t = 13.8\text{Gyr} - 4.5\text{Gyr} = 9.3\text{Gyr}$. Evaluating this for the given values of the cosmological parameters (We have to plug in $h = 0.71$), we see that in this case, $1 + z = a^{-1} = 1.42$, where the flat universe allows us to make the choice $a_0 = 1$. Now we can go away and calculate things using Eqs. (19) & (20). Note that w does not change as a function of time in the models under consideration. I tabulate the results in Tables V and VI. The re-calculated distance moduli have some implicit h-dependence because we had to evaluate the redshift of the LGM to evaluate the cosmological parameters when they were doing cosmology. Note that wherever an h appears in a numerical answer, it is h measured today. l superscripts denote values evaluated at $z = z_l$. As discussed in the problem session, the little green martians will observe a different Hubble parameter and this has been incorporated into our calculations using Eq. (20), but we've left everything in terms of the h we observe today to avoid drowning in a notational swamp. Looking at the values in Table VI, we see that low-z, $\delta m = 0.15$ photometry is cosmologically useless, while low-z, $\delta m = 0.01$ photometry should be able to tell all these cosmologies apart. At higher z, $\delta m = 0.15$ photometry should be able to distinguish all under-dense cases from a closed, flat CDM cosmology, while $\delta m = 0.01$ photometry should be able to distinguish all cases at high enough redshift. These sobering conclusions makes sense. Since $\Omega_\Lambda \approx 0.5$ for the little green martians, they should have to work harder to discover dark energy.

Problem #3:
(a) We are interested in the optical depth due to Compton scattering, so $d\tau = n_e(z)\sigma_T dl$, where $n_e(z)$ is the epoch-dependent free electron density and $\sigma_T = 6.65 \times 10^{-25}$ cm^2 is the Thompson scattering cross section. Since we are considering photons, $dl = cdt$. We can apply the Hubble equation ($\frac{da}{dt} = aH_0E(z)$, $E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}$)

and the equality $a = \frac{1}{1+z}$ to the preceding to obtain

$$d\tau = -\frac{n_{e^-}(z)\sigma_T c}{H_0(1+z)E(z)}dz. \tag{23}$$

Since we assume instantaneous reionization at $z_{\rm reion}$ and that all the free electrons come from Hydrogen (we neglect Helium reionization), $n_e(z) = n_H(z) = \frac{\rho_b(z)(1-Y)}{m_p} = \frac{\rho_{b,0}(1+z)^3(1-Y)}{m_p} = \frac{3\Omega_{b,0}H_0^2(1+z)^3(1-Y)}{8\pi G m_p}$ at redshifts less than $z_{\rm reion}$. Consolidating, we see that

$$d\tau = -\frac{3\sigma_T c H_0(1-Y)\Omega_{b,0}(1+z)^2}{8\pi G m_p E(z)}dz \tag{24}$$

To find the optical depth between us and the CMB, we must integrate $d\tau$ from the surface of last scattering to our current location:

$$\tau = \int_{t_{\rm LS}}^{t_0} n_e(z)\sigma_T c\, dt = \int_{z_{\rm CMB}}^{0} -\frac{n_{e^-}(z)\sigma_T c}{H_0(1+z)E(z)}dz = \int_0^{z_{\rm reion}} \frac{3\sigma_T c H_0(1-Y)\Omega_{b,0}(1+z)^2}{8\pi G m_p E(z)}dz \tag{25}$$

where the last equality follows from the fact that $n_e = 0$ prior to reionization. Feeding this to a calculator with all the given numbers, we see that

$$\tau = 1.67\times 10^{-3}\int_0^{z_{\rm reion}} dz' \frac{(1+z')^2}{\sqrt{\Omega_m(1+z')^3+\Omega_\Lambda}} \tag{26}$$

$$= 1.67\times 10^{-3}\left\{\sqrt{\Omega_m(1+z_{\rm reion})^3+\Omega_\Lambda} - 1\right\}\times\frac{2}{3\Omega_m} \tag{27}$$

(b) In the limit $z_{\rm reion} \gg \Omega_m^{-1}$, we can neglect Ω_Λ:

$$\tau \approx 1.67\times 10^{-3}\times\frac{2}{3\sqrt{\Omega_m}}\times\left\{(1+z_{\rm reion})^{3/2}-1\right\}, \tag{28}$$

which clearly agrees with the expression from part (a) in the large z limit.
(c) Solving for z as a function of τ for the given ΛCDM cosmology, we obtain

$$z = \left[\frac{\left[\frac{\tau}{4.28\times 10^{-3}}+1\right]^2 - 0.74}{0.26}\right]^{1/3} - 1. \tag{29}$$

For $\tau = 1$, we obtain $z_{reion} \approx 59$. For $\tau = 0.17$, we obtain $z_{reion} \approx 18$, and for $\tau = 0.09$ (WMAP 3-year data), we obtain $z_{reion} \approx 11$.

Problem #4:
If the knots are moving out at 0.19×10^{-3} arcsec yr^{-1}, $\frac{d\theta}{dt_{\rm earth}} \approx 2.9\times 10^{-17}$ rad s^{-1}. Those years and seconds are earth time, which is time-dilated by $(1+z)$ compared to time measured by cosmic observers near 3C179. So to get the transverse speed of the knots as measured by cosmic observers near 3C179 (i.e. observers who were at rest with respect to the cosmic microwave background and saw an isotropic Hubble flow when the light we now see from 3C179 was emitted),

$$v = \frac{d\ell}{dt_{\rm cosmic\ observer\ at\ z}} = \frac{D_A d\theta}{dt_{\rm earth}/(1+z)} = (1+z)D_A\frac{d\theta}{dt_{\rm earth}} \equiv D_M\frac{d\theta}{dt_{\rm earth}} \tag{30}$$

The proper motion distance to $z = 0.846$ is given by (assuming a flat universe)

$$d_M = \frac{c}{H_0}\int_0^z \frac{dx}{\sqrt{\Omega_m(1+x)^3+\Omega_\Lambda}} \tag{31}$$

$$= 4220\ {\rm Mpc}\int_0^{0.846}\frac{dx}{\sqrt{0.26(1+x)^3+0.74}} \tag{32}$$

$$= 2930\ {\rm Mpc}, \tag{33}$$

where the last step follows from numerical integration. The apparent velocity The apparent velocity is then $v_a/c = D_M \times \frac{d\theta}{dt}/c = 8.7$! The motion appears to be superluminal. What's going on?

(Optional note for those who did not take/are not taking Ay 125) Our analysis neglected the possibility that the source is also moving along the line of sight! Consider the arrival times of two pulses emitted towards us as the source moves with some direction θ, measured with respect to the line of sight. The first pulse, emitted at $t = 0$, arrives at $t_1 = D_{obs}/c$. The second pulse, emitted at $t = \delta t$, arrives at $t = \delta t + \sqrt{\left(\frac{D_{obs} - v\delta t\cos\theta}{c}\right)^2 + \left(\frac{v\delta t\sin\theta}{c}\right)^2}$ Assuming the distance traveled in time δt is much less than the observer-source distance, we can linearly expand the quantities inside the radicand to obtain

$$\delta t_{\text{apparent}} = t_2 - t_1 \approx \delta t\left(1 - \frac{v}{c}\cos\theta\right) \tag{34}$$

The transverse displacement is $\delta x = v\delta t\sin\theta$, so the apparent velocity (in units where $c = 1$) is given by

$$v_{\text{apparent}} = \frac{v\sin\theta}{1 - v\cos\theta}. \tag{35}$$

As a function of θ, v_a is maximized at $\theta = \cos^{-1} v$, so $v_{\text{apparent},max} = \frac{v}{\sqrt{1-v^2}}$, and we see that $v_{\text{apparent}} > 8.7$ is obtained for $v > c\cos(\cot^{-1}(8.7)) = 0.9935c$.

Problem #5:

a) i) For a collection of nonrelativistic, massive particles, the temperature is proportional to the average kinetic energy of the particles: $T \propto mv^2 = p^2/m$, where p is the momentum of the particle. Since $p \propto a^{-1}$ in an expanding universe, $T \propto a^{-2}$.

ii) The expansion of the Universe is adiabatic, and so $TV^{\gamma-1}$ is conserved during the expansion, where $V \propto a^3$ is volume and $\gamma \equiv C_p/C_v$. For a nonrelativistic monatomic ideal gas, $\gamma = 5/2$, so $TV^{2/3} \propto Ta^2$ must be constant. Therefore, $T \propto a^{-2}$.

b) i) The temperature of a relativistic gas is proportional to the momentum of the particles: $T \propto p$. However, p is still proportional to a^{-1}, so it follows that $T \propto a^{-1}$ as well.

ii) For a relativistic gas, the energy density is proportional to T^4 ($\rho = \sigma_r T^4$ in the notation of Coles & Lucchin). It follows that

$$C_v = \left.\frac{\partial \rho V}{\partial T}\right|_V = 4\sigma_r T^3 V. \tag{36}$$

Meanwhile, the pressure of a relativistic gas is given by $P = \rho/3$. Therefore, the enthalpy is $H = \rho V + PV = (4/3)\sigma_r T^4 V$. It follows that

$$C_p = \left.\frac{\partial H}{\partial T}\right|_P = \frac{16}{3}\sigma_r T^3 V. \tag{37}$$

Therefore, $\gamma = 4/3$ for a relativistic gas and $TV^{1/3} \propto Ta$ is constant during the expansion, implying that $T \propto a^{-1}$.

c) A coupled mixture has a single temperature T. From the first law of thermodynamics, $dE = -PdV$ at constant entropy, we can write

$$d\left[\left(\rho_m + \frac{3}{2}kTn_m + \sigma_r T^4\right)a^3\right] = -(n_m kT + \frac{1}{3}\sigma_r T^4)da^3 \tag{38}$$

for a combination of a nonrelativistic ideal gas (number density n_m and mass density ρ_m) and a relativistic gas. Since $\rho_m a^3$ and $n_m a^3$ are constant, the left hand side simplifies to

$$\frac{3}{2}kn_m a^3 dT + 4\sigma_r T^3 a^3 dT + \sigma_r T^4 da^3 = -(n_m kT + \frac{1}{3}\sigma_r T^4)da^3 \tag{39}$$

and rearranging terms gives

$$\left(\frac{3}{2}kn_m + 4\sigma_r T^3\right)a^3 dT = -\left(\frac{4}{3}\sigma_r T^4 + n_m kT\right)da^3 \tag{40}$$

$$\frac{dT}{T} = -\left(\frac{4\sigma_r T^3 + 3kn_m}{\frac{3}{2}kn_m + 4\sigma_r T^3}\right)\frac{da}{a} \tag{41}$$

$$\frac{dT}{T} = -\left(\frac{\beta + 1}{\beta + \frac{1}{2}}\right)\frac{da}{a}. \tag{42}$$

where we have defined a dimensionless ratio

$$\beta \equiv= \frac{4\sigma_r T^4}{3kn_m} = \frac{C_{v,r}}{2C_{v,m}} \tag{43}$$

where $C_{v,r}$ and $C_{v,m}$ are the heat capacities of the relativistic and nonrelativistic gases, respectively. In summary, we see that

$$T \propto a^{-\frac{\beta+1}{\beta+\frac{1}{2}}}. \tag{44}$$

If $\beta \gg 1$ then the relativistic gas dominates and $T \propto a^{-1}$. If $\beta \ll 1$ then the nonrelativistic gas dominates and $T \propto a^{-2}$. See Coles & Lucchin pages 113-115 for an estimate of β for our Universe; it turns out that $\beta \simeq 10^8(\Omega_b h^2)^{-1}$ after decoupling and is therefore assumed to be very large prior to decoupling as well. Consequently, $T \propto a^{-1}$ is a good approximation even while the baryons are nonrelativistic and coupled to photons. (Note that $\beta = \sigma_{\rm rad}$ in Coles & Lucchin; I thought it was too confusing to have σ_r and $\sigma_{\rm rad}$ when the two are not conceptually related.)