Problem Set 2

Due **in class**, Thursday 25 January 2018

Reading: See the on-line syllabus for lecture-by-lecture readings.

Collaboration policy: See the on-line collaboration policy.

Homework Problems:

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1. **If stars were the only mass-energy, this would be a good approximation!** In general relativity, mass-energy curves space-time. So a space empty of matter and energy should not be curved. An FRW universe in which the density of matter and radiation vanishes ($\Omega_0 \to 0$) describes an empty space, and so this spacetime ("Milne (1932) spacetime") should be merely the Minkowski metric in disguise (in the jargon of GR, disguises are called "gauge transformations", or just "new coordinates"). Unmask this by finding that coordinate transformation.

   Specifying the problem more explicitly: the FRW metric of constant negative curvature is

$$ds^2 = -c^2 dT^2 + a^2(T)(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\varphi^2)) \tag{1}$$

   a) Show that the dynamical equations for an empty $\Omega_0 = 0$ universe require the Hubble parameter $H^2 \propto a^{-2}$ and thus $a(T) \propto T$.

   b) Find the coordinate transformation $r(T,\chi)$, $t(T,\chi)$ that converts this FRW metric to the familiar flat Minkowski space (in spherical coordinates):

$$ds^2 = -c^2 dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{2}$$

2. **Age and Expansion history of our universe** For a flat universe composed of cold matter (aka 'dust', today making up a fraction Ω_{m0} of the critical density) and cosmological constant (as appears to be an accurate representation of our Universe for redshifts $0 < z < 10^3$)

   a) Show that the age of the universe is

$$t_0 = \frac{2}{3H_0}\frac{\sinh^{-1}\sqrt{\frac{1-\Omega_{m0}}{\Omega_{m0}}}}{\sqrt{1-\Omega_{m0}}} = \frac{2}{3H_0}\frac{\ln\left(\frac{1+\sqrt{1-\Omega_{m0}}}{\sqrt{\Omega_{m0}}}\right)}{\sqrt{1-\Omega_{m0}}}\ . \tag{3}$$

   Evaluate this age, in years, both for the currently preferred cosmological parameters $H_0 = 70\text{km s}^{-1}\text{Mpc}^{-1}$, $\Omega_{m0} = 0.29$, $\Omega_{\Lambda 0} = 0.71$, and for the pre-1998 preferred value $\Omega_{m0} \to 1$. Already in the 1980's and 1990's it was known that the stars in globular clusters were 11 to 12 Gyr old, so there was an "age" problem unless $H_0 \simeq 50\text{km s}^{-1}\text{Mpc}^{-1}$. As preferred values of H_0 began to rise in the 1990's this became more of a concern. But since a nonzero cosmological constant of 10^{-120} in Planck units seems incredibly contrived from a particle physics point of view, it was generally accepted that the problem was more likely in the astronomical H_0 estimates than that the cosmological constant Einstein had declared "his biggest mistake" should be revived.

   b) Show that the scale factor $a(t)$ as a function of time is given by

$$a(t) = a_0\left(\sqrt{\frac{\Omega_{m0}}{1-\Omega_{m0}}}\sinh\left(\frac{3}{2}\sqrt{1-\Omega_{m0}}H_0 t\right)\right)^{2/3}\ , \tag{4}$$

   where a_0 is the present-day scale factor. Show that this increases exponentially in the future.

3. **The end of cosmology**. The nearest cluster of galaxies to us, the Virgo cluster, is 16 comoving Mpc distant from us ($\langle cz \rangle = 1100$km s^{-1}). It is not, and never will be bound to the local group, but will just continue to expand away from us in the Hubble flow.

   a) Using the standard cosmological parameters given in class (neglecting curvature and radiation terms), calculate the future redshift evolution of the Virgo cluster: i.e., give its $z(t)$ as a function of time in the future. You may choose either to do this exactly (using results from the previous problem), or in the approximation that now and in the future the cosmological constant term dominates over the matter term in the Friedmann equation.

   b) Discuss the future evolution of the appearance and brightness of galaxies in the Virgo cluster, and of more distant galaxies. Estimate the time at which cosmology would cease to exist as an observational science.

4. **Transverse velocities vs. Sergeant Einstein's limit.** The galaxy 3C 179, at redshift $z = 0.846$, is observed to have knots of radio emission moving away from its center. The separation of these knots increases by 0.19 milliarcsec per year. For standard cosmological parameters, (and $H_0 = 70$km s^{-1}Mpc^{-1}), calculate the speed at which the knots are moving out. Be sure the speed you give is that which would be measured in the frame of cosmological observers near 3C 179. Compare the speed to well-known speed limits.

5. **k-corrections at mm wavelengths.** Thermal dust emission from small grains has a spectrum $F_\nu \propto \nu^3$ for frequencies below the thermal peak, i.e. for $\nu < kT/h \sim 3$THz. A dusty starburst galaxy at redshift $z = 0.05$ is observed to have $F_\nu = 20$mJy at $\nu = 240$GHz (1.25mm wavelength).

   What F_ν would you measure at the same $\nu = 240$GHz from starburst galaxies identical to that one, but at redshifts $z = 1, 3, 9$? Adopt standard cosmological parameters and $H_0 = 70$km s^{-1}Mpc^{-1}.

   Explain any counter-intuitive features of your results.