

# Ay127: Spring 2007 Homework #1 Solutions

Solutions by Adrienne Erickcek & Daniel Grin

## Problem #1:

(a) Consider three observers in a region of space small compared to the radius of curvature of the universe (i.e. locally flat space): A, B, and C. Vector addition tells us that the velocity of C relative to A must equal the velocity of B relative to A plus the velocity of C relative to B:

$$\vec{v}_{CA} = \vec{v}_{BA} + \vec{v}_{CB}. \quad (1)$$

Since we know that  $\vec{v} = \vec{f}(\vec{r}, t)$  for all observers, we have

$$\vec{f}(\vec{r}_{CA}) = \vec{f}(\vec{r}_{BA}) + \vec{f}(\vec{r}_{CB}). \quad (2)$$

Finally,  $\vec{r}_{CA} = \vec{r}_{BA} + \vec{r}_{CB}$  allows us to rewrite the previous expression as

$$\vec{f}(\vec{r}_{BA} + \vec{r}_{CB}) = \vec{f}(\vec{r}_{BA}) + \vec{f}(\vec{r}_{CB}). \quad (3)$$

(b) From part (a) we know that for any two vectors  $\vec{r}_0$  and  $\vec{\epsilon}$ ,

$$f_i(\vec{r}_0 + \vec{\epsilon}) = f_i(\vec{r}_0) + f_i(\vec{\epsilon}), \quad (4)$$

where the subscript  $i$  runs over the cartesian components ( $x, y, z$ ). If we assume that the magnitude of  $\vec{\epsilon}$  is small, we may Taylor expand to get

$$f_i(\vec{r}_0 + \vec{\epsilon}) \simeq f_i(\vec{r}_0) + \vec{\epsilon} \cdot \vec{\nabla} f_i(\vec{r}_0), \quad (5)$$

where we have dropped higher-order terms in  $\vec{\epsilon}$ . Since  $\vec{\nabla} f_i(\vec{r})$  does not depend on  $\vec{\epsilon}$ , we may define  $H_{ij}(\vec{r}_0, t) \equiv \partial_j f_i(\vec{r}_0, t)$ . Homogeneity further requires  $H_{ij}(\vec{r}_0, t) = H_{ij}(t)$ . Comparing the previous two equations, we have

$$f_i(\vec{\epsilon}) = v_i(\vec{\epsilon}) = \vec{\epsilon} \cdot \vec{\nabla} f_i(\vec{r}_0) = \sum_{j=1}^3 H_{ij}(t) \epsilon_j \quad (6)$$

for any short vector  $\vec{\epsilon}$ . Finally, any vector may be expressed as the sum of many shorter vectors,  $\vec{r} = N\vec{\epsilon}$  for some integer  $N$ . From part (a), we know that  $\vec{f}(N\vec{\epsilon}) = N\vec{f}(\vec{\epsilon})$ . Therefore,

$$v_i(\vec{r}) = \sum_{j=1}^3 H_{ij}(t) r_j. \quad (7)$$

(c) Isotropy implies that the curl of  $\vec{v}$  must vanish (zero vorticity).

$$0 = \vec{\nabla} \times \vec{v} = \sum_{l,i=1}^3 \epsilon^{kli} \partial_l v_i = \sum_{l,i=1}^3 \epsilon^{kli} \partial_l \sum_{j=1}^3 H_{ij}(t) r_j = \sum_{l,i=1}^3 \epsilon^{kli} \sum_{j=1}^3 H_{ij}(t) \delta_{lj} = \sum_{l,i=1}^3 \epsilon^{kli} H_{il}, \quad (8)$$

where the subscripts  $i, j, k, l$  run over the cartesian components ( $x, y, z$ ),  $\epsilon^{kli}$  is the three-dimensional anti-symmetric tensor and  $\delta_{lj}$  is a Kronecker delta. Since  $H_{il}$  vanishes when contracted with an anti-symmetric tensor, the  $H$  matrix must be symmetric. Therefore, it is diagonalizable. Finally, isotropy demands that  $H_{xx} = H_{yy} = H_{zz}$ . Thus we have shown that  $v_i = H(t)r_i$ .

**Alternate solution for differential topologists:** The component of  $\vec{v}$  that is perpendicular to  $\vec{r}$  is a two-dimensional vector on the sky. The Brouwer ‘‘hairy ball theorem’’ (aka ‘‘you can’t comb the hair on a billiard ball’’) says that any continuous two-dimensional tangent-vector field on a sphere must vanish at at least one point (that whorl on the back of your head where the hair stubbornly sticks up). Consider a thin shell of points equidistant from some cosmic observer. The hairy ball theorem says that on that shell, there must be at least one special point where  $\vec{v}$  has no component that is perpendicular to  $\vec{r}$ . If the Universe is isotropic, then there can be no special point on that shell. So if  $\vec{v}$  is parallel to  $\vec{r}$  at one point on the shell, then it must be parallel to  $\vec{r}$  everywhere on the shell. Repeating for all possible shells shows that the Hubble flow must be radial on all of them. Equation (3) shows that the velocity of the radial flow on each shell must be proportional to distance, and thus  $v_i = H(t)r_i$ .

**Problem #2:**

(a) In an empty universe with negative curvature, the Hubble equation reads

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{C}{a^2}, \quad (9)$$

where  $C$  is the constant with appropriate units, and dots denote derivatives with respect to the time-like coordinate  $T$ . This equation is trivially integrated to obtain  $a \propto T$ .

(b) For dimensionless  $\chi$  and dimensionful scale factor  $a$  in an empty, negatively curved universe, the metric can be rewritten

$$ds^2 = -c^2 dT^2 + c^2 T^2 (d\chi^2 + \sinh^2 \chi d\Omega^2). \quad (10)$$

Staring at this metric suggests the substitution  $r = cT \sinh \chi$  (we want to make the factor multiplying the solid angle into a radius), and correspondingly, we guess that  $t = T \cosh \chi$ . Then taking differentials and expanding binomials yields

$$-c^2 dt^2 + dr^2 = -c^2 dT^2 (\cosh^2 \chi - \sinh^2 \chi) + c^2 T^2 d\chi^2 (\cosh^2 \chi - \sinh^2 \chi) = -c^2 dT^2 + c^2 T^2 d\chi^2 \quad (11)$$

Therefore, with this substitution,

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2, \quad (12)$$

and we have brought the metric into standard Minkowski form in spherical coordinates. What's the lesson? Basically, we chose a lousy coordinate system, foliating our space-time with spatial hypersurfaces of constant negative curvature. By doing so, we hid the fact that this spacetime is flat. This has to be the case because by hypothesis, this spacetime is empty, and has no matter in it! Einstein's equation tells us that such a spacetime has to be flat. By performing the coordinate transformation, we have made manifest the flatness of this spacetime.

**Problem #3:**

(a) As seen by a moving observer, the frequency  $\nu$  of the isotropic photons in the isotropic cosmic frame will be Doppler-shifted so that the frequency observed by a moving observer is  $\nu_{\text{obs}} = \nu / [\gamma(1 - \beta \cos \theta)]$ , where  $\beta = v/c$ , and  $\gamma = (1 - \beta^2)^{-1/2}$ . This follows from the special relativistic Doppler formula. As a result, the Planck function may be written as

$$f(\nu, T_0) = \frac{1}{e^{\frac{h\nu}{kT_0}} - 1} = \frac{1}{e^{\frac{h\nu_{\text{obs}} \gamma (1 - \beta \cos \theta)}{kT_0}} - 1} = \frac{1}{e^{\frac{h\nu_{\text{obs}}}{kT}} - 1} \quad (13)$$

where we have defined a redshifted temperature of

$$T(\theta) = T_0 (1 - \beta^2)^{1/2} (1 - \beta \cos \theta)^{-1} = \quad (14)$$

$$T_0 \times \left(1 + \beta \cos \theta - \frac{\beta^2}{2} + \beta^2 \cos^2 \theta + \dots\right) \quad (15)$$

$$\simeq T_0 \times \left(1 + \beta \cos \theta + \left(\frac{\beta^2}{2}\right) \cos 2\theta\right), \quad (16)$$

where the last step follows from the identity  $\cos 2\theta = 2 \cos^2 \theta - 1$ .

(b) The dipole anisotropy is quoted as if observed from the solar system barycenter because the earth's heliocentric velocity would lead to a modulation with time-scale of a year in the temperature measured from the dipole anisotropy. The mean velocity of the earth around the sun is  $30 \text{ km s}^{-1}$ , leading to a modulation of  $\frac{\Delta T}{T} \approx 0.0001 \rightarrow \Delta T \approx 0.27 \text{ mK}$  (Over the course of a year,  $|\Delta v| \sim v$ ). Assuming the dipole anisotropy is aligned with the direction of motion of the galaxy with respect to the CMB ( $\theta \approx 0$ ),  $v_{g,CMB} = 368 \text{ km s}^{-1}$ .

(c) Since the sun is moving in roughly the opposite direction with  $v \simeq 235 \text{ km s}^{-1}$ , the velocity of the solar system barycenter with respect to the CMB is roughly  $235 \text{ km s}^{-1} + 368 \text{ km s}^{-1} \simeq 604 \text{ km s}^{-1}$ . Now we treat this more carefully, actually calculating the direction of the galactic motion with respect to the CMB. We begin by

calculating the magnitude of the solar motion with respect to the CMB. We assume that the dipole is due entirely to this motion, and not due to primordial temperature fluctuations. This is reasonable given that primordial fluctuations are of order  $\frac{\Delta T}{T} \approx 10^{-5}$ . We neglect the angular dependence in the Doppler formula and take the non-relativistic limit. Thus  $|\vec{v}| = c \frac{\Delta T}{T} = c \frac{3.36 \times 10^{-3} \text{ K}}{2.76 \text{ K}} \simeq 3.65 \times 10^7 \text{ cm s}^{-1}$ . The observed direction of motion is  $\alpha = 11\text{h } 09\text{min}$ ,  $\delta = -7^\circ$  in celestial spherical coordinates. Now we convert this to galactic polar coordinates using the formula on pg. (31) of the galactic astronomy text by Binney & Merrifield:

$$\sin b = \sin \delta_{\text{gp}} \sin \delta + \cos \delta_{\text{gp}} \cos \delta \cos (\alpha - \alpha_{\text{gp}}) \quad (17)$$

$$\cos b \sin (l_{\text{cp}} - l) = \cos \delta \sin (\alpha - \alpha_{\text{gp}}) \quad (18)$$

$$\cos b \cos (l_{\text{cp}} - l) = \cos \delta_{\text{gp}} \sin \delta - \sin \delta_{\text{gp}} \cos \delta \cos (\alpha - \alpha_{\text{GP}}), \quad (19)$$

where GP and CP refer to the galactic and celestial poles;  $l_{\text{cp}} = 122.932^\circ$ ,  $\alpha_{\text{gp}} = 192.85948^\circ$ , and  $\delta_{\text{gp}} = 27.12825^\circ$ . Plugging in numbers, we see that

$$\sin b = 0.741004 \quad (20)$$

$$\cos b \sin (122.932 - l) = -0.429013 \quad (21)$$

$$\cos b \cos (122.932 - l) = -0.516586 \quad (22)$$

Since the signs of these bottom two equations in this set tell us that  $l$  has to be in the third quadrant, we conclude that  $b = 48.817^\circ$ , and  $l = 263.223^\circ$ . In galactic Cartesian coordinates,  $v_x = |\vec{v}| \cos b \cos l$ ,  $v_y = |\vec{v}| \cos b \sin l$ , and  $v_z = |\vec{v}| \sin b$ . Plugging in numbers again, we see that

$$v_x = -2.89 \times 10^6 \text{ cm s}^{-1} \quad (23)$$

$$v_y = -2.43 \times 10^7 \text{ cm s}^{-1} \quad (24)$$

$$v_z = 2.73 \times 10^7 \text{ cm s}^{-1} \quad (25)$$

Meanwhile, the sun moves with respect to the LSR with velocity

$$v_{\odot,x} = 1.000 \times 10^6 \text{ cm s}^{-1} \quad (26)$$

$$v_{\odot,y} = 2.25 \times 10^7 \text{ cm s}^{-1} \quad (27)$$

$$v_{\odot,z} = 7.2 \times 10^5 \text{ cm s}^{-1} \quad (28)$$

So when we subtract out the motion of the sun, the velocity of the galactic center with respect to the CMB is

$$v_{\text{g-CMB},x} = v_x - v_{\odot,x} = -3.89 \times 10^6 \text{ cm s}^{-1} \quad (29)$$

$$v_{\text{g-CMB},y} = v_y - v_{\odot,y} = 4.68 \times 10^7 \text{ cm s}^{-1} \quad (30)$$

$$v_{\text{g-CMB},z} = v_z - v_{\odot,z} = 2.66 \times 10^7 \text{ cm s}^{-1} \quad (31)$$

So  $|\vec{v}_{\text{g-CMB}}| = 543 \text{ km s}^{-1}$ . Meanwhile, since  $\sin b_{\text{g-CMB}} = \frac{v_{\text{g-CMB},z}}{|\vec{v}_{\text{g-CMB}}|} = 0.5109 \rightarrow b_{\text{g-CMB}} = 30.72^\circ$ . Also,  $\cos l_{\text{g-CMB}} = \frac{v_{\text{g-CMB},x}}{|\vec{v}_{\text{g-CMB}}| \cos b_{\text{g-CMB}}} = -0.082295 \rightarrow l_{\text{g-CMB}} = 265.2$  once we properly take note of what quadrant we are in. Finally, we want to convert this direction back into celestial coordinates. Using

$$\sin \delta = \sin \delta_{\text{gp}} \sin b + \cos \delta_{\text{gp}} \cos b \cos (l_{\text{cp}} - l) \quad (32)$$

$$\cos \delta \sin (\alpha - \alpha_{\text{gp}}) = \cos b \sin (l_{\text{cp}} - l) \quad (33)$$

$$\cos \delta \cos (\alpha - \alpha_{\text{gp}}) = \cos \delta_{\text{gp}} \sin b - \sin \delta_{\text{gp}} \cos b \cos (l_{\text{cp}} - l), \quad (34)$$

we obtain  $\delta = -23^\circ$  and  $\alpha = 157.5^\circ$ .

(d) If we want to be able to measure the Hubble constant to 10% accuracy, we want peculiar velocities to be of order 10% the Hubble flow. That is  $v_p \leq \frac{1}{10} v_H = \frac{1}{10} H_0 d \simeq 7 \text{ d}_{\text{Mpc}} \text{ km s}^{-1}$ . Assuming galactic peculiar velocities with respect to the Hubble flow are of order  $500 \text{ km s}^{-1}$ , this gives the lower bound of  $d \geq 70 \text{ Mpc}$  for the Hubble constant to be measured with the accuracy desired.

#### Problem #4:

(a) i. The average density of a homogeneous sphere is given by  $\bar{\rho}_{m,\text{local}} = \frac{3M}{4\pi R^3}$ , where  $M$  is the mass contained within the sphere and  $R$  its radius. Plugging in the given numbers, we obtain  $\bar{\rho}_{m,\text{local}} = 2.55 \times 10^{-24} \text{ g cm}^{-3}$ . Meanwhile, the

mean density of the universe is obtained via  $\bar{\rho}_m = \Omega_m \rho_{\text{crit}}$ , where the critical density  $\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}$ . Plugging in constants and the relation  $H_0 \equiv 100 \text{ km s}^{-1} \text{ Mpc}^{-1} h$ , we obtain  $\bar{\rho}_m = \Omega_m h^2 \times 1.886 \times 10^{-29} \text{ g cm}^{-3} \approx 2.47 \times 10^{-30} \text{ g cm}^{-3}$  for the given values of the parameters ( $\Omega_m = 0.26, h = 0.71$ ). Comparing numbers, we see that the local density exceeds the mean density of matter in the universe by a factor of  $\sim 10^6$ .

ii. If 70% of the local matter density is accounted for by baryons, multiplying the first result in (a) yields  $\bar{\rho}_{b,\text{local}} = 1.79 \times 10^{-24} \text{ g cm}^{-3}$ . Meanwhile, on larger scales,  $\bar{\rho}_b = \Omega_b \rho_{\text{crit}}$ , yielding  $\bar{\rho}_b = \Omega_b h^2 \times 1.886 \times 10^{-29} \text{ g cm}^{-3}$ , and so  $\bar{\rho}_b = 4.18 \times 10^{-31} \text{ g cm}^{-3}$  for the given values of the cosmological parameters. The local baryon density thus exceeds the mean value in the universe by a factor of about  $4.3 \times 10^6$ . Because baryons couple to electrons through Coulomb forces and these electrons can scatter, accelerate, and radiate, baryons are able to cool, and thus contract more efficiently than dark matter. Simulations of large-scale structure bear this intuitive conclusion out.

(b) Shoving the total mass of the Milky Way and Andromeda halos into the given sphere, the average density is trivially obtained,  $\bar{\rho}_{780\text{kpc}} \simeq 1.10 \times 10^{-28} \text{ g cm}^{-3}$ . This is a factor of  $\sim 40$  larger than the mean density in the universe. As we push to larger and larger length scales, the density will approach the mean density!

### Problem #5:

a) Here we are asked to evaluate the age of a flat universe consisting of dust and a cosmological constant. For computational ease, we will evaluate the age at some arbitrary scale factor  $a_*$ , apply the result to the case of interest, and hold onto the expression desired for use in part b). In this case the Friedmann equation reads

$$\frac{1}{a^2(t)} \left( \frac{da}{dt} \right)^2 = H_0^2 \left[ \Omega_m \left( \frac{a_0}{a} \right)^3 + (1 - \Omega_m) \right], \quad (35)$$

where  $a_0$  is the scale factor today. Solving for time, we see that

$$t_* = \frac{1}{H_0} \int_0^{a_*} \frac{da}{\sqrt{\Omega_m \left( \frac{a_0^3}{a} \right) + (1 - \Omega_m) a^2}}, \quad (36)$$

where  $*$  just denotes the time evaluated at some arbitrary value of the scalar factor. We now make the substitution  $u \equiv \left( \frac{a}{a_0} \right)^{3/2}$  to obtain

$$t_* = \frac{2}{3H_0} \int_0^{\left( \frac{a_*}{a_0} \right)^{3/2}} \frac{du}{\sqrt{\Omega_m + (1 - \Omega_m) u^2}} \quad (37)$$

I don't know about you, but when I see an integral like this, my heart beckons for the comfort of hyperbolic functions. I'm not joking. This sort of looks like an arcsine integral, but with a sign reversed, so it stands to reason that hyperbolic functions will have something to do with the solution. The natural guess is to make the substitution  $u \equiv \sqrt{\frac{\Omega_m}{1 - \Omega_m}} \sinh x$ . As  $d \sinh x = \cosh x$  and  $\cosh^2 x - \sinh^2 x = 1$ , Eq. (37) can be simplified to

$$t_* = \frac{2}{3H_0 \sqrt{1 - \Omega_m}} \int_0^{\sinh^{-1} \sqrt{\frac{1 - \Omega_m}{\Omega_m}} \left( \frac{a_*}{a_0} \right)^{3/2}} dx. \quad (38)$$

Evaluating this today,  $a_* = a_0$ , and we have

$$t_0 = \frac{2}{3H_0 \sqrt{1 - \Omega_m}} \sinh^{-1} \sqrt{\frac{1 - \Omega_m}{\Omega_m}}. \quad (39)$$

Now we apply the identity  $\sinh^{-1} x = \ln x + \sqrt{x^2 - 1}$  to Eq. (39) (ask your favorite math methods book, use Mathematica, or verify the identity yourself!) to yield the last part of the desired result:

$$t_0 = \frac{2}{3H_0 \sqrt{1 - \Omega_m}} \ln \left( \frac{1 + \sqrt{1 - \Omega_m}}{\sqrt{\Omega_m}} \right). \quad (40)$$

Plugging in the given numbers and converting units from seconds to years, and Mpc to km appropriately, we see that  $t_0 = 13.8 \text{ Gyr}$ . Note that for  $\Omega_m \rightarrow 1$  equation (39) gives  $t_0 = 2/(3H_0) = 9 \text{ Gyr}$ . Thus until the 1998 'discovery' of

dark energy, there was a long-standing discrepancy between the age of the universe as estimated from cosmological indicators (and the inflation-predicted flat universe) and the  $\sim 12\text{Gyr}$  ages of the oldest stars in globular clusters estimated from stellar evolution models. Now there is no need to invoke  $H_0 = 50$ , stars surviving from previous bounces of the universe, or wild modifications to nuclear physics. Turns out we did understand stellar evolution better than cosmology!

(b) More generally, we can just leave  $a_*$  unspecified and invert Eq. (38) to solve for  $a_*$  in terms of the time:

$$a_* = a_0 \left[ \sqrt{\frac{\Omega_m}{1 - \Omega_m}} \sinh \left( \frac{3H_0 t_* \sqrt{1 - \Omega_m}}{2} \right) \right]^{2/3} \quad (41)$$