ACM95/100b
Problem Set 6 Solutions
02/23/04

Problem 1 (4×5 points)

(a)

$$u_{tt} = c^2 u_{xx} \tag{1}$$

Let u=X(x)T(t) and simplify

$$\frac{T''}{T} = c^2 \frac{X''}{X} \tag{2}$$

Since the left side of the equation is a function of t only and the right side is a function of x only, the equation can only hold if each side is the same constant.

$$\begin{aligned} T'' + \lambda c^2 T = 0 \\ X'' + \lambda X = 0 \end{aligned} \tag{3}$$

(b)

$$u_{xx} + u_{yy} + u_{zz} = 0 \tag{4}$$

Let u=X(x)Y(y)Z(z) and simplify.

$$\frac{Y''}{Y} + \frac{Z''}{Z} = -\frac{X''}{X} \tag{5}$$

Since the right side depends only on x and the left side is a function of y and z, it must be that both sides are the same constant

$$\begin{aligned} X'' + \lambda X = 0 \\ \frac{Y''}{Y} = \lambda - \frac{Z''}{Z} \end{aligned} \tag{6}$$

A similar idea applies to this last equation

$$\begin{aligned} Y'' + \gamma Y = 0 \\ Z'' - (\lambda + \gamma) Z = 0 \end{aligned} \tag{7}$$

(c)

$$u_t = k u_{xx} - c u_x \tag{8}$$

Let u=X(x)T(t) and simplify.

$$\frac{T'}{T} = k \frac{X''}{X} - c \frac{X'}{X} \tag{9}$$

Since the left side is a function of t onlyand the right side is a function of x only, each side must the same constant

$$\begin{aligned} T' - \lambda T = 0 \\ k X'' - c X' - \lambda X = 0 \end{aligned} \tag{10}$$

(d)

$$v_t + r x v_x + \frac{1}{2} \sigma^2 x^2 v_{xx} = r v \tag{11}$$

Let v=X(x)T(t) and simplify.

$$r\,x\,\frac{X'}{X} + \frac{1}{2}\sigma^2 x^2 \frac{X''}{X} = r - \frac{T'}{T} \qquad (12$$

Since the left side is a function of x only and the right side is a function of T only, each side must be the same constant.

$$\begin{aligned} &T' + (\lambda - r)\,T = 0 \\ &\frac{1}{2}\sigma^2 x^2 X'' + r\,x\,X' - \lambda X = 0 \end{aligned} \qquad (13)$$

Problem 2 (3×5 points)

(a)

$$u_t = \kappa\,(u_{xx} + u_{yy} + u_{zz}) \qquad (14)$$

Let u=X(x)Y(y)Z(z)T(t) and simplify.

$$\frac{T'}{T} = \kappa\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) \qquad (15)$$

By the reasoning of problem 1 we have

$$\begin{aligned} &T' - \lambda\,\kappa\,T = 0 \\ &\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \lambda \end{aligned} \qquad (16)$$

Applying the reasoning iteratively to this last equation gives

$$\begin{aligned} &X'' + \gamma X = 0 \\ &Y'' + \mu Y = 0 \\ &Z'' - (\lambda + \gamma + \mu)\,Z = 0 \end{aligned} \qquad (17)$$

(b)

$$u_t = \kappa\,\frac{1}{r^2}\,(r^2 u_r)_r \qquad (18)$$

Let u=R(r)T(t) and simplify

$$\frac{T'}{T} = \kappa\,\frac{1}{r^2}\,\frac{1}{R}\,(r^2 R')' \qquad (19)$$

By the reasoning of problem 1 we have

$$\begin{aligned} &T' - \lambda\,\kappa\,T = 0 \\ &(r^2 R')' - \lambda\,r^2 R = 0 \end{aligned} \qquad (20)$$

another way to write this last equation is

$$R'' + \frac{2}{r}R' - \lambda R = 0 \qquad (21)$$

(c)

$$u_t = \kappa\left(\frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2 \operatorname{Sin}\theta}(\operatorname{Sin}\theta\, u_\theta)_\theta + \frac{1}{r^2 \operatorname{Sin}^2\theta}u_{\phi\phi}\right) \qquad (22)$$

Let u=R(r)$\Xi(\theta)\Phi(\phi)$T(t) and simplify

$$\frac{T'}{T} = \kappa\left(\frac{1}{R\,r^2}(r^2 R')' + \frac{1}{r^2 \operatorname{Sin}\theta}\frac{(\operatorname{Sin}\theta\,\Xi')'}{\Xi} + \frac{1}{r^2 \operatorname{Sin}^2\theta}\frac{\Phi''}{\Phi}\right) \qquad (23)$$

By the same reasoning as problem 1 we have

$$T' - \lambda \kappa T = 0$$
$$\frac{1}{R r^2} (r^2 R')' + \frac{1}{r^2 \operatorname{Sin}\theta} \frac{(\operatorname{Sin}\theta\, \Xi')'}{\Xi} + \frac{1}{r^2 \operatorname{Sin}^2 \theta} \frac{\Phi''}{\Phi} = \lambda \quad (24$$

Multiplying by r^2 and simplifying gives

$$\frac{1}{\operatorname{Sin}\theta} \frac{(\operatorname{Sin}\theta\, \Xi')'}{\Xi} + \frac{1}{\operatorname{Sin}^2 \theta} \frac{\Phi''}{\Phi} = \lambda r^2 - \frac{1}{R} (r^2 R')' \quad (25)$$

Since the right side is a function of r and the left side is a function of θ and ϕ, both sides must be the same constant, call it γ.

$$(r^2 R')' + (\gamma - \lambda r^2) R = 0$$
$$\operatorname{Sin}\theta \frac{(\operatorname{Sin}\theta\, \Xi')'}{\Xi} - \gamma \operatorname{Sin}^2 \theta = -\frac{\Phi''}{\Phi} \quad (26)$$

Again, each side must be the same constant, call it μ.

$$\Phi'' + \mu \Phi = 0$$
$$\operatorname{Sin}\theta (\operatorname{Sin}\theta\, \Xi)' - (\gamma \operatorname{Sin}^2 \theta + \mu) \Xi = 0 \quad (27)$$

Problem 3 (5×5 points)

$$\frac{1}{c^2} u_{tt} = \frac{1}{r} (r u_r)_r + \frac{1}{r^2} u_{\varphi\varphi}$$
$$u(r, \varphi, 0) = f(r, \varphi)$$
$$u_t(r, \varphi, 0) = 0 \quad (28)$$
$$u(a, \varphi, t) = 0$$
$$u(0, \varphi, t) = \text{finite}$$

(a)

Let u=R(r)Ψ(φ)T(t) and simplify

$$\frac{1}{c^2} \frac{T''}{T} = \frac{1}{r R} (r R')' + \frac{1}{r^2} \frac{\Psi''}{\Psi} \quad (29)$$

Each side must be the same constant, call it -λ.

$$T'' + \lambda c^2 T = 0$$
$$\frac{\Psi''}{\Psi} = -\lambda r^2 - \frac{r}{R} (r R')' \quad (30)$$

Each side must be the same constant, call it -γ.

$$\Psi'' - \gamma \Psi = 0$$
$$r (r R')' + (\lambda r^2 + \gamma) R = 0 \quad (31)$$

(b)

$$\Psi'' = \gamma \Psi \quad (32)$$

In order for our solution to be physically meaningful, it must be 2π periodic in φ.

$$u(r, \varphi, t) = u(r, \varphi + 2\pi, t)$$
$$u_\varphi(r, \varphi, t) = u_\varphi(r, \varphi + 2\pi, t) \quad (33)$$

This implies

$$\Psi'' - \gamma \Psi = 0$$
$$\Psi(0) = \Psi(2\pi) \quad (34)$$
$$\Psi'(0) = \Psi'(2\pi)$$

This ODE is in Sturm-Liouville form (p=1, q=0, w=1) and the boundary conditions are periodic giving us a periodic S-L eigenvalue problem. There will be solutions of the form $e^{r\varphi}$, pluggin this in gives

$$r^2 = \gamma \tag{35}$$

So the solution is of the form

$$\Psi = A\, e^{\gamma^{1/2}\varphi} + B\, e^{-\gamma^{1/2}\varphi} \tag{36}$$

The periodic conditions give

$$\begin{aligned} A + B &= A\, e^{\gamma^{1/2} 2\pi} + B\, e^{-\gamma^{1/2} 2\pi} \\ A - B &= A\, e^{\gamma^{1/2} 2\pi} - B\, e^{-\gamma^{1/2} 2\pi} \end{aligned} \tag{37}$$

This can most easily be written in matrix form

$$\begin{pmatrix} 1 - e^{\gamma^{1/2} 2\pi} & 1 - e^{-\gamma^{1/2} 2\pi} \\ 1 - e^{\gamma^{1/2} 2\pi} & -1 + e^{-\gamma^{1/2} 2\pi} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{38}$$

If the matrix were invertible then the system would only have the trivial solution A=B=0. So, to get a non-trivial solution we require that the matrix not be invertible which is equivalent to requiring that the determinant of the matrix be zero

$$\left(1 - e^{\gamma^{1/2} 2\pi}\right)\left(-1 + e^{-\gamma^{1/2} 2\pi}\right) - \left(1 - e^{\gamma^{1/2} 2\pi}\right)\left(1 - e^{-\gamma^{1/2} 2\pi}\right) = 0 \tag{39}$$

Simplifying

$$\left(e^{2\pi\gamma^{1/2}} - 1\right)^2 = 0 \tag{40}$$

Solving this gives

$$2\pi\gamma^{1/2} = 2\pi\, n\, i \tag{41}$$

or

$$\gamma_n = -n^2 \tag{42}$$

where n is any integer. A quick check shows that n and -n both give the same eigenvalue and eigenfunction. So we have

$$\begin{aligned} \gamma_n &= -n^2 \text{ for } n = 1, 2, \ldots \\ \Psi_n &= A_n \operatorname{Sin}(n\varphi) + B_n \operatorname{Cos}(n\varphi) \end{aligned} \tag{43}$$

and

$$\begin{aligned} \gamma_0 &= 0 \\ \Psi_0 &= B_0 \end{aligned} \tag{44}$$

(c)

From parts (a) and (b) we have

$$\begin{aligned} r\,(r\,R')' + (\gamma + \lambda r^2)\, R &= 0 \\ \gamma_n &= -n^2 \end{aligned} \tag{45}$$

Recall the boundary conditions

$$\begin{aligned} u(a, \varphi, t) &= 0 \\ u(0, \varphi, t) &= \text{finite} \end{aligned} \tag{46}$$

These imply

$$\begin{aligned} R(a) &= 0 \\ R(0) &= \text{finite} \end{aligned} \tag{47}$$

The ODE can be rewritten

$$(r\,R')' - \frac{n^2}{r}\,R + \lambda\,r\,R = 0 \tag{48}$$

This in is S-L form with

$$\begin{array}{l} p(r) = r \\ q(r) = -n^2/r \\ w(r) = r \end{array} \tag{49}$$

From S-L theory we expect the orthogonality condition to be

$$\int_0^a r\,R_m(r)\,R_k(r)\,dr = \begin{array}{cc} 0 & m \neq k \\ \neq 0 & m = k \end{array} \tag{50}$$

Where m and k refer to some way of indexing the eigenfunctions.

(d)

Write the equation as

$$r^2\,R'' + r\,R' - n^2\,R + \lambda\,r^2\,R = 0 \tag{51}$$

Since the origin is a regular singular point, there will be at least one solution of the form

$$R = \sum_{k=0}^{\infty} a_k\,r^{k+\nu} \tag{52}$$

Pluggin this in and simplifying gives

$$\sum_{k=0}^{\infty} a_k\,((k+\nu)^2 - n^2)\,r^k + \lambda \sum_{k=0}^{\infty} a_k\,r^{k+2} = 0 \tag{53}$$

So the indicial equation is

$$\nu^2 = n^2 \tag{54}$$

With solutions

$$\nu = \pm n \tag{55}$$

For ν=n we get a solution of the form

$$R_1 = \sum_{k=0}^{\infty} a_k\,r^{k+n} \tag{56}$$

Since n≥0 these solutions will be finite at r=0. According to Fuch's theorem, when n>0 a second linearly independent solution will be of one of the following forms

$$\begin{array}{l} R_2 = \sum_{k=0}^{\infty} b_k\,r^{k-n} \\ \text{or} \\ R_2 = R_1 \ln|r| + \sum_{k=0}^{\infty} b_k\,r^{k-n} \end{array} \tag{57}$$

Regardless of which form the solution takes, it will be singular at r=0. For n=0 the second linearly independent solution will be of the form

$$R_2 = R_1 \ln|r| + \sum_{k=0}^{\infty} b_k r^k \qquad (58$$

Which will also be singular at r=0 since

$$\lim_{r\to 0} R_2 = a_0 \lim_{r\to 0} \ln|r| + b_0 = \infty \qquad (59)$$

Hence, for any n there will be only one solution satisfying the condition of finiteness at r=0 and it will be of the form

$$R = \sum_{k=0}^{\infty} a_k r^{k+n} \qquad (60)$$

This will be some multiple of the well known Bessel function

$$J_n\left(\sqrt{\lambda}\, r\right) \qquad (61)$$

Hence we define the eigenfunction

$$R_{nm} = J_n\left(\sqrt{\lambda_{nm}}\, r\right) \qquad (62)$$

where m is some label for the eigenvalues which we have not yet determined. The other boundary condition we have is that R(a)=0. Requiring this gives

$$J_n\left(\sqrt{\lambda_{nm}}\, a\right) = 0 \qquad (63)$$

It turns out that these Bessel functions have infinitely many roots. This becomes apparent if for example we consider the following plot of J_0

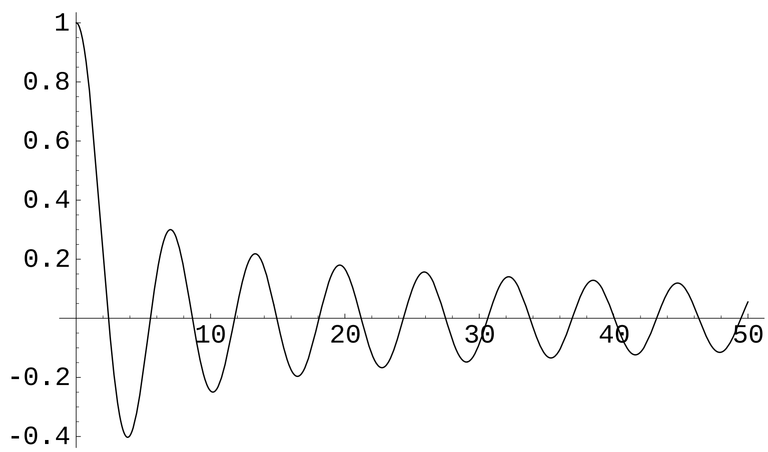

Figure 1

Call the mth root of the nth Bessel function α_{nm}.

$$J_n(\alpha_{nm}) = 0 \qquad (64)$$

So we have the eigenvalues

$$\lambda_{nm} = (\alpha_{nm} / a)^2 \qquad (65)$$

(e)

Recall from parts (a) and (d)

$$\begin{aligned} &T'' + \lambda c^2 T = 0 \\ &\lambda_{nm} = (\alpha_{nm} / a)^2 \end{aligned} \qquad (66)$$

Since the λ's are non-negative, the solutions will be of the form

$$T_{nm} = C_{nm} \operatorname{Sin}(\alpha_{nm}\, c\, t/a) + D_{nm} \operatorname{Cos}(\alpha_{nm}\, c\, t/a) \tag{67}$$

or if one of these Bessel function roots is zero, then the corresponding solution will be of the form

$$T_{nm} = D_{nm} + C_{nm}\, t \tag{68}$$

Recall the initial condition

$$u_t(r, \varphi, 0) = 0 \tag{69}$$

This implies

$$T'(0) = 0 \tag{70}$$

which tells us that C_{nm}=0.

$$T_{nm} = D_{nm} \operatorname{Cos}(\alpha_{nm}\, c\, t/a) \tag{71}$$

writing it this way includes the possibility that α_{nm}=0. From the previous parts we conclude

$$u = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} D_{nm} (A_n \operatorname{Sin}(n\varphi) + B_n \operatorname{Cos}(n\varphi)) J_n\left(\sqrt{\lambda_{nm}}\, r\right) \operatorname{Cos}(\alpha_{nm}\, c\, t/a) \tag{72}$$

by redefining the coefficients we get the desired form

$$u = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} A_{nj} (\operatorname{Sin}(n\varphi) + B_n \operatorname{Cos}(n\varphi)) J_n\left(\sqrt{\lambda_{nj}}\, r\right) \operatorname{Cos}(\alpha_{nj}\, c\, t/a) \tag{73}$$

as suggested in the problem we write this instead as

$$u = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} A_{nj} \operatorname{Sin}(n\varphi) J_n\left(\sqrt{\lambda_{nj}}\, r\right) \operatorname{Cos}(\alpha_{nj}\, c\, t/a) + \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} C_{nj} \operatorname{Cos}(n\varphi) J_n\left(\sqrt{\lambda_{nj}}\, r\right) \operatorname{Cos}(\alpha_{nj}\, c\, t/a) \tag{74}$$

To find these coefficients, recall the only condition we have yet to satisfy

$$u(r, \varphi, 0) = f(r, \varphi) \tag{75}$$

This is

$$f(r, \varphi) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} A_{nj} \operatorname{Sin}(n\varphi) J_n\left(\sqrt{\lambda_{nj}}\, r\right) + \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} C_{nj} \operatorname{Cos}(n\varphi) J_n\left(\sqrt{\lambda_{nj}}\, r\right) \tag{76}$$

Recall also the orthogonality conditions

$$\begin{gathered}
\int_0^a r\, J_m\left(\sqrt{\lambda_{mj}}\, r\right) J_k\left(\sqrt{\lambda_{ki}}\, r\right) dr = \begin{matrix} 0 & m \neq k \text{ or } j \neq i \\ \neq 0 & m = k \text{ and } j = i \end{matrix} \\
\int_0^{2\pi} \operatorname{Sin}(n\varphi) \operatorname{Cos}(n\varphi)\, d\varphi = 0 \\
\int_0^{2\pi} \operatorname{Sin}(n\varphi) \operatorname{Sin}(m\varphi)\, d\varphi = \begin{matrix} 0 & n \neq m \\ \pi & 0 < n = m \end{matrix} \\
\int_0^{2\pi} \operatorname{Cos}(n\varphi) \operatorname{Cos}(m\varphi)\, d\varphi = \begin{matrix} 0 & 0 < n \neq m \\ \pi & 0 < n = m \\ 2\pi & n = m = 0 \end{matrix}
\end{gathered} \tag{77}$$

Multiply the expression for f by Sin(k φ) and integrate over (0,2π) to get

$$\int_0^{2\pi} f(r,\varphi)\,\mathrm{Sin}(k\varphi)\,d\varphi = \pi \sum_{j=1}^{\infty} A_{kj}\, J_k\left(\sqrt{\lambda_{kj}}\ r\right) \tag{78}$$

Now multiply this by r $J_k(\sqrt{\lambda_{ki}}\,r)$ and integrate over (0,a)

$$A_{ki} = \frac{\int_0^a \int_0^{2\pi} f(r,\varphi)\,\mathrm{Sin}(k\varphi)\,d\varphi\, r\, J_k\left(\sqrt{\lambda_{ki}}\ r\right) dr}{\pi \int_0^a r\, J_k{}^2\left(\sqrt{\lambda_{ki}}\ r\right) dr} \tag{79}$$

Now, for k≠0 multiply the expression for f by Cos(k φ) and integrate over (0,2π)

$$\int_0^{2\pi} f(r,\varphi)\,\mathrm{Cos}(k\varphi)\,d\varphi = \pi \sum_{j=1}^{\infty} C_{kj}\, J_k\left(\sqrt{\lambda_{kj}}\ r\right) \tag{80}$$

then multiply this by r $J_k(\sqrt{\lambda_{ki}}\ r)$ and integrate over (0,a)

$$C_{ki} = \frac{\int_0^a \int_0^{2\pi} f(r,\varphi)\,\mathrm{Cos}(k\varphi)\,d\varphi\, r\, J_k\left(\sqrt{\lambda_{ki}}\ r\right) dr}{\pi \int_0^a r\, J_k{}^2\left(\sqrt{\lambda_{ki}}\ r\right) dr} \quad \text{for } k > 0 \tag{81}$$

finally, for k=0, multiply both sides by Cos(k φ)=1 and integrate over (0,2π)

$$\int_0^{2\pi} f(r,\varphi)\,d\varphi = 2\pi \sum_{j=1}^{\infty} C_{0\,j}\, J_0\left(\sqrt{\lambda_{0\,j}}\ r\right) \tag{82}$$

Now multiply by r $J_0(\sqrt{\lambda_{0\,i}}\,r)$ and integrate over (0,a)

$$C_{0\,i} = \frac{\int_0^a \int_0^{2\pi} f(r,\varphi)\,d\varphi\, r\, J_0\left(\sqrt{\lambda_{0\,i}}\ r\right) dr}{2\pi \int_0^a r\, J_0{}^2\left(\sqrt{\lambda_{0\,i}}\ r\right) dr} \tag{83}$$

Problem 4 (5×5 points)

$$\begin{aligned} &\mu\, y_{tt} = T\, y_{xx} + f(x,t) \\ &y(0) = y(L) = 0 \end{aligned} \tag{84}$$

(a)

From problem set 5 problem 4c we know that the spatial eigenfunctions are

$$\begin{aligned} &\lambda_n = -\left(\frac{n\pi}{L}\right)^2 \\ &y_n = A_n\, \mathrm{Sin}\left(\frac{n\pi x}{L}\right) \end{aligned} \tag{85}$$

with orthogonality condition

$$\frac{2}{L}\int_0^L \mathrm{Sin}\left(\frac{n\pi x}{L}\right)\mathrm{Sin}\left(\frac{m\pi x}{L}\right) dx = \begin{matrix} 0 & n \neq m \\ 1 & n = m \end{matrix} \tag{86}$$

Since the eigenfunctions are complete, we look for a solution to the PDE of the form

$$y(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \mathrm{Sin}\left(\frac{n\pi x}{L}\right) \tag{87}$$

and also expand f(x,t) as

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\, \mathrm{Sin}\left(\frac{n\pi x}{L}\right) \tag{88}$$

Using orthogonality conditions and multiplying these expressions by Sin(m π x/L) and integrating over (0,L) gives

$$a_m(t) = \frac{2}{L}\int_0^L \text{Sin}\left(\frac{m\pi x}{L}\right) y(x,t)\, dx$$
$$f_m(t) = \frac{2}{L}\int_0^L \text{Sin}\left(\frac{m\pi x}{L}\right) f(x,t)\, dx \qquad (89)$$

Multiply the PDE by Sin(mπx/L) and integrate over (0,L)

$$\mu\left(\int_0^L y\,\text{Sin}\left(\frac{m\pi x}{L}\right) dx\right)_{tt} = T\int_0^L y_{xx}\,\text{Sin}\left(\frac{m\pi x}{L}\right) dx + \int_0^L f(x,t)\,\text{Sin}\left(\frac{m\pi x}{L}\right) dx \qquad (90)$$

From the definition of the coefficients we may write

$$\mu\, a_m''(t) = T\,\frac{2}{L}\int_0^L y_{xx}\,\text{Sin}\left(\frac{m\pi x}{L}\right) dx + f_m(t) \qquad (91)$$

The remaining integral can be evaluated by integrating by parts twice

$$\int_0^L y_{xx}\,\text{Sin}\left(\frac{m\pi x}{L}\right) dx =$$
$$\left(y_x\,\text{Sin}\left(\frac{m\pi x}{L}\right) - y\,\frac{m\pi}{L}\,\text{Cos}\left(\frac{m\pi x}{L}\right)\right)_{x=L} - \left(y_x\,\text{Sin}\left(\frac{m\pi x}{L}\right) - y\,\frac{m\pi}{L}\,\text{Cos}\left(\frac{m\pi x}{L}\right)\right)_{x=0} - \qquad (92)$$
$$\left(\frac{m\pi}{L}\right)^2\int_0^L y\,\text{Sin}\left(\frac{m\pi x}{L}\right) dx = -y(L,t)\,\frac{m\pi}{L}\,\text{Cos}(m\pi) + y(0,t)\,\frac{m\pi}{L} - \left(\frac{m\pi}{L}\right)^2\frac{L}{2}\,a_m(t)$$

This last expression simplifies since the boundary conditions are homogeneous.

$$\int_0^L y_{xx}\,\text{Sin}\left(\frac{m\pi x}{L}\right) dx = -\left(\frac{m\pi}{L}\right)^2\frac{L}{2}\,a_m(t) \qquad (93)$$

Inserting this into the equation gives

$$\mu\, a_m''(t) + T\left(\frac{m\pi}{L}\right)^2 a_m(t) = f_m(t) \qquad (94)$$

With the initial conditions given by

$$a_m(0) = \frac{2}{L}\int_0^L \text{Sin}\left(\frac{m\pi x}{L}\right) y(x,0)\, dx$$
$$a_m'(0) = \frac{2}{L}\int_0^L \text{Sin}\left(\frac{m\pi x}{L}\right) y_t(x,0)\, dx \qquad (95)$$

Case P:

As described in the solution set, and using the answer from problem set 5 problem 4a we have

$$y(x,0) = Y(x) = \begin{cases} \frac{F_0}{T L}(L-\xi)\,x & 0 \le x < \xi \\ \frac{F_0}{T L}(L-x)\,\xi & \xi < x \le L \end{cases}$$
$$y_t(x,0) = 0$$
$$f(x,t) = 0 \qquad (96)$$

Inserting these into the results above we find

$$\mu\, a_m''(t) + T\left(\frac{m\pi}{L}\right)^2 a_m(t) = 0 \qquad (97)$$

Which has solutions of the form

$$a_m = A_m \, \text{Sin}\left(\frac{m\pi}{L}\sqrt{\frac{T}{\mu}}\, x\right) + B_m \, \text{Cos}\left(\frac{m\pi}{L}\sqrt{\frac{T}{\mu}}\, x\right) \quad (98$$

With the initial conditions given by

$$a_m(0) = \frac{2}{L}\int_0^L \text{Sin}\left(\frac{m\pi x}{L}\right) Y(x)\, dx =$$
$$\frac{2}{L}\,\frac{F_0}{T L}(L-\xi)\int_0^{\xi}\text{Sin}\left(\frac{m\pi x}{L}\right) x\, dx + \frac{2}{L}\,\frac{F_0}{T L}\,\xi\int_{\xi}^{L}\text{Sin}\left(\frac{m\pi x}{L}\right)(L-x)\, dx = \quad (99)$$
$$\frac{2 L F_0}{m^2 \pi^2 T}\,\text{Sin}\left(\frac{m\pi\xi}{L}\right)$$

$$a_m'(0) = 0 \quad (100)$$

Applying these initial conditions gives the arbitrary coefficients A_m and B_m

$$a_m(t) = \frac{2 L F_0}{m^2 \pi^2 T}\,\text{Sin}\left(\frac{m\pi\xi}{L}\right)\text{Cos}\left(\frac{m\pi}{L}\sqrt{\frac{T}{\mu}}\, t\right) \quad (101)$$

So the solution to the PDE is

$$y(x, t) = \frac{2 L F_0}{\pi^2 T}\sum_{n=1}^{\infty}\frac{1}{n^2}\,\text{Sin}\left(\frac{n\pi\xi}{L}\right)\text{Cos}\left(\frac{n\pi}{L}\sqrt{\frac{T}{\mu}}\, t\right)\text{Sin}\left(\frac{n\pi x}{L}\right) \quad (102)$$

Case H:

$$y(x, 0) = 0$$
$$y_t(x, 0) = V(x) = (P_h/\mu)\,\delta(x-\xi) \quad (103)$$
$$f(x, t) = 0$$

Inserting these into the results above we find

$$\mu\, a_m''(t) + T\left(\frac{m\pi}{L}\right)^2 a_m(t) = 0 \quad (104)$$

Which has solutions of the form

$$a_m = A_m \, \text{Sin}\left(\frac{m\pi}{L}\sqrt{\frac{T}{\mu}}\, x\right) + B_m \, \text{Cos}\left(\frac{m\pi}{L}\sqrt{\frac{T}{\mu}}\, x\right) \quad (105)$$

With the initial conditions given by

$$a_m(0) = 0$$
$$a_m'(0) = \frac{2}{L}\int_0^L \text{Sin}\left(\frac{m\pi x}{L}\right) y_t(x, 0)\, dx = \frac{2 P_h}{L\mu}\,\text{Sin}\left(\frac{m\pi\xi}{L}\right) \quad (106)$$

Applying these initial conditions gives the arbitrary coefficients A_m and B_m

$$a_m(t) = \frac{2 P_h}{\mu m \pi}\sqrt{\frac{\mu}{T}}\,\text{Sin}\left(\frac{m\pi\xi}{L}\right)\text{Sin}\left(\frac{m\pi}{L}\sqrt{\frac{T}{\mu}}\, t\right) \quad (107)$$

So the solution to the PDE is

$$y(x, t) = \frac{2 P_h}{\mu\pi}\sqrt{\frac{\mu}{T}}\sum_{n=1}^{\infty}\frac{1}{n}\,\text{Sin}\left(\frac{n\pi\xi}{L}\right)\text{Sin}\left(\frac{n\pi}{L}\sqrt{\frac{T}{\mu}}\, t\right)\text{Sin}\left(\frac{n\pi x}{L}\right) \quad (108)$$

(b)

With L=1, ξ=1/7 and F_0=T, the sollution we found in part (a) for case P is

$$y(x, t) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\text{Sin}(n\pi/7)}{n^2} \text{Cos}\left(n\pi \sqrt{\frac{T}{\mu}}\, t\right) \text{Sin}(n\pi x) \tag{109}$$

If we define the frequency associated with n=1 to be the fundamental

$$f_1 = \frac{1}{2}\sqrt{\frac{T}{\mu}} = \frac{c}{2} \tag{110}$$

then the ratios of frequencies of higher normal modes are

Frequency	Ratio	Musical Interpretation
f_2	$f_2 / f_1 = 2$	one octave
f_3	$f_3 / f_2 = 1.5$	*a* major fifth above f_2
f_4	$f_4 / f_3 = 1.33$	*a* major fourth above f_3
f_5	$f_5 / f_4 = 1.25$	*a* major third above f_4
f_6	$f_6 / f_5 = 1.2$	*a* minor third above f_5
f_7	$f_7 / f_6 = 1.17$	Less than the critical bandwidth apart from f_6 so sounds unpleasantly dissonant – *a* jangly sound
f_8	$f_8 / f_6 = 1.33$	*a* major fourth above f_6
f_9	$f_9 / f_8 = 1.13$	Less than the critical bandwidth apart from f_8 so sounds unpleasant but amplitude is getting low so not as objectionable as f_7 / f_6

Thus we can minimize the unpleasant "jangly sound" by chosing to strike the string in such a way as to set the amplitude of f_7 (the first self-dissonant harmonic) equal to zero (cf solution to PS5 #4d):i.e.strike the string 1/7 of the way along its length (cf.solution to PS5 #4d optional part). This leaves f_9 nonzero, but its amplitude is low and it doesn't sound nearly so "jangly" as say a string struck at its midpoint (with large f_7 contribution).

(c)

Setting

$$T/\mu = c^2 = 1 \tag{111}$$

We find

$$y(x, t) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\text{Sin}(n\pi/7)}{n^2} \text{Cos}(n\pi t)\, \text{Sin}(n\pi x) \tag{112}$$

Plotting 50 terms of this for various times we see a wave which changes amplitude as it travels rightward from t=0 until it is reflected at t=1. So the period is P=2*1=2.

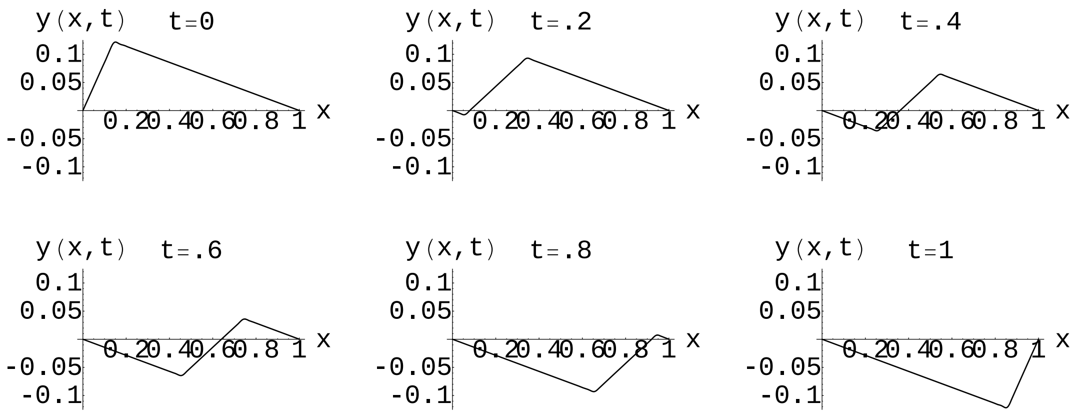

Figure 2

The wave has period 2 (2L/c in general), and its envelope is a tilted rectangle (see figure 3). The wave reflects from the boundaries with opposite amplitude to the ingoing wave. But this envelope probably doesn't look much like the envelope you sketched of the rubber band. High speed photographs of plucked strings show that real strings do look like your solution -for a little while. But real strings are more complicated than the simple mathematical one we solved here. (there is actually quite a large literature on mathematical models of more realistic strings, sponsored in part by the makers of keyboard synthesisers).

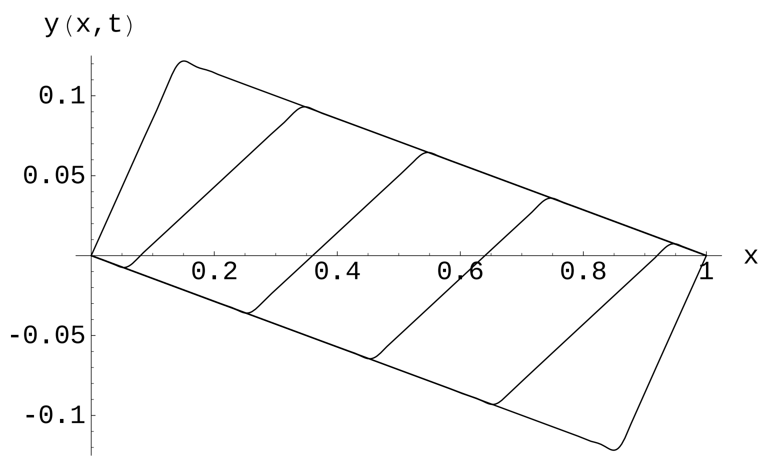

Figure 3

When a long string is plucked with small amplitude, it looks briefly somewhat like Figure 2. But in less than a second it looks more symmetric than the tilted rectangular shape. If struck with large amplitude, the string seems to never look like Figure 2. There are two explanations for these observations. First, at finite amplitude, there is dispersion (and corresponding coupling to longitudinal waves along the string), so the phases of oscillation are quickly randomized. In Figure 4 we plot the same time series as in Figure 2 with random phases on $[0,2\pi]$ added to each $\mathrm{Cos}(n\pi t)$ term.

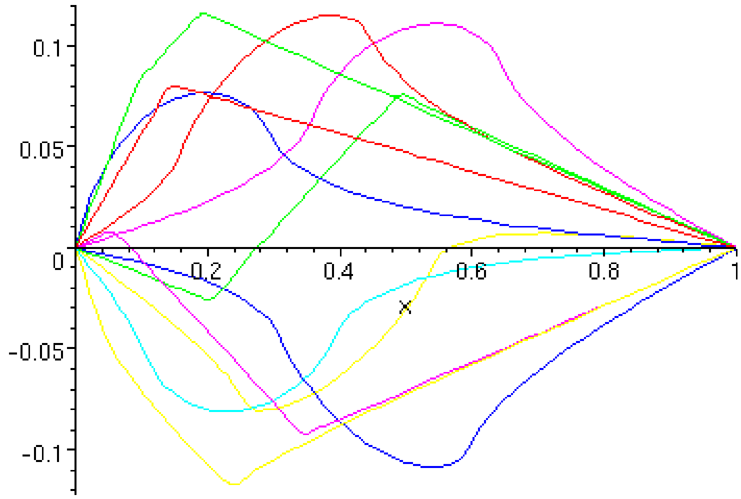

Figure 4

Second, there is damping (both in the band and at ends- fingers are flexible), and the higher modes damp much faster than the fundamental. So at the end, all one sees is the Sin(πx) term oscillating.

(d)

With L=1, P_h/μ=1, T/μ=c^2=1, and ξ=1/7 we have from the solution in part (a)

$$y(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Sin}(n\pi/7)\,\mathrm{Sin}(n\pi t)\,\mathrm{Sin}(n\pi x) \tag{113}$$

The wave has period 2 (2L/c in general), and again the 7th harmonic has zero amplitude if the string is struck 1/7 of the way along its length. So the 1/7 position is optimal for avoiding self-dissonance for both plucked and struck (by infinitely light hammers) strings.

(e)

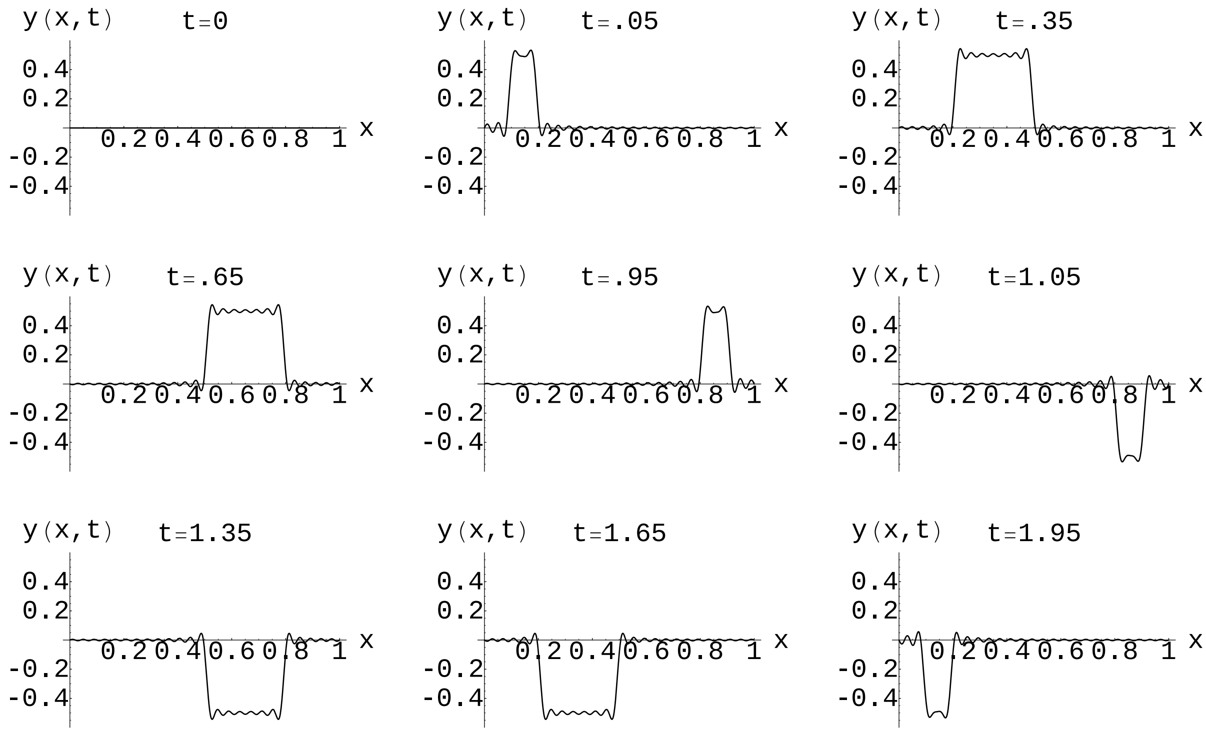

Figure 5

The plot shows that the struck rubber band starts as a delta-function pulse which becomes a spreading square-wave pulse. This spreads until its width is 2/7, and the left edge of the pulse hits the left boundary.

Then the pulse starts moving as a unit to the right, until it hits the right boundary, when it squashes and reflects as a negative amplitude pulse, and the cycle repeats. Notice that the harmonics decay only as 1/n instead of $1/n^2$ for the plucked case, because the wave pulse is discontinuous. Eyes are not fast enough to see any of this on a rubber band.

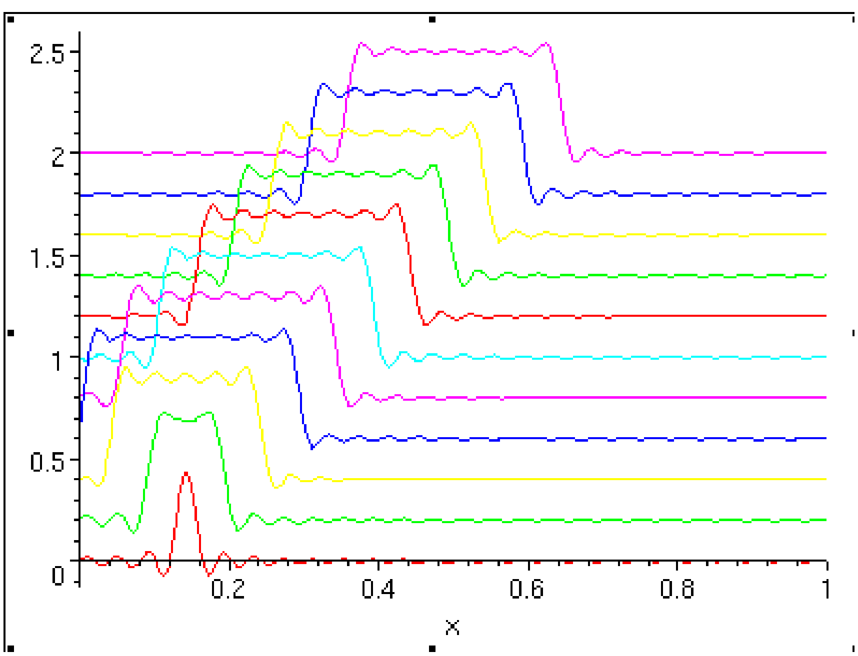

Figure 6

(f) (1 point extra credit)

The time it takes the pulse of Figure 2 to reflect off the left wall and pass the hammer position at 1/7 (leaving 0 amplitude at hammer position) is

$$\frac{2}{7}\frac{L}{c} = \frac{1}{7}\frac{2L}{c} = \frac{1}{7}\frac{1}{f_1} = \frac{1}{7}\frac{1}{261.6\,\text{Hz}} = 0.55\,\text{ms} \quad (114)$$

Since the real hammer stays in contact for 2ms, it is clearly a very BAD approximation to treat the strike as an impulse in time as we did here: the hammer really changes the length of the string and the left boundary condition to that of a driven string for 1/2 of a wave cycle. So the string is not at y(x,0)=0 and dy/dt=δ(x-ξ) when the hammer releases, as we assumed.

Problem 5 (3×4 points)

$$\begin{aligned} &u_{xx} + u_{yy} = 0 \\ &u(x, 1) = u(0, y) = u(1, y) = 0 \\ &u(x, 0) = f(x) \end{aligned} \quad (115)$$

$$u(x, y) = \sum_{n=1}^{\infty} c_n(x)\,\text{Sin}(n\pi y) \quad (116)$$

(a)

We require

$$u(1, y) = 0 \quad (117)$$

That is

$$0 = \sum_{n=1}^{\infty} c_n(1)\,\text{Sin}(n\pi y) \quad (118)$$

Since the eigenfunction are complete and orthogonal, we have

$$c_n(1) = 2\int_0^1 \text{Sin}(n\pi y)\,0\,dy = 0 \quad (119)$$

Similarly, $c_n(0)=0$

(b)

Suppose that substituting our guess into the PDE is valid, we get

$$\sum_{n=1}^{\infty} c_n''(x)\,\text{Sin}(n\pi y) - \sum_{n=1}^{\infty} c_n(x)(n\pi)^2\,\text{Sin}(n\pi y) = 0 \quad (120)$$

Since the eigenfunctions are orthogonal we find

$$c_n'' - (n\pi)^2 c_n = 0 \quad (121)$$

(c)

The solutions to the ODE in part (b) are of the form

$$c_n = A_n e^{n\pi x} + B_n e^{-n\pi x} \quad (122)$$

Fitting the initial data gives

$$\begin{aligned} &A_n + B_n = 0 \\ &A_n e^{n\pi} + B_n e^{-n\pi} = 0 \end{aligned} \quad (123)$$

The determinant of this matrix is

$$e^{-n\pi} - e^{n\pi} \quad (124)$$

Which is non-zero, so the matrix is invertible and hence the only solution is $A_n = B_n = 0$. So the only solution is the trivial solution

$$c_n = 0 \tag{125}$$

which gives the non-physical solution

$$u(x, y) = 0 \tag{126}$$

(d) (optional 10 points extra credit)

Multiplying the PDE by Sin (n π y) and integrating gives

$$\left(\int_0^1 u(x, y) \operatorname{Sin}(n \pi y)\, dy\right)_{xx} + \int_0^1 u_{yy}(x, y) \operatorname{Sin}(n \pi y)\, dy = 0 \tag{127}$$

Recall that we had

$$u(x, y) = \sum_{n=1}^{\infty} c_n(x) \operatorname{Sin}(n \pi y) \tag{128}$$

By orthogonality the coefficients are

$$c_n(x) = 2 \int_0^1 u(x, y) \operatorname{Sin}(n \pi y)\, dy \tag{129}$$

So we have

$$\frac{1}{2} c_n''(x) + \int_0^1 u_{yy}(x, y) \operatorname{Sin}(n \pi y)\, dy = 0 \tag{130}$$

Integrating by parts twice gives

$$\begin{aligned}\int_0^1 u_{yy}(x, y) \operatorname{Sin}(n \pi y)\, dy &= \left(u_y \operatorname{Sin}(n \pi y) - u\, n \pi \operatorname{Cos}(n \pi y)\right)_{y=1} - \\ &\quad \left(u_y \operatorname{Sin}(n \pi y) - u\, n \pi \operatorname{Cos}(n \pi y)\right)_{y=0} - (n \pi)^2 \int_0^1 u(x, y) \operatorname{Sin}(n \pi y)\, dy\end{aligned} \tag{131}$$

Using the boundary data and definition of the coefficients gives

$$\int_0^1 u_{yy}(x, y) \operatorname{Sin}(n \pi y)\, dy = f(x)\, n \pi - (n \pi)^2 \frac{1}{2} c_n(x) \tag{132}$$

Inserting this into our equation

$$c_n''(x) - (n \pi)^2 c_n(x) = -2\, n \pi f(x) \tag{133}$$

Here we see the essential difference between the "pluggin in" approach in part (c) and the integrating approach. The fact that the eigenfunctions are all 0 at y=1 means that the sum will not be uniformly convergent near y=1 and hence term by term differentiation of the sum isn't permissable. The correct approach used here gives a non-trivial forcing term.
Through either the use of Green's functions or variations of parameters this ODE can be solved to find the result stated in the problem set.