ACM95/100b
Problem Set 5 Solutions
02/16/04

Problem 1 (10 points)

The Gram-Schmidt procedure is as follows. Given a space, X, defined as the span of a given basis

$$X = \text{span}\{v_1, v_2, \ldots\} \tag{1}$$

and given a real inner product on this space

$$(v_i, v_j) \tag{2}$$

an orthonormal basis for X can be formed as follows. First set

$$b_1 = \frac{v_1}{\|v_1\|} \tag{3}$$

Then define

$$b_n = A\,(v_n - (v_n, b_1)\,b_1 - \ldots - (v_n, b_{n-1})\,b_{n-1}) \tag{4}$$

Finally, choose A so that b_n has unit norm:

$$1 = \|b_n\|^2 = A^2\,(\|v_n\|^2 - (v_n, b_1)^2 - \ldots - (v_n, b_{n-1})^2) \tag{5}$$

We then iterate this process.
Let's apply this procedure to the given set of functions S

$$S = \{1, x, x^2, x^3, \ldots\} \tag{6}$$

with the definition of a real inner product on these functions

$$\begin{gathered}(f, g)_w = \int_{-1}^{1} f(x)\,g(x)\,w(x)\,dx \\ w(x) = 1\Big/\sqrt{1-x^2}\end{gathered} \tag{7}$$

By the definition of the inner product, the norm of a function in this space is

$$\|f\|_w = \sqrt{(f, f)_w} = \sqrt{\int_{-1}^{1} \frac{(f(x))^2}{\sqrt{1-x^2}}\,dx} \tag{8}$$

The first orthonormal polynomial will be

$$T_0(x) = \frac{1}{\|1\|_w} = \frac{1}{\sqrt{\pi}} \tag{9}$$

The next orthonormal basis element will be of the form

$$T_1 = A\,(x - (T_0, x)_w\,T_0) \tag{10}$$

Simplifying this gives

$$T_1 = A\,x \tag{11}$$

We now choose A to give T_1 unit norm

$$1 = \|A\,x\|_w = |A|\sqrt{\frac{\pi}{2}} \tag{12}$$

The sign of A is arbitrary, we choose it positive and set

$$T_1(x) = \sqrt{\frac{2}{\pi}}\, x \tag{13}$$

The third orthonormal basis polynomial will be of the form

$$T_2 = A\,(x^2 - (T_0, x^2)_w\, T_0 - (T_1, x^2)_w\, T_1) \tag{14}$$

Inserting the expressions for the other polynomials and simplifying gives

$$T_2 = A\left(x^2 - \frac{1}{2}\right) \tag{15}$$

We then choose A to give this function unit norm

$$1 = \left\|A\left(x^2 - \frac{1}{2}\right)\right\|_w = \frac{|A|}{2}\sqrt{\frac{\pi}{2}} \tag{16}$$

The sign of A is arbitrary. We conclude

$$\begin{aligned} T_0(x) &= \sqrt{\frac{1}{\pi}} \\ T_1(x) &= \sqrt{\frac{2}{\pi}}\, x \\ T_2(x) &= \sqrt{\frac{2}{\pi}}\,(2x^2 - 1) \end{aligned} \tag{17}$$

Problem 2 (20 points)

A second order ODE is in Sturm-Liouville form if it is written as

$$\begin{aligned} &(p(x)\, y')' + q(x)\, y + \lambda\, r(x)\, y = 0 \\ &a \le x \le b \end{aligned} \tag{18}$$

We'll have a regular Sturm-Liouville eigenvalue problem if the boundary conditons are of the form

$$\begin{aligned} &\beta_1\, y(a) + \beta_2\, y'(a) = 0 \\ &\beta_3\, y(b) + \beta_4\, y'(b) = 0 \end{aligned} \tag{19}$$

We'll have a singular Sturm-Liouville eigenvalue problem if p(x) vanishes at an endpoint and we have conditions of the form:

$$\begin{aligned} &p(a) = 0 \\ &|y(a)| < \infty \\ &\text{and / or} \\ &p(b) = 0 \\ &|y(b)| < \infty \end{aligned} \tag{20}$$

We'll have a periodic Sturm-Liouville eigenvalue problem if we're given boundary conditions of the form

$$\begin{aligned} &y(a) = y(b) \\ &y'(a) = y'(b) \end{aligned} \tag{21}$$

A general second order ODE of the form

$$f\, y'' + g\, y' + h\, y = 0 \tag{22}$$

can be put into Sturm-Liouville form by the use of an integration factor

$$\left(e^{\int \frac{g}{f} dx} y'\right)' + \frac{h}{f} e^{\int \frac{g}{f} dx} y = 0 \tag{23}$$

(a)

$$\begin{aligned} & x^2 y'' + x y' + (k^2 x^2 - n^2) y = 0 \\ & 0 \le x \le c \\ & y(c) = 0 \\ & |y(0)| < \infty \end{aligned} \tag{24}$$

We rewrite the ODE in Sturm-Liouville form using an integration factor

$$\begin{aligned} & (x y')' - \frac{n^2}{x} y + k^2 x y = 0 \\ & p(x) = x \\ & q(x) = -n^2 / x \\ & w(x) = x \end{aligned} \tag{25}$$

Because p(x) vanishes at one of the endpoints, p(0)=0, and because of the boundary condition at x=0, this is a singular Sturm-Liouville eigenvalue problem.

(b)

$$\begin{aligned} & y'' + k^2 y = 0 \\ & 0 \le x \le 2\pi \\ & y(0) = y(2\pi) \\ & y'(0) = y'(2\pi) \end{aligned} \tag{26}$$

This ODE is already in Sturm-Liouville form with

$$\begin{aligned} & p(x) = 1 \\ & q(x) = 0 \\ & w(x) = 1 \end{aligned} \tag{27}$$

The boundary conditions make this a periodic Sturm-Liouville eigenvalue problem. If k=0 then setting y(x)=c for any constant c gives a solution. So k=0 is an eigenvalue with eigenfunction y(x)=1

(c)

$$\begin{aligned} & y'' + k^2 y = 0 \\ & 0 \le x \le 1 \\ & y(0) = 0 \\ & y(1) = 1 \end{aligned} \tag{28}$$

This is the same ODE as in part (b). The boundary condition y(1)=1 makes this not a Sturm-Liouville eigenvalue problem. This can be seen as follows. Recall from lecture that with the S-L operator

$$\mathbf{Ly \equiv (p\, y')' + q\, y}$$

the follwing equality can be produced using integration by parts

$$(L f, g) - (f, L g) = (\overline{g}\, p\, f' - \overline{g}'\, p\, f)_{x=b} - (\overline{g}\, p\, f' - \overline{g}'\, p\, f)_{x=a} \tag{29}$$

If the boundary conditions are prescibed such that the right side of this vanishes, then th operator L is said to be self-adjoint and we have a S-L eigenvalue problem. For the present problem the boundary conditions give

$$(L f, g) - (f, L g) = (\overline{g}\, p\, f' - \overline{g}'\, p\, f)_{x=1} - (\overline{g}\, p\, f' - \overline{g}'\, p\, f)_{x=0} = f'(1) - \overline{g'(1)} \tag{30}$$

Since the right side doesn't vanish, this problem is not a S-L eigenvalue problem.

(d)

$$(1-x^2)\,y'' - 2\,x\,y' + \lambda\,y = 0$$
$$-1 \le x \le 1 \tag{31}$$
$$y(-1),\ y(1),\ y'(-1),\ y'(1)\ \text{finite}$$

No integrating factor is needed to put this into Sturm-Liouville form, we simply collect terms

$$((1-x^2)\,y')' + \lambda\,y = 0$$
$$p(x) = 1 - x^2$$
$$q(x) = 0 \tag{32}$$
$$w(x) = 1$$

p(x)=0 at both boundary points x=±1. This fact, together with the boundary conditions, makes this a singular Sturm-Liouville eigenvalue problem.

(e)

$$(1-x^2)\,y'' - x\,y' + \lambda\,y = 0$$
$$-1 \le x \le 1 \tag{33}$$
$$y(-1),\ y(1),\ y'(-1),\ y'(1)\ \text{finite}$$

Use of an integration factor gives

$$\left(\sqrt{1-x^2}\ y'\right)' + \frac{\lambda}{\sqrt{1-x^2}}\ y = 0$$
$$p(x) = \sqrt{1-x^2}$$
$$q(x) = 0 \tag{34}$$
$$w(x) = 1\Big/\sqrt{1-x^2}$$

p(x)=0 at both boundary points x=±1. This fact, together with the boundary conditions, makes this a singular Sturm-Liouville eigenvalue problem.

Problem 3 (15 points)

(a)

If f is an element of the space spanned by an orthonormal set $\{v_i\}$ then there are unique coefficients a_i such that

$$f = \sum_{n=0}^{\infty} a_n\,v_n \tag{35}$$

Since the basis is orthonormal we have

$$(f,\ v_i) = \left(\sum_{n=0}^{\infty} a_n\,v_n,\ v_i\right) = \sum_{n=0}^{\infty} a_n\,(v_n,\ v_i) = a_i \tag{36}$$

So f can be written

$$f = \sum_{n=0}^{\infty} (f,\ v_n)\,v_n \tag{37}$$

(i)

As given in class, the normalized Legendre polynomials are

$$\{l_0,\ l_1,\ l_2,\ \ldots\} = \left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\ x,\ \sqrt{\frac{5}{2}}\ \frac{1}{2}\,(3\,x^2-1),\ \ldots\right\} \tag{38}$$

defined on $-1\le x\le 1$ with an inner product weighted by $w(x)=1$. If $f(x)$ is expandable in terms of these functions we write

$$f(x) = \sum_{n=0}^{\infty} (f, l_n)\, l_n(x) \tag{39}$$

For our given f we calculate

$$\begin{aligned}
(f, l_0) &= \int_{-1}^{1} (1 - x^2)\, \frac{1}{\sqrt{2}}\, dx = \frac{2\sqrt{2}}{3} \\
(f, l_1) &= \int_{-1}^{1} (1 - x^2) \sqrt{\frac{3}{2}}\, x\, dx = 0 \\
(f, l_2) &= \int_{-1}^{1} (1 - x^2) \sqrt{\frac{5}{2}}\, \frac{1}{2} (3x^2 - 1)\, dx = -\frac{2}{3} \sqrt{\frac{2}{5}}
\end{aligned} \tag{40}$$

So we write:

$$f(x) = \frac{2\sqrt{2}}{3}\, l_0 - \frac{2}{3} \sqrt{\frac{2}{5}}\, l_2 + \ldots \tag{41}$$

By plugging in the expressions for l_0 and l_2 we see that f is exactly equal to the sum of the first two non-trivial terms written above. So we may write

$$\boxed{f(x) = \frac{2\sqrt{2}}{3}\, l_0 - \frac{2}{3} \sqrt{\frac{2}{5}}\, l_2} \tag{42}$$

By orthogonality of the Legendre polynomials we see from this expression that

$$(f, l_i) = \begin{cases} 2\sqrt{2}/3 & \text{if } i = 0 \\ -2\sqrt{2/5}/3 & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

So orthogonality assures us that the remaining coefficients should be zero.

(ii)

As found in problem 1, the normalized Chebyshev polynomials are

$$\{T_0, T_1, T_2, \ldots\} = \left\{\sqrt{\frac{1}{\pi}},\ \sqrt{\frac{2}{\pi}}\, x,\ \sqrt{\frac{2}{\pi}}\, (2x^2 - 1),\ \ldots\right\} \tag{44}$$

defined on $-1\le x\le 1$ with an inner product weighted by $w(x)=1/\sqrt{1-x^2}$. If $f(x)$ is expandable in terms of these functions we write

$$f(x) = \sum_{n=0}^{\infty} (f, T_n)\, T_n(x) \tag{45}$$

For our given f we calculate

$$(f, T_0) = \int_{-1}^{1} (1 - x^2) \sqrt{\frac{1}{\pi}} \frac{1}{\sqrt{1 - x^2}} \, dx = \frac{\sqrt{\pi}}{2}$$

$$(f, T_1) = \int_{-1}^{1} (1 - x^2) \sqrt{\frac{2}{\pi}} \, x \frac{1}{\sqrt{1 - x^2}} \, dx = 0 \qquad (46$$

$$(f, T_2) = \int_{-1}^{1} (1 - x^2) \sqrt{\frac{2}{\pi}} \, (2 x^2 - 1) \frac{1}{\sqrt{1 - x^2}} \, dx = -\frac{1}{2} \sqrt{\frac{\pi}{2}}$$

So we write:

$$f(x) = \frac{\sqrt{\pi}}{2} T_0 - \frac{1}{2} \sqrt{\frac{\pi}{2}} T_2 + \ldots \qquad (47)$$

By plugging in the expressions for T_0 and T_2 we see that f is exactly equal to the sum of the first two non-trivial terms written above. So we may write

$$f(x) = \frac{\sqrt{\pi}}{2} T_0 - \frac{1}{2} \sqrt{\frac{\pi}{2}} T_2 \qquad (48)$$

By orthogonality of the Chebyshev polynomials we see from this expression that

$$(f, l_i) = \begin{cases} \sqrt{\pi} / 2 & \text{if } i = 0 \\ -\sqrt{\pi/2} / 2 & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases} \qquad (49)$$

So orthogonality assures us that the remaining coefficients should be zero.

(b)

If an element of a real inner product space f is expandable in a series of orthonormal basis elements, then the expression derived in part (a)

$$f = \sum_{n=0}^{\infty} (f, v_n) v_n \qquad (50)$$

can be used to find an expression for the norm of f.

$$\|f\|^2 = (f, f) = \left(\sum_{n=0}^{\infty} (f, v_n) v_n, \sum_{m=0}^{\infty} (f, v_m) v_m \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (f, v_m)(f, v_n)(v_n, v_m) = \sum_{n=0}^{\infty} (f, v_n)^2 \qquad (51)$$

written another way, if f is

$$f = \sum_{n=0}^{\infty} a_n v_n \qquad (52)$$

then

$$\|f\|^2 = \sum_{n=0}^{\infty} a_n^2 \qquad (53)$$

Specifying this to function spaces we have

$$\int_a^b (f(x))^2 w(x) \, dx = \sum_{n=0}^{\infty} \left(\int_a^b f(x) v_n(x) w(x) \, dx \right)^2 \qquad (54)$$

These results are known as Parseval's equality (here specialized for orthonormal sets of basis functions rather than more general orthogonal ones)

We now verify Parseval's relation for each of the expansions in part (a).

(i)

We found

$$f(x) = 1 - x^2 = \frac{2\sqrt{2}}{3}\, l_0 - \frac{2}{3}\sqrt{\frac{2}{5}}\, l_2 \tag{55}$$

The norm of f is

$$\|1 - x^2\|^2 = \int_{-1}^{1} (1 - x^2)^2\, dx = \frac{16}{15} \tag{56}$$

The sum of the squares of the coefficients is the same

$$\sum_{n=0}^{\infty} {a_n}^2 = \left(\frac{2\sqrt{2}}{3}\right)^2 + \left(-\frac{2}{3}\sqrt{\frac{2}{5}}\right)^2 = \frac{16}{15} \tag{57}$$

(ii)

We found

$$1 - x^2 = \frac{\sqrt{\pi}}{2}\, T_0 - \frac{1}{2}\sqrt{\frac{\pi}{2}}\, T_2 \tag{58}$$

The norm of f is

$$\|1 - x^2\|^2 = \int_{-1}^{1} (1 - x^2)^2 \frac{1}{\sqrt{1 - x^2}}\, dx = \frac{3\pi}{8} \tag{59}$$

The sum of the squares of the coefficients is the same

$$\sum_{n=0}^{\infty} {a_n}^2 = \left(\frac{\sqrt{\pi}}{2}\right)^2 + \left(-\frac{1}{2}\sqrt{\frac{\pi}{2}}\right)^2 = \frac{3\pi}{8} \tag{60}$$

Problem 4 (30 points)

$$\begin{aligned} &y'' + \frac{1}{T} f(x) = 0 \\ &y(0) = y(L) = 0 \end{aligned} \tag{61}$$

(a)

$$\begin{aligned} &G'' = -\frac{F_0}{T}\, \delta(x - \xi) \\ &G(0) = G(L) = 0 \end{aligned} \tag{62}$$

Since the forcing is zero for all x except $x=\xi$, the solution will be of the form

$$G(x \mid \xi) = \begin{array}{ll} a + b\,x & 0 \le x < \xi \\ c + d\,x & \xi < x \le L \end{array} \tag{63}$$

We choose the coefficients to satisfy the two boundary conditions

$$0 = G(0 \mid \xi) = a$$
$$0 = G(L \mid \xi) = c + d\,L \quad (64$$

This gives

$$G(x \mid \xi) = \begin{cases} b\,x & 0 \le x < \xi \\ d\,(x - L) & \xi < x \le L \end{cases} \quad (65)$$

The Green's function will be continuous with a discontinuous first derivative. The magnitude of the jump is given by

$$G_x(\xi^+ \mid \xi) - G_x(\xi^- \mid \xi) = \lim_{\epsilon \to 0} \int_{\xi-\epsilon}^{\xi+\epsilon} G_{xx}(x \mid \xi)\, dx = \lim_{\epsilon \to 0} \int_{\xi-\epsilon}^{\xi+\epsilon} -\frac{F_0}{T}\,\delta(x - \xi)\, dx = -\frac{F_0}{T} \quad (66)$$

This condition together with the continuity condition gives

$$d - b = -\frac{F_0}{T}$$
$$b\xi = d(\xi - L) \quad (67)$$

Solving these we find

$$\boxed{G(x \mid \xi) = \begin{cases} \frac{F_0}{T\,L}(L - \xi)\,x & 0 \le x < \xi \\ \frac{F_0}{T\,L}(L - x)\,\xi & \xi < x \le L \end{cases}} \quad (68)$$

(b)

$$f(x) = \mu\,g \quad (69)$$

By the principle of superposition, we expect that our solution will be found using the Green's function found above with F_0=1

$$y(x) = \int_0^L G(x \mid \xi)\, f(\xi)\, d\xi =$$
$$\mu\,g \int_0^L G(x \mid \xi)\, d\xi = \mu\,g \left(\int_0^x \frac{1}{T\,L}(L - x)\,\xi\, d\xi + \int_x^L \frac{1}{T\,L}(L - \xi)\,x\, d\xi \right) = \frac{\mu\,g\,(L - x)\,x}{2\,T} \quad (70)$$

Checking this we find

$$y(0) = y(L) = 0$$
$$y'' = \left(\frac{\mu\,g\,(L - x)\,x}{2\,T} \right)'' = -\frac{\mu\,g}{T} = -\frac{f(x)}{T} \quad (71)$$

So it is indeed the solution.

(c)

$$y'' + \lambda\,y = 0$$
$$y(0) = y(L) = 0 \quad (72)$$

This linear constant coefficient ODE has solutions of the form $e^{r\,x}$. Plugging this in gives

$$r^2 + \lambda = 0 \quad (73)$$

So the general solution will be

$$y = A\,e^{(-\lambda)^{1/2}\,x} + B\,e^{-(-\lambda)^{1/2}\,x} \quad (74)$$

To determine λ (which could in principle be any complex number), we apply the boundary conditions

$$0 = y(0) = A + B$$
$$0 = y(L) = A\, e^{(-\lambda)^{1/2} L} + B\, e^{-(-\lambda)^{1/2} L} \quad (75$$

This system has non trivial solutions, A and B, only if the determinant is zero

$$e^{(-\lambda)^{1/2} L} - e^{-(-\lambda)^{1/2} L} = 0 \quad (76)$$

This is easily solved if we rewrite it

$$e^{2(-\lambda)^{1/2} L} = 1 = e^{2 n \pi i} \quad (77)$$

So we find

$$\lambda_n = \left(\frac{n \pi}{L}\right)^2$$
$$y_n = A_n \operatorname{Sin}\left(\frac{n \pi x}{L}\right) \quad (78)$$

This was a regular Sturm-Liouville eigenvalue problem with weight function w(x)=1. So the appropriate inner product is

$$(f, g) = \int_0^L f(x)\, g(x)\, dx \quad (79)$$

Note that the eigenfunctions are orthogonal

$$\int_0^L \operatorname{Sin}\left(\frac{n \pi x}{L}\right) \operatorname{Sin}\left(\frac{m \pi x}{L}\right) dx = 0 \text{ if } n \neq m \quad (80)$$

To make them orthonormal, we need to choose the A_n to give them each unit norm

$$1 = \|y_n\|^2 = A_n^2 \int_0^L \operatorname{Sin}^2\left(\frac{n \pi x}{L}\right) dx = A_n^2\, \frac{L}{2} \quad (81)$$

So our orthonormal set of eigenfunctions is

$$y_n = \sqrt{\frac{2}{L}}\, \operatorname{Sin}\left(\frac{n \pi x}{L}\right) \quad (82)$$

(d)

Plugging in the given constants to the results of parts (a) and (c) gives

$$G(x) = \begin{array}{ll} 6x/7 & 0 \leq x < 1/7 \\ (1-x)/7 & 1/7 < x \leq 1 \end{array} \quad (83)$$

$$y_n = \sqrt{2}\, \operatorname{Sin}(n \pi x) \quad (84)$$

The easiest way to expand the Green's function in the eigenfunction basis is to use the result derived in problem 3a

$$G = \sum_{n=1}^{\infty} (G, y_n)\, y_n \quad (85)$$

The inner product is given by

$$(G, y_n) = \int_0^1 G(x)\sqrt{2}\,\text{Sin}(n\pi x)\,dx =$$
$$\frac{6\sqrt{2}}{7}\int_0^{1/7} x\,\text{Sin}(n\pi x)\,dx + \frac{\sqrt{2}}{7}\int_{1/7}^1 (1-x)\,\text{Sin}(n\pi x)\,dx = \frac{\sqrt{2}}{n^2\pi^2}\,\text{Sin}\left(\frac{n\pi}{7}\right) \quad (86$$

So we have:

$$\mathbf{G} = \sum_{n=1}^{\infty} \mathbf{a_n\,y_n} = \sum_{n=1}^{\infty} \frac{\mathbf{2}}{\mathbf{n^2\,\pi^2}}\,\mathbf{Sin}\left(\frac{\mathbf{n\,\pi}}{\mathbf{7}}\right)\mathbf{Sin}\,(\mathbf{n\,\pi\,x})$$

The first 9 coefficients are

n	1	2	3	4	5	6	7	8	9
$a_n \times 10^4$	621.7	280.1	155.2	87.31	44.81	17.27	0	−9.714	−13.83

Testing Parseval's relation.
The norm of G is

$$\|G\|^2 = \int_0^1 G^2(x)\,dx = \int_0^{1/7}(6x/7)^2\,dx + \int_{1/7}^1\left(\frac{1-x}{7}\right)^2 dx = \frac{12}{2401} \quad (87)$$

The difference between this and the sum of the first 9 squared coefficients is very small

$$\frac{12}{2401} - \sum_{n=1}^{9}\left(\frac{\sqrt{2}}{n^2\pi^2}\,\text{Sin}\left[\frac{n\pi}{7}\right]\right)^2 = 5.191\times 10^{-6} \quad (88)$$

Optional Part
Setting ξ=1/n gives

$$G(x) = \begin{array}{ll} (1-\frac{1}{n})\,x & 0 \le x < \frac{1}{n} \\ (1-x)\,\frac{1}{n} & \frac{1}{n} < x \le L \end{array} \quad (89)$$
$$y_m = \sqrt{2}\,\text{Sin}(m\pi x)$$

The coefficients of the expansion are given by

$$a_m =$$
$$(y_m, G) = \int_0^{\frac{1}{n}}\left(L-\frac{1}{n}\right)x\sqrt{2}\,\text{Sin}(m\pi x)\,dx + \int_{\frac{1}{n}}^1 (1-x)\,\frac{1}{n}\sqrt{2}\,\text{Sin}(m\pi x)\,dx = \frac{\sqrt{2}\,\text{Sin}(\frac{m\pi}{n})}{m^2\pi^2} \quad (90)$$

When m=kn for integer k we find

$$a_m = a_{kn} = \frac{\sqrt{2}\,\text{Sin}(k\pi)}{(kn\pi)^2} = 0 \quad (91)$$

(e)

We plot the exact solution (blue) versus four approximations (red)

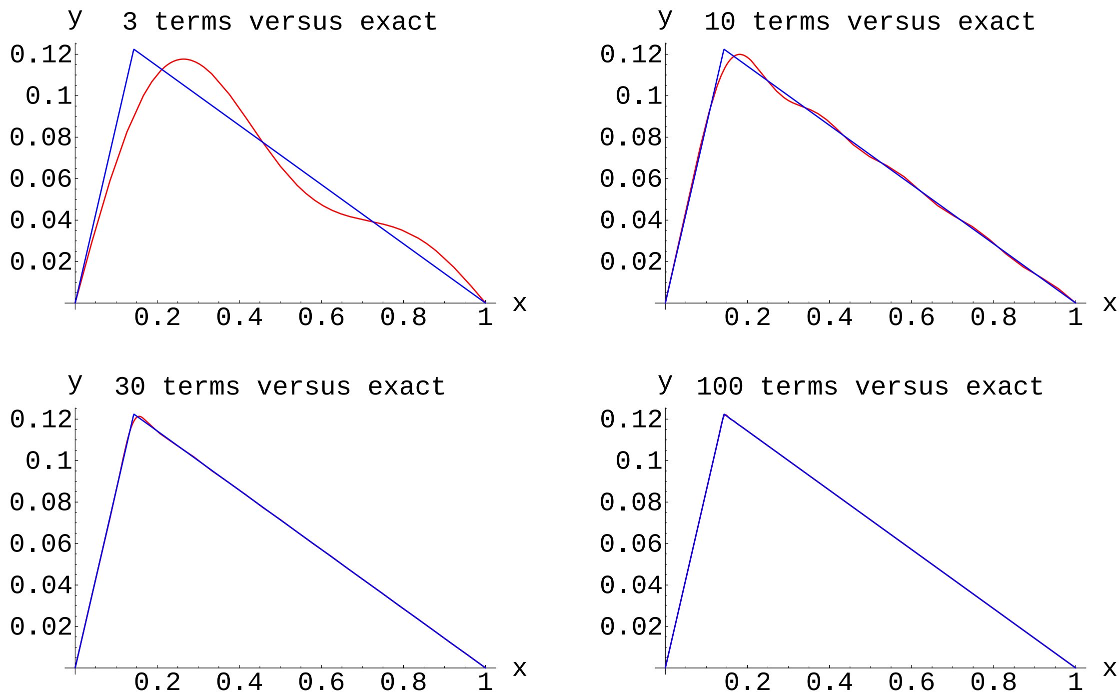

Figure 1

The approximation becomes nearly indistinguishable from the exact solution for n>10 terms. We now plot the residual=approximation-exact for these four approximations.

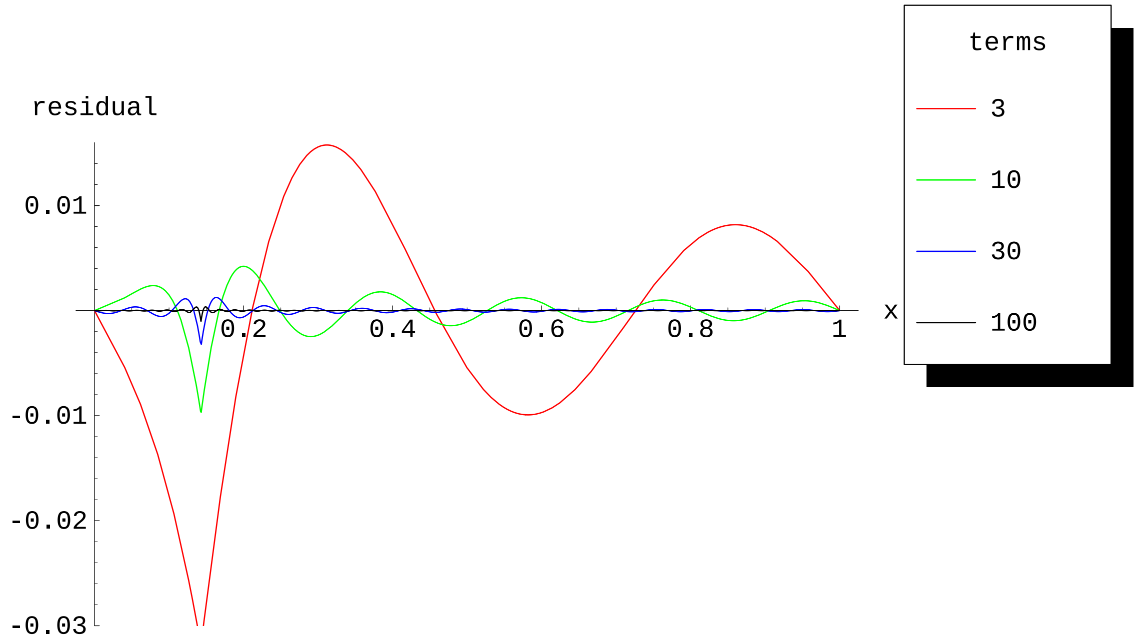

Figure 2

This reaffirms that for n>10 terms the approximation is very good. From the previous two plots, it appears as though the maximum residual, for a given number of terms, occurs at the point where G has a discontinuous derivative, i.e. x=1/7. Below we plot the absolute value of the residual at x=1/7 for n=1 to 50 terms.

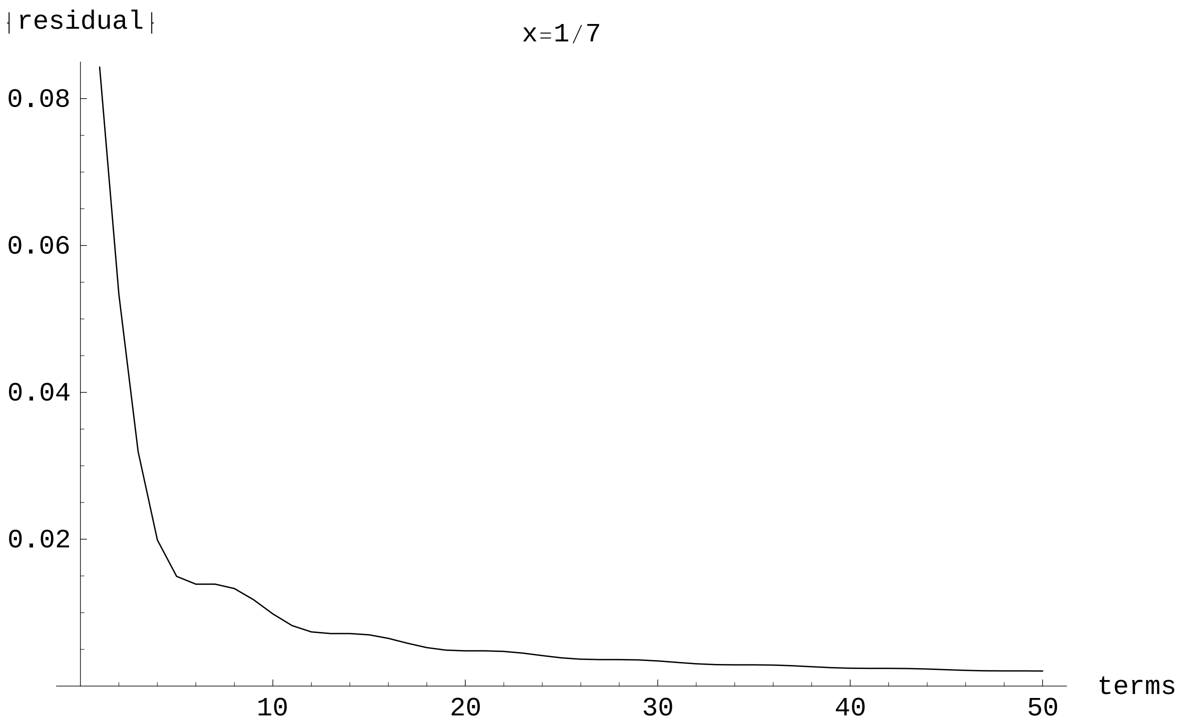

Figure 3

The maximum residual appears to vanish as $n \to \infty$.

Problem 5 (20 points)

(a)

$$y = \begin{cases} 1 & 1/4 < x < 3/4 \\ 0 & \text{otherwise} \end{cases} \tag{92}$$

As in problem 4

$$y = \sum_{n=1}^{\infty} (y, y_n)\, y_n \tag{93}$$

The inner product is given by

$$(y, y_n) = \int_0^1 y(x) \sqrt{2}\, \mathrm{Sin}(n \pi x)\, dx = \sqrt{2} \int_{1/4}^{3/4} \mathrm{Sin}(n \pi x)\, dx = \frac{\sqrt{2}}{n \pi} (1 - (-1)^n)\, \mathrm{Cos}\left(\frac{n \pi}{4}\right) \tag{94}$$

This vanishes whenever n is an even number. So we might instead write

$$\begin{aligned} z_k &= \sqrt{2}\, \mathrm{Sin}((2k-1)\pi x) \\ a_k &= \frac{2\sqrt{2}}{(2k-1)\pi} \mathrm{Cos}\left(\frac{(2k-1)\pi}{4}\right) \end{aligned} \tag{95}$$

So we have:

$$\mathbf{y = \sum_{k=1}^{\infty} a_k z_k = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \mathrm{Cos}\left(\frac{(2k-1)\pi}{4}\right) \mathrm{Sin}((2k-1)\pi x)}$$

The first 9 coefficients are

k	1	2	3	4	5	6	7	8	9
$a_k \times 10^4$	6366	−2122	−1273	909.5	707.3	−578.7	−489.7	424.4	374.5

Observe that these are decreasing more slowly than the coefficients found in problem 4. The norm of y is

$$\|y\|^2 = \int_0^1 y^2(x)\,dx = \int_{1/4}^{3/4} 1\,dx = \frac{1}{2} \tag{96}$$

The difference between this and the sum of the first 9 squared coefficients is

$$\frac{1}{2} - \sum_{n=1}^{9} \left(\frac{2\sqrt{2}}{(2n-1)\pi} \text{Cos}\left[\frac{(2n-1)\pi}{4}\right] \right)^2 = 1.125 \times 10^{-2} \tag{97}$$

This error is four orders of magnitude larger than that found in problem 4 resulting from the slower decay of the coefficients. The slower decay is a result of the discontinuity in y(x).

(e)

We plot the exact solution (blue) versus four approximations (red)

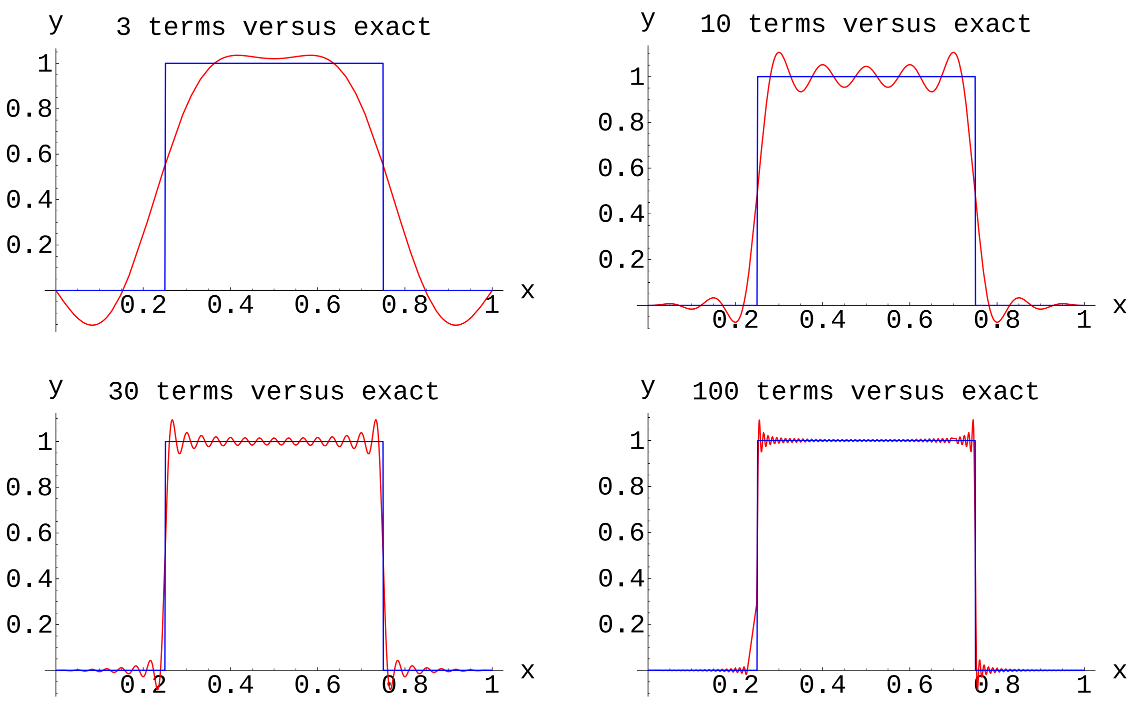

Figure 4

Notice that even for n=100 terms, the error at the discontinuity is substantial. We now plot the residual=approximation-exact for these four approximations.

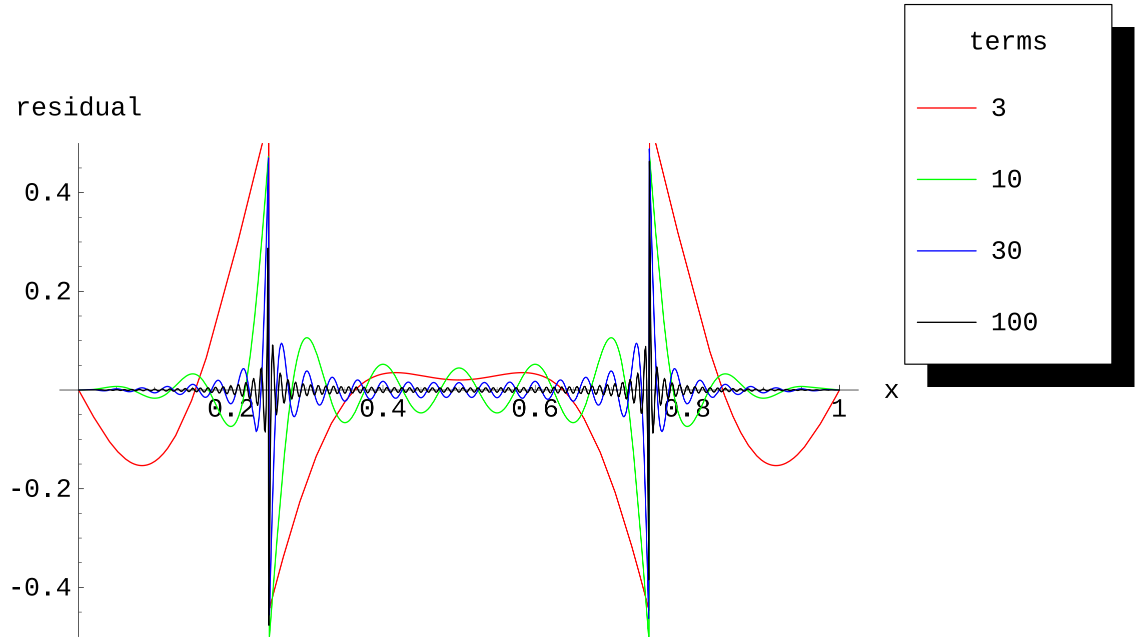

Figure 5

This reaffirms that even for n>100 the error is non-negligible From the previous two plots, it appears as though the maximum residual, for a given number of terms, occurs at the points where y is discontinuous, i.e. x=1/4 and x=3/4. Below we plot the absolute value of the residual at x=1/4 for n=1 to 50 terms.

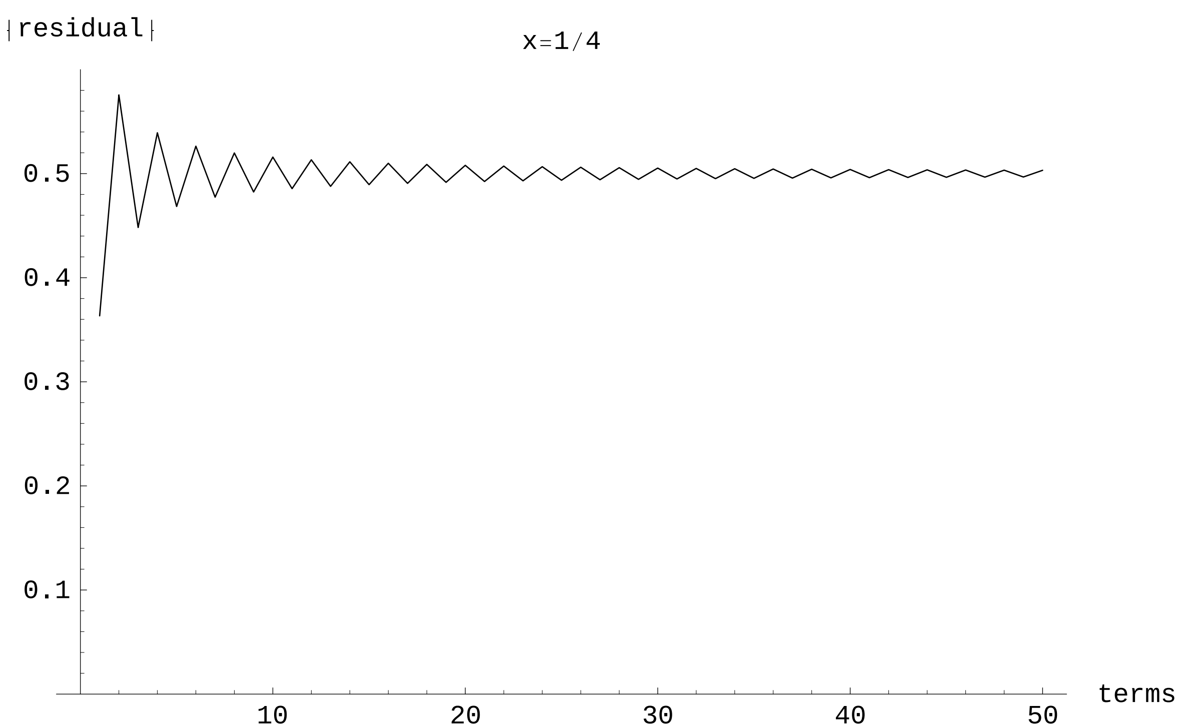

Figure 6

This residual appears to approach 0.5 as $n \to \infty$. This was expected since Sturm-Liouville theory guarantess that our series should converge to (1+0)/2 at x=1/4 (convergence in the mean). There is another interesting feature of our plots. It appears that the approximations have a maxima near x=1/4 and x=3/4 that is larger than 1 and remains so for all n. This is known as the Gibbs Phenomena. It is seen easily from the following zoomed in plot for n=30, 100 and 200 terms.

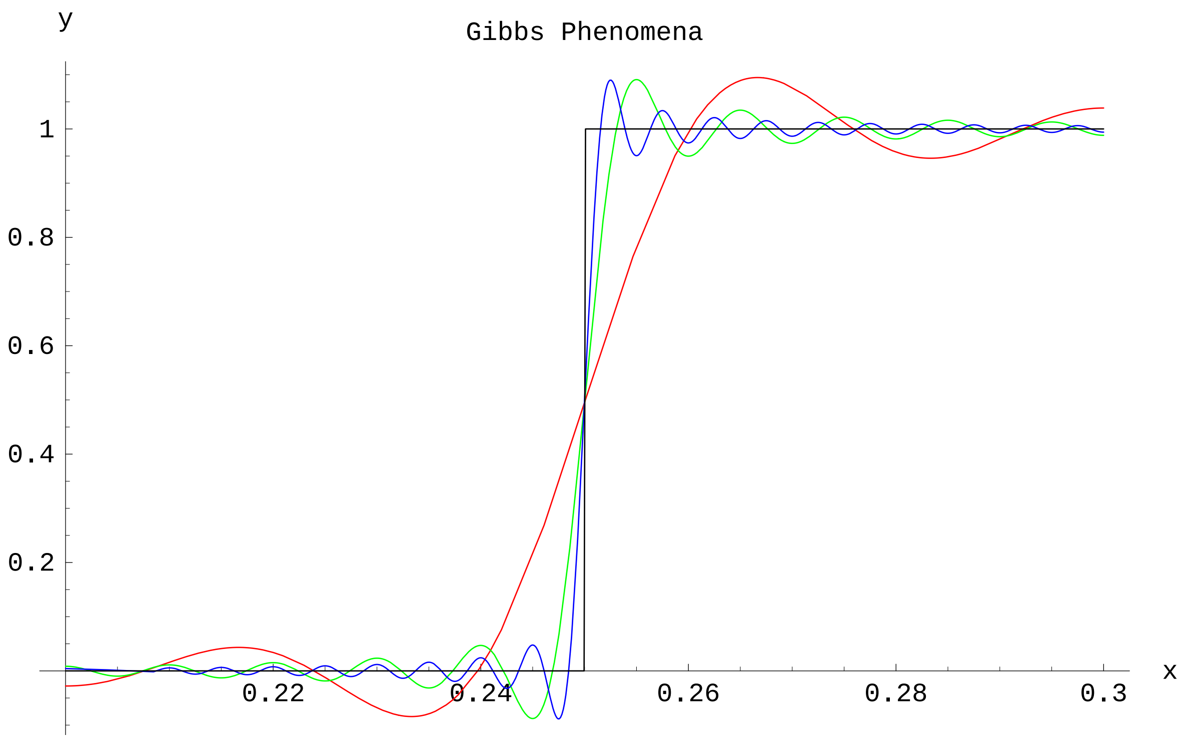

Figure 7

As n increases from 30 to 200 the height of the overshoot doesn't diminish but the width of the peak narrows.

(b)

$$y = 1/\mathrm{Sin}(\pi x) \tag{98}$$

As in the previous part

$$y = \sum_{n=1}^{\infty} (y, y_n)\, y_n \tag{99}$$

The inner product is given by

$$(y, y_n) = \int_0^1 y(x)\sqrt{2}\,\mathrm{Sin}(n\pi x)\,dx = \sqrt{2}\int_0^1 \frac{\mathrm{Sin}(n\pi x)}{\mathrm{Sin}(\pi x)}\,dx = \begin{matrix} \sqrt{2} & n\text{ odd} \\ 0 & n\text{ even} \end{matrix} \tag{100}$$

See Appendix 1 for how this integral is calculated. The first 9 non-trivial coefficients are all $\sqrt{2}$=1.414... These do not decay. For this reason we expect the Parseval relation to be void. Indeed, the norm of y is infinite

$$\|y\|^2 = \int_0^1 \frac{1}{\mathrm{Sin}^2(\pi x)}\,dx = \infty \tag{101}$$

Since y isn't square integrable, we shouldn't expect that our eigenfunctions can approximate it. Sturm-Liouville theory guarantees convergence in the mean of series of eigenfunctions to piecewise smooth functions. That is, if f(x) is continuous at x=a then

$$\lim_{N\to\infty} \sum_{n=1}^{N} (f, y_n)\, y_n(a) = f(a) \tag{102}$$

and if f has a discontinuity at x=a then

$$\lim_{N\to\infty} \sum_{n=1}^{N} (f, y_n)\, y_n(a) = \frac{f(a^+) + f(a^-)}{2} \tag{103}$$

y(x) is not piecewise smooth since

$$\lim_{x\to 0} |y(x)| = \lim_{x\to 1} |y(x)| = \infty \tag{104}$$

So we are not guaranteed convergence in the mean. The lack of convergence is seen readily by plotting n=3, 10, 30 and 100 terms.

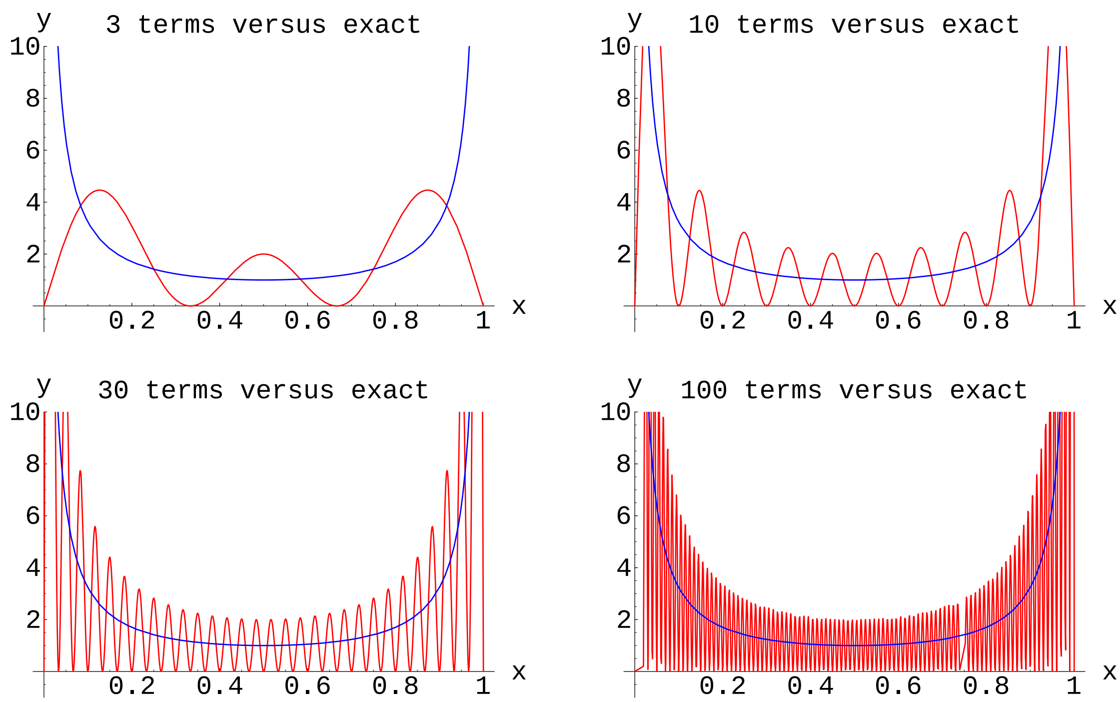

Figure 8

Possible bonus stuff(?): show that even though $y_n(0)= y_n(1)=0$ the series diverges "near" x=0 or x=1.
The Fourier sine series we've constructed is summable in some sense.

$$y = \sum_{n=1}^{\infty} 2\,\mathrm{Sin}((2\,n-1)\,\pi\,x) = 2\,\mathrm{Im}\left(e^{-\pi x i} \sum_{n=1}^{\infty} e^{2 n \pi x i}\right) \tag{105}$$

The Nth partial sum of this is easy to find since the series, when written in the latter form, is geometric.

$$S_N = 2\,\mathrm{Im}\left(e^{-\pi x i} \sum_{n=1}^{N} e^{2 n \pi x i}\right) = 2\,\mathrm{Im}\left(e^{-\pi x i}\, \frac{e^{2 \pi x i} - e^{2 (N+1) \pi x i}}{1 - e^{2 \pi x i}}\right) = 2\,\frac{\mathrm{Sin}^2(N\,\pi\,x)}{\mathrm{Sin}(\pi\,x)} \tag{106}$$

This doesn't have a well defined limit as N→∞, but we can still learn from this expression a few things about the behaviour of the sum. Firstly, the sum remains finite unless x=0,1. By L'Hopital's rule and some trig identities, as x→0 we have

$$2\,\frac{\mathrm{Sin}^2(N\,\pi\,x)}{\mathrm{Sin}(\pi\,x)} \to 2\,\frac{2\,N\,\pi\,\mathrm{Sin}(N\,\pi\,x)\,\mathrm{Cos}(N\,\pi\,x)}{\pi\,\mathrm{Cos}(\pi\,x)} = 2\,N\,\frac{\mathrm{Sin}(2\,N\,\pi\,x)}{\mathrm{Cos}(\pi\,x)} \tag{107}$$

Notice that for any small non-zero values of x, we can always choose N so large as to make this last expression as large as we want. So despite the fact that $y_n(0) = 0$, for x "near" 0 the sum is arbitrarily large. For example, let x=1/4N and let N be large:

$$S_N(x) = 2\,\frac{\mathrm{Sin}^2(N\,\pi\,x)}{\mathrm{Sin}(\pi\,x)} = 2\,\frac{\mathrm{Sin}^2(\pi/4)}{\mathrm{Sin}(\pi/4\,N)} = \frac{1}{\mathrm{Sin}(\pi/4\,N)} \to \frac{4\,N}{\pi} \to \infty \tag{108}$$

A similar phenomena occurs at the other boundary point x=1. The bottom two plots in figure 8 demonstrate these results.

Appendix1: Calculation of a tricky integral in problem 5

$$I_n = \int_0^1 \frac{\sin(n\pi x)}{\sin(\pi x)}\,dx \quad (109)$$

First observe that using some trig identities gives

$$\begin{aligned} I_1 &= \int_0^1 \frac{\sin(\pi x)}{\sin(\pi x)}\,dx = \int_0^1 1\,dx = 1 \\ I_2 &= \int_0^1 \frac{\sin(2\pi x)}{\sin(\pi x)}\,dx = \int_0^1 2\cos(\pi x)\,dx = 0 \end{aligned} \quad (110)$$

We will now prove by induction that $I_{2k}=0$ and $I_{2k-1}=1$. Note the following trig identity

$$\sin((n+2)\pi x) - \sin(n\pi x) = 2\sin(\pi x)\cos((n+1)\pi x) \quad (111)$$

This gives

$$I_{n+2} - I_n = 2\int_0^1 \cos((n+1)\pi x)\,dx = 0 \quad (112)$$

Hence, by induction, all the odd indexed integrals are 1 and all the even indexed intergals are 0.

Appendix2: a final interesting footnote about the sum in problem 5

Observe in figure 8 that the partial sums appear to oscillate about the function which they supposedly represent. We can make this mathematically precise in two ways.

Method 1: The Cesaro Sum

As we just mentioned it seems that the partial sums converge to the function in some average sense. The average of the partial sums is known as the Cesaro sum

$$\text{Cesaro Sum} = \frac{1}{N}\sum_{n=1}^{N} S_n = \frac{2}{N\sin(\pi x)}\sum_{n=1}^{N} \sin^2(n\pi x) \quad (113)$$

Using trig identities we can write this as

$$\frac{2}{N\sin(\pi x)}\,\mathrm{Re}\left(\sum_{n=1}^{N} \frac{1 - e^{2 n\pi x i}}{2}\right) \quad (114)$$

This is easily summed

$$\frac{1}{\sin(\pi x)} - \frac{\sin(N\pi x)\cos((N+1)\pi x)}{N\sin^2(\pi x)} \quad (115)$$

Letting $N \to \infty$ gives the original function our series is supposedly representing. So the Cesaro Sum of our series converges to the appropriate function uniformly.

Method 2: Generalized Riemann-Lebesgue Lemma

Pick any square integrable test function f(x) with compact support on [0,1] and compute

$$\int_0^1 S_n(x)\,f(x)\,dx = \int_0^1 2\,\frac{\sin^2(N\pi x)}{\sin(\pi x)}\,f(x)\,dx \quad (116)$$

Since $\sin^2(N\pi x)$ is periodic with a wave number that becomes infinite as $N \to \infty$, by a generalization of the Riemann-Lebesgue lemma we have

$$\lim_{N\to\infty}\int_0^1 2\,\frac{\sin^2(N\pi x)}{\sin(\pi x)}\,f(x)\,dx = 2\,\overline{\sin^2(\pi x)}\int_0^1 \frac{1}{\sin(\pi x)}\,f(x)\,dx \quad (117)$$

where the overbar denotes the average over one period. This average equals 1/2, so evidently

$$\lim_{N\to\infty} \int_0^1 S_N(x)\, f(x)\, dx = \int_0^1 \frac{1}{\mathrm{Sin}(\pi x)} f(x)\, dx \tag{118}$$

In this peculiar sense, the partial sums converge to the appropriate function (similar to the way a delta sequence converges to the delta "function").