Greens Functions

Greens function $G(x; \xi)$ is solution to

$$L G = \delta(x-\xi)$$

with

\begin{align*}
B_y &= 0 \\
B_x y &= 0
\end{align*}

Notice that it depends only on the homogeneous solution $y_1$, and $y_2$ with parameters $\xi$.

Can linearly superpose $G(x; \xi)$ for different $\xi$ to get solution to (1) for any $f(x)$:

$$y(x) = \int_a^b f(\xi) G(x; \xi) \, d\xi$$

Check:

\begin{align*}
L y &= \int_a^b f(\xi) L G(x; \xi) \, d\xi \\
&= \int_a^b f(\xi) \left[ \delta(x-\xi) \right] \, d\xi \\
&= f(x)
\end{align*}
Ways to find $G(x)$

**1. (best for $2^\text{nd}$ order ODEs) — express solution to (1) as linear combinations of solutions to homogeneous equation, matches $f$ ($a < b$) 
   - but $G$ discontinuity from derivative of $G(x)$

2. Eigenfunction expansion
   - (usually for $y''$ but not for $y''$)
   - where usual form solution of type, (1) available

3. Exact solve (1) by variation of parameters, compare to (3) and identify $G$.
   - (usually more work than method 1/2)

Detail of method 1. For $i^\text{th}$ order ODEs of form (1).

Two cases:

Case 1: \( \lambda \) is not an eigenvalue. Then (1) has unique solution, and the homogeneous equation $Ly = 0$ has only the trivial solution $y = 0$.

In this case:

\[ \begin{align*}
&\text{let } y_1(x) \text{ be solution to } Ly_1 = 0 \quad \Rightarrow y_1 = 0 \\
&\text{let } y_2(x) \text{ be solution to } Ly_2 = 0 \quad \Rightarrow y_2 = 0
\end{align*} \]

$y_1$ and $y_2$ are linearly independent unique up to multiplicative constants

Clearly $G(x) = \theta \langle 5 \rangle y_1(x), x > 3$

$\theta \langle 5 \rangle y_1(x), x > 3$
we choose $c_i(3)$ and $c_2(3)$ to satisfy (2):

(6) \[ G(x, 3) \text{ must be continuous at } 3, \text{ otherwise } \frac{dG}{dx} = \frac{d}{dx} \left( e^{\frac{x^2}{2}} \right) \]

(7) \[ \frac{dG(x, 3)}{dx} \text{ must be discontinuous, with } p(x) \frac{dG}{dx} \bigg|_{x=3} - p(x) \frac{dG}{dx} \bigg|_{x=3} = 1 \]

so that \[ \frac{d}{dx} \left( p(x) \frac{dG}{dx} \right) \bigg|_{x=3} = 1 \]

Solving the resulting pair of equations for $c_i(3)$ and $c_2(3)$ gives

(8) \[
\begin{cases}
G(x, 3) = \left( \begin{array}{c}
\frac{y_1(3) y_2(3)}{p(3) W(3)} \\
\frac{y_1(3) y_2(3)}{p(3) W(3)}
\end{array} \right) & x < 3 \\
\left( \begin{array}{c}
y_1(3) y_2(3) \\
y_1(3) y_2(3)
\end{array} \right) & x > 3
\end{cases}
\]

As this is true for any $J + c(3)$, we have $J = 0$ from (1).

For the $J = 0$ from (1), the simplifying:

(9) \[ \text{Abel's Theorem: } J = -1 \Rightarrow \quad e^\rho \leq \quad e^\rho \cdot \sqrt{p(x) W(x)} \text{ constant,} \quad J = \text{constant} \]

The constant can be evaluated by

Case II. $\lambda$ is an eigenvalue. Then the homogeneous equation $y$ has

a nontrivial solution $y$, which satisfies

\[ Ly = 0 \quad \Rightarrow \quad y_1 = 0 \quad y_2 = 0 \]
The derivation of \( c \) for case I fails now:

If chosen \( y_1 \) is a solution of \( \mathbf{y}_1 = 0 \), \( \mathbf{b}_1 \mathbf{y}_1 = 0 \),

\[ y_1 = \cos \alpha y_1 \]

and the Wronskian \( \mathbf{W}(\mathbf{y}_1, y_2) = 0 \) so (3) holds.

(11)

We can now take as follows the derivation of (8) using \( y_1 \) for

the second (linearly independent of \( y_1 \)) solution to \( \mathbf{b}_1 \mathbf{y}_1 \).

Suppose \( \mathbf{b}_1 \mathbf{y}_1 \neq 0 \) since it is linearly independent of \( y_1 \), and \( \mathbf{b}_1 \mathbf{y}_1 = 0 \).

\( y_1 \) and \( y_1(6) \) are fixed.

\[ \int \mathbf{b}_1(\mathbf{y}) = \frac{\mathbf{b}_1(y)}{\mathbf{y}} \int_0^a \frac{\mathbf{b}_1(\mathbf{y})}{\mathbf{b}_1(\mathbf{y})} \, d\mathbf{s} \]

for S-L problem \( \{ \mathbf{p}, \mathbf{w} \} = \mathbf{c} \mathbf{y} \)

If \( \lambda \) is an eigenvalue and \( y, \mathbf{y} \) the associate eigenfunction,

each solution to (11) exists uniquely if \( \mathbf{b}_1 \) satisfies:

\[ \int_0^a f(\mathbf{y}) \mathbf{y}_1(\mathbf{y}) \, d\mathbf{s} = 0 \]

in. there is no solution unless \( \mathbf{b}_1 \) is orthogonal to the eigenfunction.

If \( \mathbf{b}_1 \) does satisfy this orthogonality constraint, the solution \( \mathbf{y} \) is not unique (cyclic) and generally:

\[ \mathbf{y} = c_1 \mathbf{y}_1(\mathbf{x}) + \int_b^a \mathbf{w}(\mathbf{x}, \mathbf{y}) f(\mathbf{x}) \, d\mathbf{s} \]

which solves (11) by orthogonality of \( \mathbf{y}_1 \) using (11), not (8).