Network coding – introduction

- Network coding is fundamentally different from traditional forwarding/routing in that it allows algebraic operations across packets in a network
- Can be applied at the physical layer, network layer or application layer
- Potential usefulness includes increasing network capacity, reducing optimization/coordination complexity, improving robustness, distributed compression, etc.

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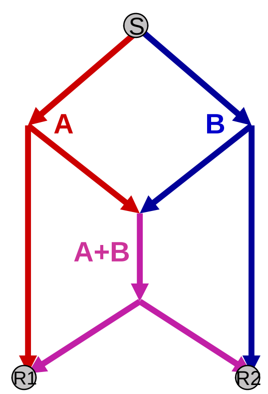
Wired multicast capacity

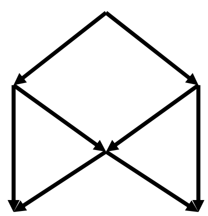


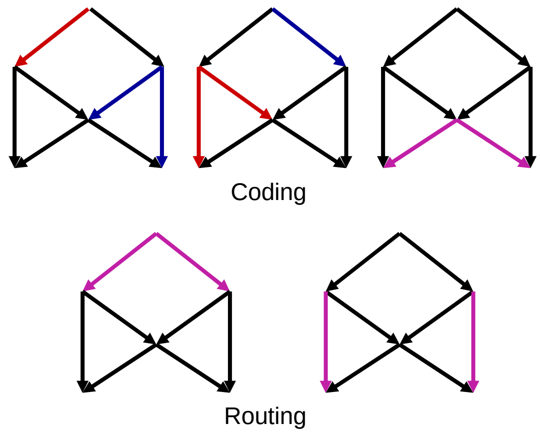
R. Ahlswede et al., ``Network information flow,'' *IEEE Trans. Inform. Theory*, IT-46: 1204-1216,2000.

2

Wireless multicast

- Static wireless model with fixed link rates and a half-duplex constraint
- E.g. [SaEp05]





$$\frac{\text{Coding rate}}{\text{Routing rate}} = \frac{4}{3}$$

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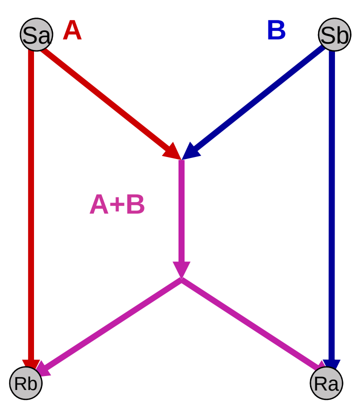
Wired unicasts



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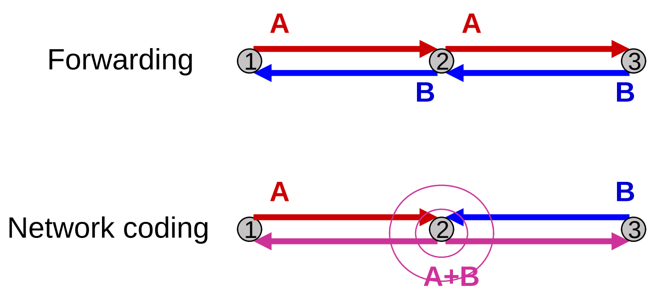
Wireless unicasts



R. W. Yeung and Z. Zhang, ``Distributed source coding for satellite communications,'' *IEEE Trans. Inform. Theory*, IT-45: 1111-1120, 1999.

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Basic problem formulation

- Directed graph G=(V,E) with unit edge capacities
- Source nodes with unit entropy rate exogenous source processes, sink nodes with output processes, random processes transmitted on edges
- Set of connections C={(source v, sink w, source processes from v demanded by w)}
- A network coding problem is defined by a pair (G,C)

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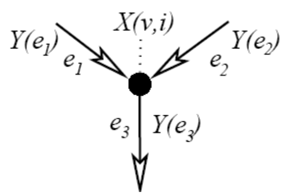
Basic questions

- Given a network problem (G,C),
  - is the problem solvable?
  - if so, how to construct a solution?
- Multicast/intra-session network coding
  - Coding across packets from a single unicast or multicast session
- Non-multicast/inter-session network coding
  - Coding across packets from different communication sessions
  - More complex than multicast network coding

Delay-free scalar linear network coding

- Group bits into vectors of length m, each vector representing a symbol in the finite field of 2^m elements
- Each successive output symbol is calculated from the corresponding input symbols using the same linear function -> can analyze transmission of one symbol from each source

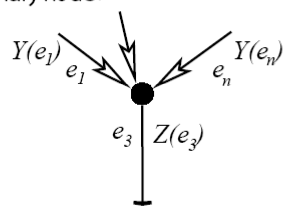


$$Y(e_3) = \sum_i \alpha_i X(v,i) + \sum_{j=1,2} \beta_j Y(e_j)$$

R. Koetter and M. Medard, "An algebraic approach to network coding". *IEEE Transactions on Networking,* October 2003

Delay-free scalar linear network coding

At a receiver (terminal) node:



$$Z(v, j) = \sum_{j=1}^{n} \varepsilon_j Y(e_j).$$

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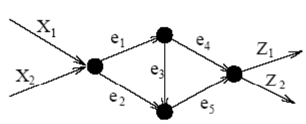
A simple example



$$\begin{aligned}
Y(e_1) &= \alpha_{1,e_1} X_1 + \alpha_{2,e_1} X_2 \\
Y(e_2) &= \alpha_{1,e_2} X_1 + \alpha_{2,e_2} X_2 \\
Y(e_3) &= \beta_{e_1,e_3} Y(e_1) \\
Y(e_4) &= \beta_{e_1,e_4} Y(e_1) \\
Y(e_5) &= \beta_{e_2,e_5} Y(e_2) + \beta_{e_3,e_5} Y(e_3) \\
Z_1 &= \varepsilon_{e_4,1} Y(e_4) + \varepsilon_{e_5,1} Y(e_5) \\
Z_2 &= \varepsilon_{e_4,2} Y(e_4) + \varepsilon_{e_5,2} Y(e_5)
\end{aligned}$$

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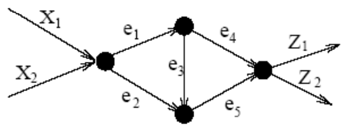
Transfer matrix



$$F = \begin{pmatrix} 0 & 0 & \beta_{e_1,e_3} & \beta_{e_1,e_4} & 0 \\ 0 & 0 & 0 & 0 & \beta_{e_2,e_5} \\ 0 & 0 & 0 & 0 & \beta_{e_3,e_5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \alpha_{1,e_1} & \alpha_{1,e_2} & 0 & 0 & 0 \\ \alpha_{2,e_1} & \alpha_{2,e_2} & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \varepsilon_{1,e_1} & \varepsilon_{1,e_2} & 0 & 0 & 0 \\ \varepsilon_{2,e_1} & \varepsilon_{2,e_2} & 0 & 0 & 0 \end{pmatrix}$$

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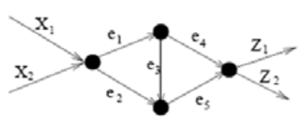
Transfer matrix



$$[X_1 \quad X_2]\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \beta_{e_1e_3} & \beta_{e_1e_4} & 0 \\ 0 & 0 & 0 & 0 & \beta_{e_2e_5} \\ 0 & 0 & 0 & 0 & \beta_{e_3e_5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \right.$$

$$\left. + \begin{bmatrix} 0 & 0 & 0 & 0 & \beta_{e_1e_3}\beta_{e_3e_5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = [Z_1 \quad Z_2]$$

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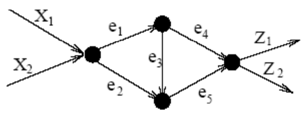
Transfer matrix



$$F = \begin{pmatrix} 0 & 0 & \beta_{e_1,e_3} & \beta_{e_1,e_4} & 0 \\ 0 & 0 & 0 & 0 & \beta_{e_2,e_5} \\ 0 & 0 & 0 & 0 & \beta_{e_3,e_5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Summing the "path gains":

$$I + F + F^2 + \ldots = (I-F)^{-1} = \begin{pmatrix} 1 & 0 & \beta_{e_1,e_3} & \beta_{e_1,e_4} & \beta_{e_1,e_3}\beta_{e_3,e_5} \\ 0 & 1 & 0 & 0 & \beta_{e_2,e_5} \\ 0 & 0 & 1 & 0 & \beta_{e_3,e_5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Observe that $G = (I-F)^{-1}$ is polynomial and, for an acyclic graph, is upper triangular with each non-diagonal entry corresponding to the total gain of all paths between two edges

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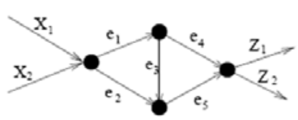
Transfer matrix



$$[X_1 \quad X_2]\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & \beta_{e_1e_3} & \beta_{e_1e_4} & \beta_{e_1e_3}\beta_{e_3e_5} \\ 0 & 1 & 0 & 0 & \beta_{e_2e_5} \\ 0 & 0 & 1 & 0 & \beta_{e_3e_5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = [Z_1 \quad Z_2]$$

$$[X_1 \quad X_2]\begin{bmatrix} \beta_{e_1e_4} & \beta_{e_1e_3}\beta_{e_3e_5} \\ 0 & \beta_{e_2e_5} \end{bmatrix} = [Z_1 \quad Z_2]$$

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Transfer matrix

Input vector: $\underline{x}^T = (X(v,1), X(v,2), \ldots, X(v', \mu(v')))$

Output vector: $\underline{z}^T = (Z(u,1), Z(u,2), \ldots, Z(u', \nu(u')))$

Transfer matrix: M

$$\underline{z} = M\underline{x} = B^T \cdot \underbrace{(I - F^T)^{-1}}_{G^T} \cdot A^T \underline{x}$$

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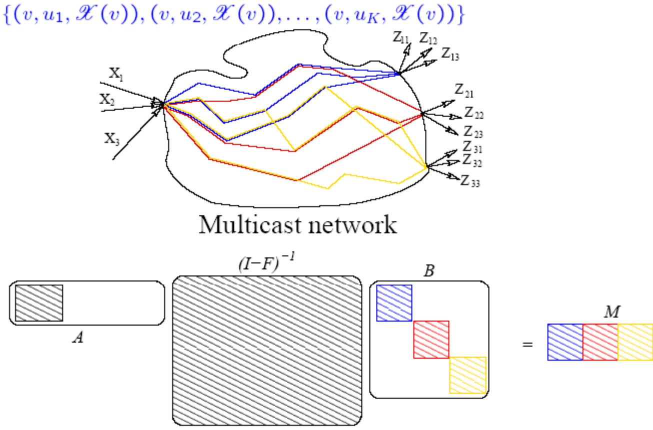
Multicast

$\mathscr{C} = \{(v, u_1, \mathscr{X}(v)), (v, u_2, \mathscr{X}(v)), \ldots, (v, u_K, \mathscr{X}(v))\}$



M is a $|\mathscr{X}(v)| \times K|\mathscr{X}(v)|$ matrix.

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Multicast solvability

Theorem: Let (G,C) be a multicast network coding problem. There exists a network coding solution for (G,C) if and only if there exists a flow of sufficient capacity between the source and each sink individually.

R. Ahlswede et al., ``Network information flow,'' *IEEE Trans. Inform. Theory*, IT-46: 1204-1216,2000.

Theorem: For a feasible multicast network coding problem, there exists a scalar linear network coding solution over a sufficiently large finite field of any characteristic

S.-Y. R. Li, R. W. Yeung, and N. Cai. "Linear network coding". *IEEE Transactions on Information Theory* , Februray, 2003

R. Koetter and M. Medard, “An algebraic approach to network coding”. *IEEE Transactions on Networking,* October 2003

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Proof outline

- For a point to point connection
  - The capacity is given by the max flow min cut bound
  - A particular communication rate is feasible if and only if the determinant of the corresponding transfer matrix is nonzero
- For a multicast connection
  - If the max flow min cut bound is satisfied for each sink, the product of the transfer matrices is a nonzero polynomial in coding coefficients, and there exists a solution in a sufficiently large field (and vice versa)

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Delays and cycles in networks

- Delay-free network coding implicitly assumes that coding occurs over an acyclic topology
- For networks with cycles, we can
  - do convolutional network coding, or
  - code over a time-expanded network, as in packet networks
    - source process <-> source packet
    - link <-> transmitted packet