Generalized Processor Sharing

- let $Z_j(t) = \begin{cases} 1 & \text{if queue } j \text{ nonempty at time } t \\ 0 & \text{else} \end{cases}

- let $Z(t) = \begin{cases} 1 & \text{if at least 1 queue is nonempty} \\ 0 & \text{else} \end{cases}

- let $C$ be the link service rate, eg in \text{bits/sec}

- let $\Phi_j$ be the weight of class $j$, $1 \leq j \leq N$

- service rate applied to queue $j$ at time $t$
  (rate at which fluid is drained from queue $j$) is
  $$\frac{\Phi_j Z_j(t) C}{\sum_{i=1}^{N} \Phi_i Z_i(t)} (0 \text{ if } Z(t) = 0)$$

  eg. if at time $t$ queues 3 & 7 are nonempty, queue 3 is being served at rate $\frac{\Phi_3}{\Phi_3 + \Phi_7} C$ & queue 7 is being served at rate $\frac{\Phi_7}{\Phi_3 + \Phi_7} C$

- average service rate applied to queue $j$ over interval $[0,t]$ is
  \[\text{cumulative service over } [0,t] \text{ \over total time in } [0,t] \text{ during which queue is nonempty}\]
  $$\frac{\int_0^t \sum_{i=1}^{N} \Phi_i Z_i(t) C \, dt}{\int_0^t Z_j(t) \, dt}$$
- Each queue has a constant service per virtual time unit.

- The weight of a nonempty queue is inversely proportional to total queueing time for every "round" of virtual time.

- $V(t)$ is a piecewise linear nondecreasing function of virtual time.

- Think of virtual time as clocking with bits from each nonempty queue.

- For class 1 fluid, arrive the GFS server offers a lower service curve.

- $V(0) = 0$

- $q = \frac{1}{\phi_i / C + t}$

- $\sum_{i=1}^{\infty} \phi_i = \frac{M^2}{c}$

- $\int \frac{c}{\phi_i / C + t} \, dt = \frac{c}{\phi_i / C} \ln(1 + C \phi_i / t)$

- Since $2 \phi(t) \leq 1$
→ easier to compute virtual compared to actual packet departure times

- let $a_{k}^{(j)} = \text{arrival time}$
  
  $L_{k}^{(j)} = \text{length}$
  
  $s_{k}^{(j)} = \text{service initiation time}$
  
  $d_{k}^{(j)} = \text{departure time}$

- $V(d_{k+1}^{(j)}) = V(s_{k+1}^{(j)}) + \frac{L_{k}^{(j)}}{\Phi_{j}}$
  
  $= \max (V(d_{k}^{(j)}), V(a_{k+1}^{(j)})) + \frac{L_{k}^{(j)}}{\Phi_{j}}$

  ▲ because service virtual time taken
  
  is FCFS within to serve packet
  
  classes

- scaling the weights $\Phi_{j}$ by the same factor scales $V(t)$, does not change the ordering of packet finish times

- GPS can be used to define practical packet-scheduling policies (where entire packets are served sequentially) which have bounded performance relative to GPS
Weighted Fair Queuing

1. The virtual time process $V(t)$ is simulated as if there were a GPS scheduler.

2. Each arriving packet is marked with its virtual finish time in the GPS scheduler.

3. A packet that is selected for transmission is transmitted completely.

4. Upon completion of a packet transmission, the next packet chosen for transmission is the one with the smallest virtual finish time among all queued packets.

Example:

- Diagram showing the processes with weights (WT) and capacities at different times.
- GPS (fluid) process with no other arrivals.
- WFQ process with packets at different capacities.
- WFWFQ process with packets at different capacities.
Worst case fair weighted fair queuing (WFWFQ)

- similar to WFQ except that the packet chosen for transmission is the one with the earliest finish time among all packets that would already have started service in GPS
- busy period: time period during which buffer is nonempty
- idle period: time period during which buffer is empty

- for a work-conserving scheduler, the evolution of $X(t)$ depends only on the arrival times & packet lengths, does not depend on the order in which packets are scheduled for transmission

$\Rightarrow$ busy & idle periods are invariant with the scheduling policy, & the same packets depart in each busy period under any scheduling policy
Lemma 1: If 2 packets $i_1$ & $i_2$ are present in the multiplexer at a scheduling instant, then packet $i_1$ is selected to be served in WFA ahead of $i_2$ iff the same order of departure occurs in GPS (or $i_1$ & $i_2$ have the same departure time in GPS & $i_1$ satisfies the tie-breaking criterion).

Proof: Let $d_1$ & $d_2$ be the departure times under GPS.

- By the WFA rule, packet $i_1$ is selected iff
  
  $V(d_1) < V(d_2)$, or $V(d_1) = V(d_2)$ & $i_1$ wins the tie-breaker.

- $V(d_1) < V(d_2) \iff d_1 < d_2$

- $V(d_1) = V(d_2) \iff d_1 = d_2$

  since during a busy period $V(t)$ is strictly increasing.

- Note: for packets that are not present at the same time, the GPS & WFA service orders can be different.

□
- let $l_{\text{max}}$ be an upper bound on the packet lengths

- let $d_k$ be the departure time of a packet $k$ under GFS, & $\hat{d}_k$ the WFQ departure time of the same packet

**Theorem 1:** $d_k \leq \hat{d}_k + \frac{l_{\text{max}}}{c}$ for any packet $k$ where $c$ is the full link rate

**Proof:** Since WFQ & GFS are work-conserving, it suffices to consider a single busy period (same duration, same packets are served by both disciplines)

- index packets according to their order of departure under WFQ
- assume wlog that the busy period starts at time 0
- note that $\hat{d}_1 = \frac{L_1}{c}$, $\hat{d}_2 = \frac{L_1 + L_2}{c}$, ..., $\hat{d}_k = \frac{\sum_{i=1}^{k} L_i}{c}$

**Case 1:** Under GFS, packets $P_1, \ldots, P_{k-1}$ all depart earlier or at the same time as $P_k$

$\Rightarrow d_k \geq \frac{\sum_{i=1}^{k} L_i}{c} = \hat{d}_k$

**Case 2:** There is some $m \in \{1, k-1\}$ such that $d_m > d_k$, $d_n \leq d_k \forall m+1 \leq n \leq k-1$
- let $\hat{s}_m$ be the time at which packet $p_m$ began service under WFQ
- since in GPS $p_m$ departs after $p_{m+1}, \ldots, p_k$, packets $p_{m+1}, \ldots, p_k$ cannot be present at time $\hat{s}_m$, or they would have been chosen for service by WFQ ahead of $p_m$
- $p_{m+1}, \ldots, p_k$ arrived after $\hat{s}_m$ & in GPS departed by $d_k$
-  
- $d_k \geq \hat{s}_m + \sum_{i=m+1}^{k} \frac{L_i}{c}$
-  
- $\leq \hat{d}_{m-1} + \sum_{i=m+1}^{k} \frac{L_i}{c}$
-  
- $\geq \hat{d}_k - \frac{L_m}{c}$
-  
- $\geq \hat{d}_k - \frac{L_{\text{max}}}{c}$