Generalized Processor Sharing

- let $z_j(t) = \begin{cases} 1 & \text{if queue } j \text{ nonempty at time } t \\ 0 & \text{else} \end{cases}$

  $z(t) = \begin{cases} 1 & \text{if at least 1 queue is nonempty} \\ 0 & \text{else} \end{cases}$

- let c be the link service rate, eg in bits/sec
- let ϕ_j be the weight of class j, $1 \le j \le N$
- service rate applied to queue j at time τ (rate at which fluid is drained from queue j) is

$$\frac{\phi_j z_j(\tau)}{\sum_{i=1}^{N} \phi_i z_i(\tau)} c \qquad (0 \text{ if } z(\tau) = 0)$$

eg. if at time τ queues 3 & 7 are nonempty, queue 3 is being served at rate $\frac{\phi_3}{\phi_3 + \phi_7} c$ & queue 7 is being served at rate $\frac{\phi_7}{\phi_3 + \phi_7} c$

- average service rate applied to queue j over interval $[0, t]$ is

$$\frac{\text{cumulative service over } [0,t]}{\text{total time in } [0,t] \text{ during which queue is nonempty}}$$

$$= \frac{\int_0^t \frac{\phi_j z_j(\tau)}{\sum_{i=1}^{N} \phi_i z_i(\tau)} c \, d\tau}{\int_0^t z_j(\tau) \, d\tau}$$

$$\geq \frac{\phi_j \frac{\int_0^t Z_j(\tau)}{\sum_{i=1}^N \phi_i} c \, d\tau}{\int_0^t Z_j(\tau) \, d\tau} \qquad \text{Since } Z_i(\tau) \leq 1$$

$$= \frac{\phi_j}{\sum_{i=1}^N \phi_i} c$$

→ for class j fluid arrivals the GPS server offers a lower service curve

$$S_{GPS}(t) = \frac{\phi_j}{\sum_{i=1}^N \phi_i} c \, t^+$$

- Virtual time
  - think of the server as clocking 1 unit of virtual time for every "round" of serving ϕ_i bits from each nonempty queue i

$$V(t) = \int_0^t \frac{c}{\sum_{i=1}^N \phi_i Z_i(\tau)} Z(\tau) \, d\tau$$

  - $V(t)$ is a piecewise linear nondecreasing function
    - $V(0) = 0$
    - slope $= 0$ when all queues empty, o/w slope is inversely proportional to total weight of nonempty queues
  - each queue sees constant service per virtual time unit

→ easier to compute virtual compared to actual packet departure times

- let $a_k^{(j)}$ = arrival time, $L_k^{(j)}$ = length, $s_k^{(j)}$ = service initiat^n time, $d_k^{(j)}$ = departure time $\Bigg\}$ of kth packet in jth queue

- $$V(d_{k+1}^{(j)}) = V(s_{k+1}^{(j)}) + \frac{L_{k+1}^{(j)}}{\phi_j}$$

$$= \max\left(V(d_k^{(j)}), V(a_{k+1}^{(j)})\right) + \frac{L_{k+1}^{(j)}}{\phi_j}$$

↑ because service is FCFS within classes

↗ virtual time taken to serve packet

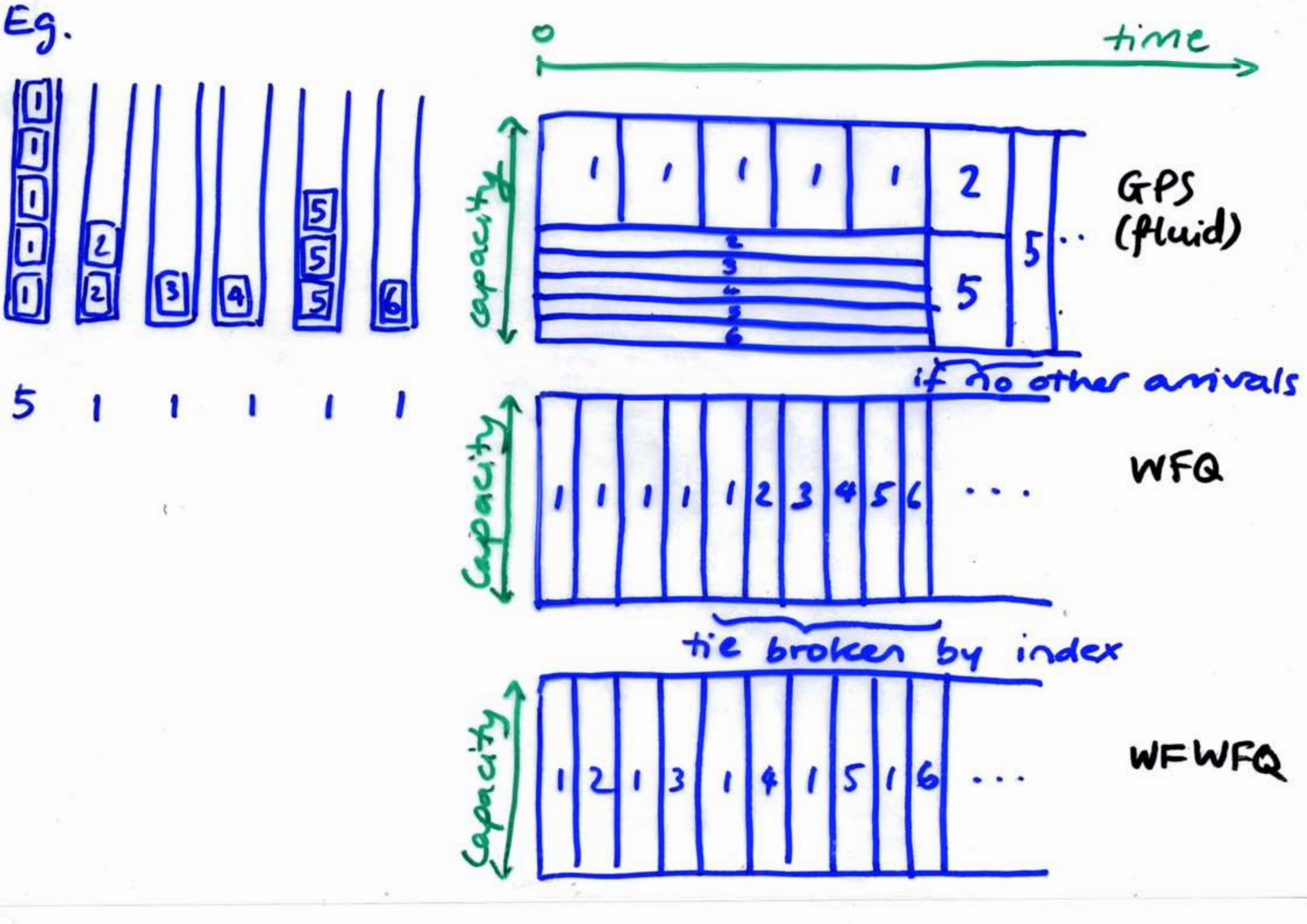
- scaling the weights ϕ_i by the same factor scales $V(t)$, does not change the ordering of packet finish times

- GPS can be used to define practical packet scheduling policies (where entire packets are served sequentially) which have bounded performance relative to GPS

Weighted Fair Queuing

1. The virtual time process $V(t)$ is simulated as if there were a GPS scheduler
2. Each arriving packet is marked with its virtual finish time in the GPS scheduler
3. A packet that is selected for transmission is transmitted completely
4. Upon completion of a packet transmission, the next packet chosen for transmission is the one with the smallest virtual finish time among all queued packets

Eg.



Worst case fair weighted fair queuing (WFWFQ)

- similar to WFQ except that the packet chosen for transmission is the one with the earliest finish time among all packets that would already have started service in GPS

- busy period: time period during which buffer is nonempty
  idle period: time period during which buffer is empty
- for a work-conserving scheduler, the evolution of $X(t)$ depends only on the arrival times & packet lengths, does not depend on the order in which packets are scheduled for transmission
  → busy & idle periods are invariant with the scheduling policy, & the same packets depart in each busy period under any scheduling policy

Lemma 1: If 2 packets i_1 & i_2 are present in the multiplexer at a scheduling instant, then packet i_1 is selected to be served in WFQ ahead of i_2 iff the same order of departure occurs in GPS (or i_1 & i_2 have the same departure time in GPS & i_1 satisfies the tie-breaking criterion)

Proof: Let d_1 & d_2 be the departure times under GPS

- By the WFQ rule, packet i_1 is selected iff $V(d_1) < V(d_2)$, or $V(d_1) = V(d_2)$ & i_1 wins the tie-breaker
- $V(d_1) < V(d_2) \Leftrightarrow d_1 < d_2$
  $V(d_1) = V(d_2) \Leftrightarrow d_1 = d_2$
  since during a busy period $V(t)$ is strictly increasing

$\square$

- Note: for packets that are not present at the same time, the GPS & WFQ service orders can be different

- let L_{max} be an upper bound on the packet lengths
- let d_k be the departure time of a packet k under GPS, & $\hat{d}_k$ the WFQ departure time of the same packet

Theorem 1: $\hat{d}_k \le d_k + \frac{L_{max}}{C}$ for any packet k where C is the full link rate

Proof: Since WFQ & GPS are work-conserving, it suffices to consider a single busy period (same duration, same packets are served by both disciplines)

- index packets according to their order of departure under WFQ
- assume wlog that the busy period starts at time 0
- note that $\hat{d}_1 = \frac{L_1}{C}$, $\hat{d}_2 = \frac{L_1 + L_2}{C}$, ..., $\hat{d}_k = \frac{\sum_{i=1}^{k} L_i}{C}$

Case 1: Under GPS, packets $P_1, \dots, P_{k-1}$ all depart earlier or at the same time as P_k

$$\rightarrow d_k \ge \frac{\sum_{i=1}^{k} L_i}{C} = \hat{d}_k$$

Case 2: There is some $m \in [1, k-1]$ such that $d_m > d_k$, $d_n \le d_k$ $\forall$ $m+1 \le n \le k-1$

- let $\hat{s}_m$ be the time at which packet p_m began service under WFQ
- since in GPS p_m departs after $p_{m+1}, \ldots, p_k$, packets $p_{m+1}, \ldots, p_k$ cannot be present at time $\hat{s}_m$, or they would have been chosen for service by WFQ ahead of p_m

→ $p_{m+1}, \ldots, p_k$ arrived after $\hat{s}_m$ & in GPS departed by d_k

$$\rightarrow \quad d_k \geq \hat{s}_m + \frac{\sum_{i=m+1}^{k} L_i}{C}$$

$$= \hat{d}_{m-1} + \frac{\sum_{i=m+1}^{k} L_i}{C}$$

$$= \hat{d}_k - \frac{L_m}{C}$$

$$\geq \hat{d}_k - \frac{L_{max}}{C}$$

□