Lemma 1: If $\underline{r}^*$ is the optimal solution to (P) under (3), then for any feasible rate vector $\underline{r}$,

$$\sum_{\sigma \in S} \frac{r_\sigma - r_\sigma^*}{r_\sigma^{*\alpha}} \leq 0$$

($\underline{r}^*$ is called α-proportionally fair)

Proof:

$$\sum_\ell P_\ell \sum_{\sigma \in S_\ell} r_\sigma \leq \sum_\ell P_\ell C_\ell \quad \text{fr } (1)$$

$$= \sum_\ell P_\ell \sum_{\sigma \in S_\ell} r_\sigma^* \quad \text{fr } (5)$$

$$\Rightarrow 0 \geq \sum_\ell P_\ell \sum_{\sigma \in S_\ell} (r_\sigma - r_\sigma^*)$$

$$= \sum_\sigma \sum_{\ell \in \mathcal{L}_\sigma} P_\ell (r_\sigma - r_\sigma^*)$$

$$= \sum_\sigma (r_\sigma - r_\sigma^*) \left.\frac{\partial U}{\partial r_\sigma}\right|_{\underline{r}^*} \quad \text{fr } (4)$$

$$= \sum_\sigma (r_\sigma - r_\sigma^*)(r_\sigma^*)^{-\alpha} \quad \text{fr } (3) \quad \square$$

- to show $\alpha \to \infty$ gives the MMF alloc$^{\underline{n}}$, let $\underline{r}^*(\alpha)$ be the optimal solution to (P), & let $\underline{r}$ be a feasible rate vector s.t. $\underline{r} \neq \underline{r}^*$ & for some session j, $r_j > r_j^*(\alpha)$ for $\alpha \to \infty$

- from Lemma 1,

$$\sum_{\sigma \in S} \frac{r_\sigma - r_\sigma^*(\alpha)}{(r_\sigma^*(\alpha))^\alpha} \leq 0$$

$$\Rightarrow \frac{r_j - r_j^*(\alpha)}{(r_j^*(\alpha))^\alpha} \leq -\sum_{\sigma \neq j} \frac{r_\sigma - r_\sigma^*(\alpha)}{(r_\sigma^*(\alpha))^\alpha}$$

$$\Rightarrow 1 \leq -\sum_{\sigma \neq j} \frac{r_\sigma - r_\sigma^*(\alpha)}{(r_\sigma^*(\alpha))^\alpha} \frac{(r_j^*(\alpha))^\alpha}{r_j - r_j^*(\alpha)}$$

since $r_j - r_j^*(\alpha) > 0$

$$\leq \underbrace{\sum_{\sigma \neq j}}_{\text{positive terms only}} - \frac{r_\sigma - r_\sigma^*(\alpha)}{r_j - r_j^*(\alpha)} \left(\frac{r_j^*(\alpha)}{r_\sigma^*(\alpha)}\right)^\alpha$$

$\Rightarrow r_j^*(\alpha) \geq r_k^*(\alpha)$ for $\alpha \to \infty$ for some $k \neq j$, otherwise all RHS terms would go to 0 as $\alpha \to \infty$.

Also, since we consider only strictly positive terms on the RHS, & $r_j > r_j^*(\alpha)$ for $\alpha \to \infty$,

$$r_k < r_k^*(\alpha) \leq r_j^*(\alpha)$$

$\Rightarrow \underline{r}^*$ is MMF

Congestion control as a decentralized solution to resource allocation problem

- different types of congestion control mechanisms correspond to different utility functions & different distributed optimization algorithms
- TCP can be viewed as approximating a primal penalty function algorithm with utility function $\approx -\frac{1}{r_\sigma d_\sigma^2}$ for small RTT d_σ & small loss probability.
- Primal penalty function algorithm
  - Problem (P′):

$$\max_{\underline{r} \geq 0} V(\underline{r}) = \sum_{\sigma \in S} U_\sigma(r_\sigma) - \sum_{l \in \mathcal{L}} \int_0^{f_l(\underline{r})} h_l(y, \gamma_l)\, dy$$

  - $h_l(\cdot)$ is the penalty (or price) function for link l
  - assumption: for each $l \in \mathcal{L}$, $h_l(y, \gamma_l)$ is a continuous nondecreasing function of y & a decreasing function of γ_l, st.

$$\int_0^f h_l(y, \gamma_l)\, dy \to \infty \text{ as } f \to \infty$$

    → since we have assumed $U_\sigma(\cdot)$ is strictly concave, $V(\cdot)$ is strictly concave in $\underline{r}$
  - parameter γ_l is used to adapt the penalty according to congestion
  - assumption: $\forall \sigma \in S$, $U_\sigma(r_\sigma) \to -\infty$ as $r_\sigma \to 0$
    → optimal solution satisfies $r_\sigma > 0 \;\; \forall \sigma \in S$
  - optimal $\underline{r}$ satisfies

$$\frac{\partial V}{\partial r_\sigma} = 0 \quad \forall \sigma \in S$$

$$\Rightarrow \frac{\partial U}{\partial r_\sigma} - \underbrace{\sum_{l \in \mathcal{L}_\sigma} \underbrace{h_l(f_l(\underline{r}), \gamma_l)}_{p_l \text{ : link (marginal) price}}}_{q_\sigma \text{: session } \sigma \text{ route price}} = 0 \quad \forall \sigma \in S$$

  - decentralized iterative algorithm:
    - let $p_l(t) = h_l(f_l(\underline{r}(t)), \gamma_l(t))$

$$q_\sigma(t) = \sum_{l \in \mathcal{L}_\sigma} p_l(t)$$

      be the link & route prices at time t
    - for each $\sigma \in S$,

$$\frac{dr_\sigma}{dt} = k_\sigma(r_\sigma)\left(\frac{\partial U}{\partial r_\sigma} - q_\sigma(t)\right)^{+}_{r_\sigma} \;^{*} \quad (1)$$

      where $k_\sigma(r_\sigma)$ is any nondecreasing continuous function st. $k_\sigma(r_\sigma) > 0 \;\forall\; r_\sigma > 0$
    - for each $l \in \mathcal{L}$,

$$\frac{d\gamma_l}{dt} = \alpha_l\left(C_l - f_l(r(t))\right)^{+}_{\gamma_l}$$

      where $\alpha_l > 0$, &

$$^{*}\left(g(\gamma)\right)^{+}_{\gamma} = \begin{cases} g(\gamma) & \text{if } \gamma > 0, \text{ or } \gamma = 0 \;\&\; g(\gamma) \geq 0 \\ 0 & \text{if } \gamma = 0 \;\&\; g(\gamma) < 0 \end{cases}$$

      (if $C_l < f_l(r)$, $\gamma_l \downarrow$ → link price $\uparrow$s)