algorithm for computing the MMF rate vector

- start w/ all-zero rate vector $\underline{r} = \underline{0}$.
- for each step $k = 1, 2, \ldots$, define:
  - $\mathcal{L}^k$: set of links that are unsaturated (ie. $f_\ell(\underline{r}) < C_\ell$) at the start of step k
  - S^k: set of sessions that are unsaturated (do not pass through any saturated link) at the start of step k
  - n_ℓ^k: no. of sessions in S^k that use link ℓ
- at each step $k = 1, 2, \ldots$, all unsaturated sessions are incremented equally in rate until one or more additional links becomes saturated, i.e.

$$r_\sigma^k := \begin{cases} r_\sigma^{k-1} + \min\limits_{\ell \in \mathcal{L}^k} \left(\dfrac{C_\ell - f_\ell(\underline{r}^{k-1})}{n_\ell^k} \right) & \text{if } \sigma \in S^k \\ r_\sigma^{k-1} & \text{otherwise} \end{cases}$$

- end when S^k is empty

- At each step k, all sessions in S^k have the same rate; all sessions in S^k passing through a link that saturates in step k have at least as much rate as any other session on that link & hence are bottlenecked by that link

→ upon termination, each session has a bottleneck link, so by the earlier proposition, the final rate vector is MMF

Network Utility Optimization

- different types of applications have different utility vs rate functions

- let the utility of session σ a function of its allocated rate r_σ be $U_\sigma(r_\sigma)$

- Network resource allocation problem (P)

$$\max \sum_{\sigma \in S} U_\sigma(r_\sigma)$$

$$\text{subj to: } \sum_{\sigma \in S_\ell} r_\sigma \leq C_\ell \quad \forall \ell \in \mathcal{L} \quad (1)$$

$$r_\sigma \geq 0 \quad \forall \sigma \in S \quad (2)$$

- unique solution if utility functions are strictly concave
  - also, concavity leads to fairness
  - assumption: for each session $\sigma \in S$, $U_\sigma(r_\sigma)$ is a strictly concave, non-decreasing, continuously differentiable function
  - different choices for the utility functions lead to different resource allocations, eg.
    - $U(r) = r$ → maximum efficiency (total rate) allocation
    - $U(r) = -(h(r))^\alpha$, $\alpha \to \infty$ where $h(r)$ is a differentiable decreasing convex positive function for $r \geq 0$

      → max-min fair allocation

- General class of utility functions

$$U_\sigma(r_\sigma) = \frac{w_\sigma r_\sigma^{1-\alpha}}{1-\alpha} \qquad \alpha \geq 0, \ \alpha \neq 1$$

  - $w_\sigma = 1$, $\alpha = 0$ → $U_\sigma(r_\sigma) = r_\sigma$

    → maximum efficiency allocation
  - $w_\sigma = 1$, $\alpha \to \infty$

    → max-min fair allocation
  - α controls trade-off between efficiency & fairness: small α emphasizes efficiency & large α emphasizes fairness

- Analysis for $U_\sigma(r_\sigma) = \frac{r_\sigma^{1-\alpha}}{1-\alpha}$ (3)

  (Mo & Walrand 00, informal version)
  - the optimal solution of (P) exists & is unique (strictly concave objective function, linear constraints)
  - Let $L(\underline{r}, p) = \sum_{\sigma \in S} U(r_\sigma) + \sum_{\ell \in \mathcal{L}} p_\ell (C_\ell - \sum_{\sigma \in S_\ell} r_\sigma)$

    where p_ℓ are the dual variables

  - A feasible value $\underline{r}^*$ is optimal for (P) iff the Karush-Kuhn-Tucker (KKT) conditions hold

$$\left.\frac{\partial U}{\partial r_\sigma}\right|_{\underline{r}^*} - \sum_{\ell \in \mathcal{L}_\sigma} p_\ell = 0 \qquad \forall \sigma \in S \quad (4)$$

$$p_\ell (C_\ell - \sum_{\sigma \in S_\ell} r_\sigma^*) = 0 \qquad \forall \ell \in \mathcal{L} \quad (5)$$

$$p_\ell \geq 0 \qquad \forall \ell \in \mathcal{L} \quad (6)$$