Chapter 3.

Advanced Polarimetric Concepts

Since the introduction of polarimetric radar data in the 1980’s, many different analysis techniques have been investigated. Many of these are application specific. Here we shall discuss the theoretical background for many of these techniques, and compare the information that is provided by the different approaches. As we shall see, many of these techniques provide very similar information, with the result that the choice of analysis technique becomes more one of personal preference.

3.1. Vector-Matrix Duality of Scatterer Representation

In the previous Chapter we showed that the received power can be written in terms of the scatterer covariance matrix as follows

\[ P = \mathbf{A} \cdot [\mathbf{C}] \mathbf{A}^*; \quad [\mathbf{C}] = \mathbf{T} \mathbf{T}^* \]  

(3.1)

If we restrict ourselves to the backscatter direction where \( S_{hv} = S_{vh} \) then the usual forms for the antenna and scatterer vectors are \( \mathbf{\tilde{A}} = \left( \begin{array}{c} p_h^{rec} \ p_h^{tr} \\
 \sqrt{2} \left\{ p_h^{rec} \ p_v^{tr} \right. \end{array} \right) \) and \( \mathbf{\tilde{T}} = \left( \begin{array}{cc} S_{hh} & S_{hv} \\
 \sqrt{2} S_{hv} & S_{vv} \end{array} \right) \). The superscript * denotes complex conjugation and \( \sim \) denotes the transpose operation. The vector \( \mathbf{T} \) contains the same information as the original scattering matrix.

The covariance matrix as defined in this expression is a positive semi-definite Hermitian matrix. This means that all the eigenvalues are real, and that the eigenvectors are orthogonal. To prove that the eigenvalues have to be non-negative, recall that we can diagonalize the covariance matrix using a unitary transformation as follows

\[ [\mathbf{\Lambda}] = [\mathbf{U}]^{-1} [\mathbf{C}] [\mathbf{U}] \]  

(3.2)

Here \( [\mathbf{\Lambda}] \) is a 3x3 diagonal matrix containing the non-negative real eigenvalues of the covariance matrix, and \( [\mathbf{U}] \) is a 3x3 complex matrix with columns equal to the normalized eigenvectors of the covariance matrix. But since the covariance matrix is Hermitian, the matrix \( [\mathbf{U}] \) also satisfies

\[ [\mathbf{U}]^\dagger [\mathbf{U}] = [\mathbf{I}] \Rightarrow [\mathbf{U}]^\dagger = [\mathbf{U}]^{-1} \]  

(3.3)

where the \( \dagger \) sign denotes the adjoint (complex conjugate transpose) of the matrix. The normalized eigenvectors (or their complex conjugates) form an orthonormal basis, so we can write any antenna vector as a linear combination of these vectors.
Using (3.4) in (3.1), we find the following expression for the power

\[ P = [U]^{\dagger} \mathbf{B} \cdot [C][U] \mathbf{B} = [U]^{\dagger} [C][U] \mathbf{B} = |b_1|^2 \lambda_1 + |b_2|^2 \lambda_2 + |b_3|^2 \lambda_3 \geq 0 \]  

(3.5)

The received power must be non-negative for all antenna vectors, which means that all the eigenvalues must be non-negative.

In the case where the covariance matrix represents a single scatterer, \textit{i.e.} it was calculated from a scattering matrix as shown in (3.1), it is easy to show that the eigenvalues are

\[ \lambda_1 = S_{hh} S_{hh}^* + S_{vv} S_{vv}^* + 2S_{hv} S_{hv}^*, \lambda_2 = \lambda_3 = 0 \]  

(3.6)

In this case, two of the three eigenvalues are zero. This, in fact, is the test of whether one could calculate an equivalent scattering matrix from any given covariance matrix.

The covariance matrix characterization is particularly useful when analyzing multi-look radar images, since the covariance matrix of a multi-look pixel is simply the average covariance matrix of all the individual measurements contained in the multi-look pixel. Recall that multi-looking is performed by averaging the power from adjacent pixels together in order to reduce speckle. This averaging process can be written as

\[ \langle P \rangle = \frac{1}{MN} \sum_{j=1}^{M} \sum_{i=1}^{N} P_{ij} = \frac{1}{MN} \sum_{j=1}^{M} \sum_{i=1}^{N} \mathbf{A} \cdot [C_{ij}] \mathbf{A}^* = \mathbf{A} \cdot \langle [C] \rangle \mathbf{A}^* \]  

(3.7)

where the two subscripts denote averaging in the range and azimuth directions, respectively. The angular brackets \( \langle \rangle \) denote this spatial averaging. In general this average covariance matrix will have more than one non-zero eigenvalue. All eigenvalues must still be non-negative, however.

Cloude (1988) was the first to use the orthonormality of the eigenvectors of the covariance matrix (in the context of radar polarimetry) to propose the decomposition of the covariance matrix in terms of its eigenvalues and eigenvectors:

\[ \langle [C] \rangle = \sum_{i=1}^{3} \lambda_i \hat{e}_i \hat{e}_i^* \]  

(3.8)

The decomposition proposed by Cloude as shown in (3.8) is unique. That is, since the eigenvectors of the covariance matrix are orthogonal, they form a natural basis in which to express the scattering. In some sense, this breaks the covariance matrix into orthogonal components just like one normally would do for a vector. But just like a vector can be expressed in many different coordinate systems, the same can be said for a covariance matrix. For example, we could also choose to write the scattering matrix in a different basis, such as the Pauli basis, where the basis is formed by the following three matrices:
Any set of orthonormal vectors can be used when expressing the scattering matrix. For each case, the average covariance matrix can be decomposed in the form given by (3.8), but for each basis set, the eigenvectors will be different. The eigenvalues, however, are invariant under basis transformations.

Just like in the vector case, the choice of coordinate system depends on the application. In the case of the covariance matrix, this choice is often dictated by the fact that we are trying to interpret the total scattering in terms of known models or scattering mechanisms. For example, the three Pauli vectors represent scattering from a metallic trihedral corner reflector, a metallic dihedral corner reflector, and a metallic dihedral corner reflector rotated by 45 degrees about the line of sight, respectively, as discussed in Chapter 2.

The Cloude eigenvector decomposition is a special case of the general decomposition. While it is mathematically unique, its interpretation is not necessarily straightforward. The reason for this is that there is no guarantee that the eigenvectors will represent any known physical scattering mechanism directly. An additional complication comes from the fact that if this decomposition is done on every pixel in a multi-looked image, the eigenvectors that form the coordinate system for this decomposition may be different from pixel to pixel. This means that the coordinate system generally varies from pixel-to-pixel, unless all the covariance matrices have identical eigenvectors. Therefore the value of any eigenvalue may vary from pixel to pixel, and it is not easy to tell if the variation is due to the strength of the scattering, or to the fact that the eigenvectors are different. One common way to reduce this problem is to express the scattering vector in the Pauli basis and then calculate the equivalent covariance matrix and perform the decomposition in this basis. This does not really overcome the fundamental issue of the coordinate system varying from pixel to pixel, however. It only facilitates interpreting the eigenvectors in terms of Pauli vectors.

Many authors have proposed so-called target decomposition schemes in which a covariance matrix is decomposed into separate matrices based on simple models. In those cases, care must be taken to ensure that the individual matrices that are used in the decomposition all satisfy the condition that their individual eigenvalues must be non-negative. We shall discuss this further in a later Section.

Whether one thinks of a scatterer in terms of a covariance matrix or a set of scattering vectors is a matter of personal choice. In the rest of this Chapter we shall explore various interpretations of the average scattering based on different applications. Before doing so, we shall discuss a number of polarimetric parameters that are often encountered in the literature.

### 3.2. Eigenvalue and Eigenvector-Based Polarimetric Parameters

Cloude (1988) and later Cloude and Pottier (1996) introduced a number of polarimetric parameters that are derived from the eigenvalues and eigenvectors that are commonly
used today. Here we shall discuss these parameters, as well as others that are commonly encountered in the literature.

### 3.2.1 Parameters used to describe randomness in scattering

One such parameter, intended to measure target randomness, is the *entropy* defined as

\[
H_T = -\sum_{i=1}^{3} P_i \log_3 P_i; \quad P_i = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3}
\]

(3.9)

As pointed out by Cloude, the target entropy is a measure of target disorder, with \(H_T = 1\) for random targets with three equal eigenvalues and \(H_T = 0\) for simple (single, non-random) targets.

Recall from the discussion of polarization responses in Chapter 2 that the amount of variation (i.e. randomness) in the scattering properties manifests itself in the form of a “pedestal” in the polarization response. While we normally refer to the pedestal height in the context of the co-polarized response, Durden *et al.* (1988) showed that measuring the pedestal height is equivalent to measuring the ratio of the minimum eigenvalue to the maximum eigenvalue, i.e.

\[
\text{Pedestal Height} = \frac{\min(\lambda_1, \lambda_2, \lambda_3)}{\max(\lambda_1, \lambda_2, \lambda_3)}
\]

(3.10)

In reality, this ratio over-estimates the total variation in the observed radar cross-section as a function of polarization. The reason for this is that in order for this minimum and maximum to be realized, both eigenvectors corresponding to those eigenvalues must be valid antenna vectors (see (3.5)). This is not necessarily the case, possibly resulting in a smaller ratio. Nevertheless, this definition of the pedestal height is a useful measure of the randomness of the scattering process.

Using the model of randomly oriented thin cylinders (we will discuss this in more detail later) Kim and van Zyl (2001) introduced the so-called *radar thin vegetation index*

\[
RVI = \frac{4 \min(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{8\sigma_{hv}}{\sigma_{hh} + \sigma_{vv} + 2\sigma_{hv}}
\]

(3.11)

This parameter is also a measure of the randomness in the scattering, and generally varies between 0 and 1. As the cylinders become thick compared to the radar wavelength this ratio decreases. In the limiting case where the cylinders are very thick compared to the radar wavelength, this ratio approaches zero. The factor 4 in equation (3.13) is arbitrary; it was chosen so that the RVI for a cloud of randomly oriented thin cylinders would be equal to 1.

To illustrate the similarity between these parameters with real image data, we shall first consider the image of San Francisco we discussed before in Chapter 2. From the signatures shown in Figure 2-22, we expect little randomness for the ocean scattering, and significant randomness for the vegetation scattering in the Golden Gate Park area. The urban area should show intermediate randomness. Figure 3-1 shows the comparison of the three measures of randomness introduced above. What is immediately obvious is
that these three images convey the same basic information. The only real difference is the scaling from the bare ocean surface to the vegetated areas. The entropy image shows a more compressed scale with less variation in color than the other two. The pedestal height image based on the ratio of the eigenvalues shows the largest dynamic range. Overall, however, there is little reason to prefer one display over the other.

Note how both the entropy and the thin vegetation index show an increase in randomness in the ocean from left to right in the image. Recall that the radar illumination is from the left, and that the angle of incidence increases from left to right. The observed increase in randomness in the ocean is due to the decrease in signal-to-noise ratio as the angle of incidence increases. Also, note that not all urban areas show the same amount of randomness. This can be explained by the orientation of the buildings relative to the radar look direction. If a building is oriented such that it presents a large face to the radar, in general the dominant scattering mechanism is a double reflection off the street onto the face of the building and back to the radar. This is the case above and below Golden Gate Park in the image. If the building is turned slightly, however, such that the front face of the building no longer is orthogonal to the direction in which the radar waves propagate, this double reflection signal no longer travels back to the radar. The result is that other direct reflections from the street, and possibly the roofs of the buildings, begin to dominate, and the signals appear more random. This effect is visible in the middle right portion of the image.

**Figure 3-1.** L-Band randomness images of San Francisco acquired with the NASA/JPL AIRSAR system. Golden Gate Bridge is visible in the top center of the image linking the Presidio of San Francisco to the Golden Gate National Recreational Area north of the entrance to San Francisco Bay. Golden Gate Park is the rectangular feature in the lower half of the image in the left portion of the city. The image on the left shows the entropy scaled from 0 to 1, the middle image shows the pedestal height scaled from 0 to 0.5, and the image on the right shows the RVI scaled from 0 to 1.

As a second example we consider an image of a portion of the Black Forest in Germany acquired with the NASA/JPL AIRSAR system during the summer of 1991. The L-band image is shown for reference in Figure 3-2. The bright feature in the left portion of the image is the town of Vilingen. The brighter right-hand portion of the
image is a mixed forest consisting of spruce (*Picea abies*), pine (*Pinus sylvestris*) and fir (*Abies alba*) trees. The dry weight biomass ranges up to 50 kg/m². The darker areas in the upper portion and to the left in the image are mostly agricultural fields with varying amounts of biomass depending on the crop type and maturity of the plants.

![Image of a mixed forest with labels for Tannheim, Herzogenweiler, Pfaffenweiler, Clearing, and Forest, along with Agriculture, Villingen.](image)

**Figure 3-2.** L-Band total power image of a portion of the Black Forest in Germany acquired with the NASA/JPL AIRSAR system in the summer of 1991.

Figure 3-3 shows the three parameters calculated from the L-band data. In all cases, the forested areas show significant randomness, and the urban areas show little. The agricultural areas show variations consistent with the amount of vegetation present on a field scale.
Figure 3-4 compares the radar thin vegetation index for the three frequencies (C-Band, L-Band and P-band) that the AIRSAR system uses to acquire images. The C-band images show much more detail in the agricultural areas, because the shorter wavelength is more sensitive to the smaller biomass in these fields. The P-Band image, on the other hand, shows a large variation in the forested area. This is due to the increased penetration through the canopy at the longer wavelength with a resulting increase in double reflections from the ground to the trunks of the trees and back to the radar. The variation in RVI is due to the effect that the underlying topography has on the resulting mixture of scattering mechanisms as discussed by van Zyl (1993).

Figure 3-3. L-Band randomness images of the area shown in Figure 3-2. Forested areas show the highest amount of randomness.

Figure 3-4. Thin vegetation index images of the area shown in Figure 3-2 for different frequencies. See the text for a discussion of the images.
3.2.2 Alpha Angle

Cloude and Pottier (1996) proposed the following description for the eigenvectors of the covariance matrix:
\[
\mathbf{e} = \left( \cos \alpha \cos \beta e^{i\delta} \sin \alpha \sin \beta e^{i\gamma} \right)
\] (3.12)

In a basis formed by the Pauli matrices. The average angles are then calculated using
\[
\bar{\alpha} = \frac{1}{3} \sum_{i=1}^{3} \alpha_i P_i
\] (3.13)

Where \( P_i \) is defined in (3.9). The \( \alpha \) angle in particular has received significant attention, and together with the entropy has been proposed as a way to perform an unsupervised classification of polarimetric SAR images. To investigate the meaning of these angles a bit further, consider that the normalized eigenvector (expressed in Pauli basis) can be written as follows
\[
\mathbf{e} = \frac{e^{i\varphi}}{\sqrt{2(\sigma_{hh} + \sigma_{vv} + 2\sigma_{hv})}} \left( (S_{hh} + S_{vv}) \left( S_{hv} - S_{sv} \right) e^{-i\delta} \right) (3.14)
\]

where \( \sigma_{xy} = S_{xy} S_{xy}^* \) and the phase angle \( \varphi \) represents the phase of \( S_{hh} + S_{vv} \). A comparison of (3.14) and (3.12) now shows that the angle \( \delta \) is the relative phase between \( S_{hh} - S_{sv} \) and \( S_{hv} + S_{sv} \). Similarly, the angle \( \gamma \) is the relative phase between \( S_{hv} \) and \( S_{hh} + S_{sv} \). Next, notice that

\[
\tan \beta = \frac{2|S_{hv}|}{|S_{hh} - S_{sv}|} = \frac{2|S_{hv}/S_{sv}|}{|1 - S_{hv}/S_{sv}|} = \frac{2|R_s|}{|1 - R_{co}|}
\] (3.15)

The angle \( \beta \) is therefore a function of the ratio of the cross-polarized term to the co-polarized VV term in the scattering matrix, as well as the ratio of the two co-polarized terms. The angle \( \alpha \) can similarly be written as
\[
\tan \alpha = \sqrt{\left[1 - R_{co}\right]^2 + 4|R_s|^2} \quad |1 + R_{co}|
\] (3.16)

At this point it is useful to compare these parameters for a few canonical cases. These are summarized in the table below.

<table>
<thead>
<tr>
<th>Scattering</th>
<th>Matrix Elements</th>
<th>( R_{co} )</th>
<th>( R_s )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical</td>
<td>( S_{sv} \neq 0; S_{hh} = S_{hv} = 0 ),</td>
<td>0</td>
<td>0</td>
<td>( \pi/4 )</td>
<td>0</td>
</tr>
<tr>
<td>Dipole</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horizontal</td>
<td>( S_{hh} \neq 0; S_{vv} = S_{hv} = 0 ),</td>
<td>( \infty )</td>
<td>0</td>
<td>( \pi/4 )</td>
<td>0</td>
</tr>
<tr>
<td>Dipole</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$S_{hh} = S_{vv} = 1, S_{hv} = 0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>--------------------------</td>
<td>----------------------------------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td><strong>Trihedral</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Dihedral</strong></td>
<td>$S_{hh} = -S_{vv} = 1, S_{hv} = 0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$\pi/2$</td>
<td>$0$</td>
</tr>
<tr>
<td><strong>Dipole</strong></td>
<td>$S_{hh} = \cos^2 \psi$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$S_{vv} = \sin^2 \psi$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$S_{hv} = \sin \psi \cos \psi$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Slightly Rough Surface</strong></td>
<td>$S_{vv} \geq S_{hh} \cdot R_x \square 1$</td>
<td>$0 \leq R_{co} \leq 1$</td>
<td>$R_x \square 1$</td>
<td>$\approx \tan^{-1}\left(\frac{</td>
<td>1-R_{co}</td>
</tr>
<tr>
<td><strong>Dielectric Double Bounce</strong></td>
<td>$S_{vv} \leq S_{hh} \cdot R_x \square 1$</td>
<td>$0 \geq R_{co} \geq -1$</td>
<td>$R_x \square 1$</td>
<td>$\approx \tan^{-1}\left(\frac{</td>
<td>1-R_{co}</td>
</tr>
</tbody>
</table>

**Figure 3-5.** Alpha angle as a function of dielectric constant for a slightly rough dielectric surface for three different incidence angles. Note that as the incidence angle and the dielectric constant increases, the alpha angle also increases. At very large dielectric constants and incidence angles, the alpha angle will approach 45 degrees.
The results in Table 3.1 show that the angle $\alpha$ varies from zero degrees for trihedral scattering to $\pi/2$ for dihedral scattering. Dipole scattering represents an intermediate case where $\alpha = \pi/4$. Note that this value of $\alpha$ for dipole scattering does not depend on the physical orientation of the dipole. The angle $\beta$ is near zero for all cases where the cross-polarized return is small compared to the co-polarized returns. For the case of a single dipole, the $\beta$ angle is related to the physical orientation of the dipole.

Also note that when the cross-polarized term is small compared to the co-polarized ones, the angle $\alpha$ is basically proportional to the co-polarized ratio. For example, in the case of a bare slightly rough surface, this ratio is a function of the surface dielectric constant. Therefore, for such a surface, $\alpha$ would be a function of the dielectric constant. For wet surfaces at high incidence angles this ratio is very small, and $\alpha$ will approach $\pi/4$, which is the same value for dipole scattering. Figure 3-5 shows the alpha angle for different dielectric constants and different incidence angles. Note that as the surface dielectric constant increases, the alpha angle increases as discussed above. Also, as the angle of incidence increases, so does the alpha angle because of the change in the co-polarized ratio.

**Figure 3-6.** Alpha angle as a function of angle of incidence for a double reflection from two dielectric surfaces with the same dielectric constant (epsilon). At angles of incidence equal to the Brewster angle of the dielectric surfaces, the alpha angle will be 45 degrees. Note the large overlap with the values for the slightly rough surface in Figure 3-5.
In the case of double reflections from dielectric surfaces, the co-polarized ratio will approach infinity at the Brewster angle. For that case, the angle $\alpha$ will also approach $\pi/4$. Figure 3-6 illustrates this point further. Here we calculated the alpha angle for a dielectric dihedral reflection assuming that both surfaces have the same dielectric constant. When the dielectric constant becomes very large, the alpha angle approaches $\pi/2$, which is the expected value for a metallic dihedral. For low dielectric constants, however, the alpha angle is closer to 45 degrees. Note that for some dielectric constant and angle of incidence ranges the alpha angle can actually be less than $\pi/4$. This will mostly happen for low dielectric constant values at intermediate angles of incidence. However, these examples illustrate that care must be exercised when interpreting the values of $\alpha$.

![Image](image3-7)

**Figure 3-7.** Alpha angle for the San Francisco image. See the text for discussion.

To illustrate the meaning of the alpha angle in image data, we show the L-band alpha angle image for San Francisco in Figure 3-7. The alpha angle for the ocean is mostly less than 45 degrees, consistent with the expectation for a slightly rough surface. Note the increase in the alpha angle with increasing angle of incidence (the angle of incidence increases from left to right across the image) in the ocean, consistent with the predictions shown in Figure 3-5. The vegetated areas all show alpha angles near 45 degrees, consistent with dipole scattering. The urban areas consistently show alpha
angles larger than 45 degrees, consistent with the expectations for a double reflection signal from a non-metallic surface.

Figure 3-8. Multi-frequency alpha angle for the Black Forest image. See the text for discussion.

Figure 3-8 shows the alpha angles at different frequencies for the Black Forest image. As in the case of San Francisco, the urban areas consistently show alpha angles larger than 45 degrees. We also see the effect of frequency very clearly in the agricultural areas. At C-band, the alpha angle is mostly near 45 degrees, while at L- and P-band, the values are closer to zero, indicating bare (or nearly bare) surfaces. The forested areas at L- and P-band show alpha angles either near 45 degrees (especially at L-Band), or larger than 45 degrees where there is an appreciable amount of double reflection signal. The C-band image interestingly shows the alpha angle in the forested areas to be less than 45 degrees, and generally less than the values observed in the agricultural areas where there is vegetation. The explanation for this lies in the size of the branches relative to the radar wavelength. When we used the term “dipole” before, we could have substituted “cylinder that is thin compared to the radar wavelength.” Let us consider such a cylinder that is oriented vertically. In the thin cylinder limit, we will observe the scattering shown in Table 3.1. As the cylinder becomes thicker compared to the radar wavelength, we observe an increase in the HH term relative to the VV term. In fact, in the thick cylinder limit, the HH term will approach the VV term. Therefore, as the cylinder radius increase relative to the radar wavelength, the co-polarized ratio will increase from zero to 1 in the limit of a thick cylinder. Equation 3-18 predicts that this will cause the alpha angle to decrease from the 45 degree range as the cylinders increase in thickness relative to the radar wavelength. In these images, the cylinders are of fixed size, but the radar wavelength changes. At C-band, the radius of a given cylinder will be larger relative to the radar wavelength than at L-band. We would therefore expect to see a lower alpha angle at C-band than at L-band.
3.3. Decomposition of Polarimetric Scattering

Earlier in this Chapter we discussed the duality between representing a scatterer by its scattering matrix, or by an equivalent vector containing the elements of the scattering matrix. Here we shall explore this in greater detail.

First, let us consider the case of a single scatterer that can be represented by its scattering matrix. We shall further restrict our discussion to the backscatter case where reciprocity is assumed to hold. If we assume that the scattering matrix was measured in the linear basis, we can write

\[
\begin{pmatrix}
S_{hh} & S_{hv} \\
S_{hv} & S_{vv}
\end{pmatrix} = S_{hh} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + S_{hv} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + S_{vv} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

Or in the equivalent vector form:

\[
\begin{pmatrix} S_{hh} \\ S_{hv} \\ S_{vv} \end{pmatrix} = S_{hh} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + S_{hv} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + S_{vv} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

In writing the scattering matrix in vector form, we used an orthogonal basis to express the elements of the scattering vector. It should be immediately obvious that there are an infinite number of such orthogonal bases that one could choose to represent the scattering vector. Any combination of the form

\[
S = a_1 p_1 + a_2 p_2 + a_3 p_3
\]

would be permissible, as long as

\[
p_i \cdot p_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}
\]

The logical question is then whether there is any basis other than the one used to make the measurement to begin with that should be considered for such decomposition? The answer depends on the specific application, but the basis derived from the Pauli spin matrices has a nice intuitive interpretation. This basis is shown in equation Error! Reference source not found. in vector form, and in matrix form is

\[
\begin{pmatrix}
S_{hh} & S_{hv} \\
S_{hv} & S_{vv}
\end{pmatrix} = \frac{a}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{b}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

This decomposition was also used by Krogager (1993) in his thesis. The first two terms involve only the co-polarized elements of the scattering matrix, and can be interpreted as scattering by an odd number of reflections from a metallic structure, and an even number of reflections from a metallic structure, respectively. The first matrix represents therefore scattering from a flat plate, a sphere or a metallic trihedral corner reflector. The second matrix represent scattering from a metallic dihedral corner reflector.

The third matrix can be interpreted in different ways. Since it only involves the cross-polarized component of the scattering matrix, it is usually interpreted as indicating
the amount of random scattering. While this interpretation certainly has some merit in the practical sense that scattering from vegetated areas usually shows a large amount of cross-polarized return, it is not strictly correct from a theoretical point of view. The third matrix, as pointed out in Chapter 2, is also the scattering matrix of a dihedral corner reflector rotated by 45 degrees about the line of sight. The resulting scattered energy is fully polarized, but the polarization vector has been rotated. Admittedly, this is a special case. In most practical applications, a large cross-polarized component also typically is associated with significant depolarization of the scattered energy.

To illustrate the usefulness of this approach for interpreting scattering, we display a color image of San Francisco in Figure 3-9 in which we assigned the blue color as the magnitude of $a$ in equation (3.21), the red color as the magnitude of $b$ and the green color as the magnitude of $c$.

![Color overlay of San Francisco](image)

**Figure 3-9.** Color overlay of San Francisco displaying $|S_{hh} + S_{vv}|$ in blue, $|S_{hh} - S_{vv}|$ in red, and $|S_{hv}|$ in green. These three images are the magnitudes of the scattering matrix elements when they are expressed in the Pauli basis.

This image shows that the urban area shows a large fraction of the scattering in the red color, which corresponds to the dihedral component. The ocean, on the other hand, shows much more blue on the left, consistent with the single scattering mechanism. The vegetated areas, on the other hand, show significant cross-polarized return. Note the interesting change in color in the ocean from the left to the right in the image, where there is significantly more red visible in the ocean. The explanation for this lies in the fact that for a dielectric surface like the ocean, the co-polarization ratio is a function of both the dielectric constant and the angle of incidence. Figure 3-10 shows the expected co-
polarization ratio for a surface with a dielectric constant of 81, and it shows the co-
polarized ratio to be less than 1, especially at the larger angles of incidence. The Pauli
basis, however, forces the HH and VV terms to be equal. Therefore, if the co-polarized
ratio is less than 1, a dihedral component is needed to explain the difference between the
HH and VV terms. The smaller the co-polarization ratio, the stronger the dihedral
component required to explain the difference. This is shown as the dashed curve in
Figure 3-10.

![Figure 3-10](image_url)

**Figure 3-10.** Expected co-polarization ratio and the ratio of double reflection scattering
to single reflection scattering as a function of the angle of incidence for a surface with a
dielectric constant of 81. The values were calculated using the first order small
perturbation model and assuming the Pauli basis to calculate double and single reflection
components.

It is important to appreciate that the first order small perturbation model only
includes single scattering terms. The fact that the ocean scattering is *interpreted* to have
a significant amount of double reflections is only because of the basis that we have
chosen to interpret the scattering in. This basis forces the HH and VV components to be
the same for the “single scattering” term, leading to this interpretation. This is a
fundamental issue with many of the target decomposition schemes proposed in the
literature. All of these try to interpret scattering based on an assumption of an underlying
basis. If the scattering fits the basis, the interpretation is obviously appropriate. If the
basis is not consistent with the actual scattering, however, the *interpretation* should be
modified to take this fact into account.
Nevertheless, the Pauli basis, provides a good general purpose framework for interpreting polarimetric radar images, as the San Francisco example shows. Figure 3-11 shows the three-frequency Pauli images for the Black Forest image.

![Figure 3-11. Pauli basis color overlays for the Black Forest image. The color scheme is the same as that in Figure 3-9. The C-band image is on the left, the L-Band in the middle, and the P-Band on the right.](image)

These images show a consistent interpretation of the scattering from what we discussed before. The scattering from the randomly oriented vegetation is relatively strong in the cross-polarized return. The urban areas, on the other hand, are dominated by dihedral-type reflections at all wavelengths. The increased amount of penetration at P-band shows relatively speaking a stronger double reflection signal in much of the forest. The C-band signals interact more with the shorter agricultural crops than the longer wavelengths, resulting in increased cross-polarized returns in those areas.

The discussion so far was about the decomposition of the scattering matrix, or its associated vector form, into orthogonal components. As pointed out before, there is potentially an infinite set of bases we can use for this decomposition. A more important question is what about the case where we have an average covariance matrix? What is the most appropriate way to decompose this observed scattering into simpler parts? We shall discuss this in detail in the next Section.

### 3.3.1 Scattering Decomposition in the Incoherent Case using Orthonormal Bases

As mentioned in the beginning of this Chapter, after multi-looking to reduce speckle, we can write the average covariance matrix as

$$ \langle [C] \rangle = \frac{1}{MN} \sum_{j=1}^{M} \sum_{i=1}^{N} [C_{ij}] $$

(3.22)
where the two sums indicate averaging in the range and azimuth directions, respectively. We shall restrict our discussion to the backscatter case, where the individual covariance matrices are defined as

\[
[C] = \begin{pmatrix}
S_{hh} & \sqrt{2}S_{hv} & S_{vv} \\
\sqrt{2}S_{hv} & S_{hv}^* & 2S_{hv}S_{vv}^* \\
S_{vv} & \sqrt{2}S_{vh}S_{vv}^* & S_{vv}^*
\end{pmatrix} = \begin{pmatrix}
S_{hh}S_{hh}^* & \sqrt{2}S_{hh}S_{hv}^* & S_{hh}S_{vv}^* \\
\sqrt{2}S_{hv}S_{hv}^* & 2S_{hv}S_{sv}^* & \sqrt{2}S_{hv}S_{sv}^* \\
S_{vv}S_{hh}^* & \sqrt{2}S_{vh}S_{hh}^* & S_{vv}^*
\end{pmatrix}
\]

(3.23)

The covariance matrix is Hermitian. It therefore contains at most three independent complex numbers, and three real numbers, for a total of nine real numbers. The scattering matrix, on the other hand, contains at most three complex numbers. If we remove an absolute phase number from one of the elements of the scattering matrix, we would be left with one real and two complex numbers (a total of five real numbers). There must therefore be at least four relations between the elements of the covariance matrix of a single scatterer. These are

\[
\begin{align*}
C_{00}C_{11} - C_{01}C_{10} &= 2S_{hh}S_{hh}^*S_{hv}S_{hv}^* - \sqrt{2}S_{hh}S_{hv}^*\sqrt{2}S_{hv}S_{hv}^* = 0 \\
C_{00}C_{22} - C_{02}C_{20} &= S_{hh}S_{hv}^*S_{vv}S_{hv}^* - S_{hh}S_{hv}^*S_{vv}S_{hv}^* = 0 \\
C_{11}C_{22} - C_{12}C_{21} &= 2S_{hv}S_{hv}^*S_{vv}S_{hv}^* - \sqrt{2}S_{hv}S_{vv}^*\sqrt{2}S_{hv}S_{hv}^* = 0 \\
C_{00}C_{11}C_{22} - C_{01}C_{10}C_{22} &= 2S_{hh}S_{hh}^*S_{hv}S_{hv}^*S_{vv}S_{hv}^*S_{hv}S_{hv}^* - \sqrt{2}S_{hh}S_{hv}^*S_{hv}S_{hv}^*S_{hv}S_{hv}^* = 0
\end{align*}
\]

(3.24)

Once we perform the averaging process shown in equation (3.22) during the multi-looking process, these relations will no longer hold. Instead, the equal signs should be replaced with greater than or equal to signs. In fact, that is simply a statement of the Cauchy-Schwarz inequality applied to complex numbers. Therefore, in general,

\[
\begin{align*}
C_{00}C_{11} - C_{01}C_{10} &\geq 0 \\
C_{00}C_{22} - C_{02}C_{20} &\geq 0 \\
C_{11}C_{22} - C_{12}C_{21} &\geq 0 \\
C_{00}C_{11}C_{22} - C_{01}C_{10}C_{22} &\geq 0
\end{align*}
\]

(3.25)

Unless equality holds in each of the four cases, we cannot find an equivalent scattering matrix to fully represent the scattering described by the covariance matrix as shown in equation (3.23). A reasonable question then is whether we can find a set of scattering matrices which when transformed to their covariance matrices could be added to fully describe the observed covariance matrix. To answer this question, consider the decomposition proposed by Cloude (Cloude, 1988):

\[
\langle [C] \rangle = \sum_{i=1}^{3} \lambda_i \hat{e}_i \hat{e}_i^* \]

(3.26)

This decomposition is unique. That is, since the eigenvectors of the covariance matrix are orthogonal, they form a natural basis in which to express the scattering. There is one potential problem with this decomposition. Since the decomposition is performed for each covariance matrix, i.e. on a pixel-by-pixel basis in an image, the resulting basis for the decomposition changes from pixel to pixel. This could make it more difficult to
compare the meaning of a given eigenvalue in different areas without also looking at the associated eigenvectors that form the basis. To illustrate what we mean, consider the special case of the covariance matrix of terrain with reflection symmetry. In that case, the covariance matrix has the special form (cf Borgeaud et al., 1985):

\[
\begin{bmatrix}
\xi & 0 & \rho \\
0 & \eta & 0 \\
\rho^* & 0 & \zeta
\end{bmatrix}
\]

(3.27)

where

\[
\xi = \langle S_{hh}^* S_{hh} \rangle \\
\rho = \langle S_{hh}^* S_{vv} \rangle \\
\eta = 2 \langle S_{hv}^* S_{hv} \rangle \\
\zeta = \langle S_{vv}^* S_{vv} \rangle
\]

(3.28)

The parameters \( \xi, \eta, \zeta \) and \( \rho \) all depend on the size, shape and electrical properties of the scatterers, as well as their statistical angular distribution. It is easily shown that the eigenvalues of \( \langle [C] \rangle \) are

\[
\lambda_1 = \frac{1}{2} \left( \xi + \zeta + \sqrt{(\xi - \zeta)^2 + 4|\rho|^2} \right)
\]

\[
\lambda_2 = \frac{1}{2} \left( \xi + \zeta - \sqrt{(\xi - \zeta)^2 + 4|\rho|^2} \right)
\]

\[
\lambda_3 = \eta
\]

(3.29)

All these are real numbers, as expected for a Hermitian matrix. It follows from inspection that the first and third eigenvalues are positive. To show that the second is also, note that from the second relationship in (3.25)

\[
\xi \zeta \geq |\rho|^2 \Rightarrow (\xi - \zeta)^2 + 4|\rho|^2 \leq (\xi + \zeta)^2
\]

(3.30)

and hence the second eigenvalue is also positive.
The corresponding three eigenvectors are

\[
\mathbf{k}_1 = \sqrt{\frac{[\xi - \xi + \sqrt{\Delta}]^2}{[\xi - \xi + \sqrt{\Delta}]^2 + 4|\rho|^2}} \begin{pmatrix} 2\rho / [\xi - \xi + \sqrt{\Delta}] \\ 0 \\ 1 \end{pmatrix}
\]

\[
\mathbf{k}_2 = \sqrt{\frac{[\xi - \xi - \sqrt{\Delta}]^2}{[\xi - \xi - \sqrt{\Delta}]^2 + 4|\rho|^2}} \begin{pmatrix} 2\rho / [\xi - \xi - \sqrt{\Delta}] \\ 0 \\ 1 \end{pmatrix}
\]

\[
\mathbf{k}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\]

(3.31)

In these expressions, we used the shorthand notation

\[
\Delta = (\xi - \bar{\xi})^2 + 4|\rho|^2
\]

(3.32)

We note that \( \Delta \) is always positive. Also note that we can write the ratio of the first elements of the first two eigenvectors as

\[
\frac{k_{21}}{k_{31}} = \sqrt{\frac{[\xi - \xi + \sqrt{\Delta}]^2/[\xi - \xi - \sqrt{\Delta}]^2 + 4|\rho|^2}{[\xi - \xi - \sqrt{\Delta}]^2/[\xi - \xi + \sqrt{\Delta}]^2 + 4|\rho|^2}} \frac{\xi - \xi - \sqrt{\Delta}}{4|\rho|^2}
\]

(3.33)

which is always negative. This means that the first two eigenvectors represent scattering matrices that can be interpreted in terms of odd and even numbers of reflections. Without looking explicitly at the eigenvectors, however, we will not know which eigenvalue to associate with which scattering mechanism.

Figure 3-12 illustrates this with an example. On the left we display the image of San Francisco previously discussed with the three eigenvalues as defined in (3.29) colored blue for the first eigenvalue, red for the second, and green for the third. Note that the third eigenvalue and eigenvector are identical to the third Pauli element encountered before. The ocean is dominated by a blue color, indicating that the scattering is dominated by whatever scattering mechanism the first eigenvector represents. The urban areas, however, also is dominated by a blue color, indicating the same. Vegetated areas consistently show a green color indicating depolarization is relatively speaking high. On the right we show the same image, but this time we examined the eigenvectors for each
pixel, and if the co-polarized phase is in the range $[-90^\circ;90^\circ]$, we assign a blue color to the eigenvalue corresponding to that eigenvector. If the co-polarized phase is in the range $[90^\circ;270^\circ]$, we assign a red color. We already showed that the phase difference between the first two eigenvectors is 180 degrees, so once we identified the color for the first eigenvector, the color for the second is automatically determined. This image is quite different from the one on the left. In particular, the urban area is now dominated by red, indicating scattering more consistent with double reflections.

**Figure 3-12.** Two color overlays for the image of San Francisco. In the image on the left, colors are assigned to the three eigenvalues without further examining the eigenvectors. The first eigenvalue is colored blue, the second red, and the third green. In the image on the right, the co-polarized phase of the first eigenvector is used to determine the color of the first eigenvalue. If the phase is more consistent with odd numbers of reflections, the eigenvalue is colored blue. Otherwise it is colored red. The color of the second eigenvector is determined using the fact that the two co-polarized phases of the first two eigenvectors are 180 degrees different.

Comparing the results on the right in Figure 3-12 with those in Figure 3-9, we note many similarities between the Pauli basis display and the eigenvalue display. The main difference is that the eigenvalue display colors appear more pure. Also, note that the gradient from blue to red in the ocean is not as pronounced in the image on the right in Figure 3-12. The reason is that while the Pauli basis insists that the co-polarized terms have the same amplitude, the eigenvectors do not – see equation (3.31).

A natural question is whether there is a preferred basis in which to do the decomposition? In general, we could also choose to write the average covariance matrix in the following form
The Pauli and the eigenvalue bases are two special cases of this decomposition. The answer to this question is not obvious. It depends largely on the goal of the analysis. As our discussion shows, the decomposition is not the hard part; interpreting the results is. This desire to be able to interpret the results of such decomposition is what led many researchers to propose decompositions based on specific models, rather than orthogonal bases as discussed so far. We shall look at this in more detail in the next Section.

3.3.2 Model Based Scattering Decomposition in the Incoherent Case

The basic idea behind model based decompositions is to hypothesize that the measured covariance matrix can be modeled as the combination of a number of individual matrices representing scattering as predicted by models. In this Section we shall examine a number of different model based decomposition schemes in more detail.

3.3.2.1 Freeman-Durden Three Component Scattering Decomposition

For vegetated terrain, we could hypothesize that the dominant scattering mechanisms might be direct scattering from randomly oriented branches, plus double reflections from the ground/trunk combination, plus direct (although attenuated) scattering from the underlying ground surface. This is the basic idea behind the three-component scattering decomposition proposed by Freeman and Durden (1998). This decomposition can be written as follows

$$ \langle [C] \rangle = f_s [C_{\text{ground}}] + f_d [C_{\text{trunk-ground}}] + f_v [C_{\text{branches}}] $$

(3.35)

with

$$ [C_{\text{ground}}] = \begin{pmatrix} |\beta|^2 & 0 & \beta \\ 0 & 0 & 0 \\ \beta^* & 0 & 1 \end{pmatrix} $$

$$ [C_{\text{trunk-ground}}] = \begin{pmatrix} |\alpha|^2 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha^* & 0 & 1 \end{pmatrix} $$

(3.36)

$$ \langle [C_{\text{branches}}] \rangle = \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 2/3 & 0 \\ 1/3 & 0 & 1 \end{pmatrix} $$
The matrix representing branch scattering assumes that the branches are thin compared to the radar wavelength, and that the branches are uniformly randomly oriented. We discussed this case before in Chapter 2. From (3.35) one can then derive the following four equations:

\[
\begin{align*}
\langle |S_{hh}|^2 \rangle &= f_s |\beta|^2 + f_d |\alpha|^2 + f_v \\
\langle |S_{vv}|^2 \rangle &= f_s + f_d + f_v \\
\langle |S_{hv}|^2 \rangle &= f_v / 3 \\
\langle |S_{hh}^* S_{vv}^*| \rangle &= f_v \beta + f_d \alpha + f_v / 3
\end{align*}
\]

(3.37)

Freeman and Durden point out that there are four equations and five unknowns. They then make the following crucial suggestion: since neither the ground reflection nor the double reflection terms add to the predicted cross-polarized return, they can use the measured cross-polarized return to solve for the parameter \( f_v \). They continue to suggest that the volume contribution can then be subtracted from the measured matrix before solving for the other terms. In other words, we can write (3.35) as

\[
\begin{bmatrix}
3 \langle |S_{hv}|^2 \rangle & 0 & \langle |S_{hh}|^2 \rangle \\
0 & 2 \langle |S_{hv}|^2 \rangle & 0 \\
\langle |S_{hh}|^2 \rangle & 0 & 3 \langle |S_{hv}|^2 \rangle
\end{bmatrix}
= f_s \begin{bmatrix} C_{\text{ground}} \end{bmatrix} + f_d \begin{bmatrix} C_{\text{trunk-ground}} \end{bmatrix}
\]

(3.38)

Once the subtraction has been done, there are three remaining equations in four unknowns

\[
\begin{align*}
\langle |S_{hh}|^2 \rangle' &= f_s |\beta|^2 + f_d |\alpha|^2 \\
\langle |S_{hv}|^2 \rangle' &= f_s + f_d \\
\langle |S_{hh}^* S_{vv}^*| \rangle' &= f_v \beta + f_d \alpha
\end{align*}
\]

(3.39)

The primes on the left serve to remind us that these are the quantities after the volume scattering contribution have been subtracted. The phase of the remaining co-polarized component is then used to fix either \( \alpha \) or \( \beta \). The argument is that if the residual co-polarized phase is closer to zero than to \( \pi \), surface scattering dominate, and therefore we should solve for \( \beta \) explicitly. Therefore, we set \( \alpha = -1 \) (indicating a double reflection) and solve for the remaining parameters. On the other hand, if the residual co-polarized phase is closer to \( \pi \) than to zero, double reflection scattering dominate, and therefore we should solve for \( \alpha \) explicitly. In this case, we set \( \beta = 1 \) (indicating a single reflection) and solve for the remaining parameters.
Figure 3-13 shows a color overlay image of the three contributions calculated using the Freeman and Durden model for the image of the Black Forest at L-Band. The image on the left shows the relative strength of the three scattering mechanisms in the color code indicated. Overall, the image clearly shows volume scattering to dominate in the vegetated areas, double reflections to dominate in the urban areas, and some of the agricultural areas to show surface scattering. At this qualitative level, the results appear consistent with our expectations. The image is also similar to the Pauli basis image shown in the middle of Figure 3-11. But a deeper examination shows a significant flaw in this decomposition. The image on the right shows the results of an analysis of the eigenvalues of the matrix on the left in equation (3.38) after we subtracted the vegetation contribution from the original data. If any of the eigenvalues are negative, we blanked out the pixel in the image on the right in Figure 3-13. This surprising result shows that the majority of the pixels in the vegetated area end up with negative eigenvalues after we subtracted the scattering from the vegetation as suggested by Freeman and Durden (1998). But this is exactly where we expect the scattering model for scattering from the vegetation to be most applicable! We shall show later that this surprising result is the consequence of assigning all the cross-polarized return to the vegetation scattering.

**Figure 3-13.** Two color overlays displaying the results of the Freeman and Durden decomposition applied to the L-band image of the Black Forest. The image on the left displays the surface contribution in blue, the double reflection contribution in red, and the vegetation contribution in green. The image on the right shows only those pixels with non-negative eigenvalues once the vegetation contribution has been subtracted. See the text for more discussion.
3.3.2.2 Four Component Model Proposed by Yamaguchi et al.

The three component model described above assumes that the terrain has reflection symmetry by ignoring the terms in the covariance matrix that involves products of co-polarized and cross-polarized terms. While this assumption seems to be valid for most types of terrain in the sense that these components of the covariance matrix are much smaller than the others, one cannot always assume that reflection symmetry will hold. Yamaguchi et al. (2005) recognized this fact and proposed an extension of the three-component model to include a term that would account for the non-zero products of co-polarized and cross-polarized terms. Their decomposition is written as

\[
\langle [C] \rangle = f_s \left[ C_{\text{ground}} \right] + f_d \left[ C_{\text{trunk-ground}} \right] + f_c \left[ C_{\text{branches}} \right] + f_h \left[ C_{\text{helix}} \right]
\]  (3.40)

with the matrix representing the helix scattering taking one of the following two forms:

\[
\begin{bmatrix}
1 & j\sqrt{2} & -1 \\
-j\sqrt{2} & 2 & j\sqrt{2} \\
-1 & -j\sqrt{2} & 1
\end{bmatrix} ; \quad \begin{bmatrix}
1 & -j\sqrt{2} & -1 \\
j\sqrt{2} & 2 & -j\sqrt{2} \\
-1 & j\sqrt{2} & 1
\end{bmatrix}
\]  (3.41)

Note that these two matrices both predict the cross products \( \langle S_{hh}S_{hv}^* \rangle \) and \( \langle S_{hv}S_{vv}^* \rangle \) to be purely imaginary numbers. This may not be the case in observed data. To get around this, Yamaguchi et al. (2005) recommend using only the imaginary portion of the observed cross-products in the decomposition. Furthermore, the model matrices predict that \( \langle S_{hh}S_{hv} \rangle = \langle S_{hv}S_{vv} \rangle \). Again, this may not be exactly what is observed. Therefore, they recommend using

\[
\frac{f_h}{4} = \frac{1}{2} \left| \text{Im} \left( \langle S_{hh}S_{hv}^* \rangle + \langle S_{hv}S_{vv}^* \rangle \right) \right| 
\]  (3.42)

Finally, to decide which matrix in equation (3.41) to use, they propose

\[
\begin{cases}
\text{if } \text{Im} \left( \langle S_{hh}S_{hv}^* \rangle + \langle S_{hv}S_{vv}^* \rangle \right) > 0 \Rightarrow \text{use } C_{r-helix} \\
\text{Im} \left( \langle S_{hh}S_{hv}^* \rangle + \langle S_{hv}S_{vv}^* \rangle \right) < 0 \Rightarrow \text{use } C_{l-helix}
\end{cases}
\]  (3.43)

Assuming a uniformly oriented canopy, and writing out the covariance matrices in equation (3.40), we find the following five equations with six unknowns:
The unknown quantities are $\alpha, \beta, f_s, f_d, f_v$ and $f_c$. To solve for these unknowns, note that the last expression allows us to find $f_c$ as follows

$$f_c = 2 \left| \text{Im} \left( \langle S_{hh}S_{hv}^* \rangle + \langle S_{hv}S_{vv}^* \rangle \right) \right|$$

(3.45)

We can then use the second equation in (3.44) to find $f_v$.

$$f_v = 8 \left( \langle |S_v|^2 \rangle - \frac{f_c}{4} \right)$$

(3.46)

Once these two unknowns are determined, the contributions of the helix scattering and the volume scattering can be subtracted from the observed covariance matrix. Solving for the remaining unknowns than follows the same procedure as outlined by Freeman and Durden as discussed in the previous Section. Specifically,

$$f_s \left[ \mathbf{C}_{\text{ground}} \right] + f_d \left[ \mathbf{C}_{\text{trunk-ground}} \right] = \langle |\mathbf{C}| \rangle - \frac{f_c}{8} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} - \frac{f_v}{4} \begin{pmatrix} 1 & \pm j\sqrt{2} & -1 \\ \mp j\sqrt{2} & 2 & \pm j\sqrt{2} \\ -1 & \mp j\sqrt{2} & 1 \end{pmatrix}$$

(3.47)

Yamaguchi et al. (2005) went one step further than the Freeman and Durden decomposition in that they recognized that not all vegetated terrains are well represented by a uniformly oriented canopy. In some cases, the orientations are preferentially vertical, while in others it may be preferentially horizontal. For these cases, they propose to use cosine squared distributions around either vertical or horizontal directions, which leads to the following covariance matrices

$$\begin{pmatrix} 8 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 8 \end{pmatrix}$$

(3.48)

In their decomposition algorithm, they use the ratio of the VV to HH cross-sections to decide which canopy model to use. If the ratio of VV to HH power is less than -2 dB, they use the cosine squared distribution around the horizontal direction. If the ratio is
between -2 dB and +2 dB, they use the uniform distribution, and when it is larger than +2 dB, they use the cosine squared distribution around the vertical direction. In the case where we use a distribution that is preferentially horizontal, the expressions in (3.44) need to be modified as follows:

\[
\begin{align*}
\langle |S_{hh}|^2 \rangle &= f_s |\beta|^2 + f_d |\alpha|^2 + \frac{8}{15} f_v + \frac{1}{4} f_c \\
\langle |S_{vh}|^2 \rangle &= \frac{2}{15} f_v + \frac{1}{4} f_c \\
\langle |S_{vv}|^2 \rangle &= f_s + f_d + \frac{3}{8} f_v + \frac{1}{4} f_c \\
\langle S_{hh} S_{vv}^* \rangle &= f_s \beta + f_d \alpha + \frac{3}{15} f_v - \frac{1}{4} f \\
\frac{1}{2} \text{Im} \left\{ \langle S_{hh} S_{hh}^* \rangle + \langle S_{hh} S_{vv}^* \rangle \right\} &= \frac{f_c}{4}
\end{align*}
\]

The helix component is still determined by equation (3.45), but we now have to modify how we determine the volume component:

\[
f_v = \frac{15}{2} \left( \langle |S_{hh}|^2 \rangle - \frac{f_c}{4} \right)
\]  

(3.50)

The surface and double reflection components are then determined using the Freeman and Durden approach after the following subtraction

\[
f_s \left[ C_{\text{ground}} \right] + f_d \left[ C_{\text{trunk-ground}} \right] = \langle [C] \rangle - \frac{f_s}{15} \begin{pmatrix} 8 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 3 \end{pmatrix} - \frac{f_d}{4} \begin{pmatrix} 1 & \pm j\sqrt{2} & -1 \\ \mp j\sqrt{2} & 2 & \pm j\sqrt{2} \\ -1 & \mp j\sqrt{2} & 1 \end{pmatrix}
\]  

(3.51)

In the case where the ratio of VV to HH is larger than +2 dB, we assume a preferentially vertical orientation, the helix component is determined as before, and the volume component is determined by (3.50). The surface and double reflection components are determined after the following subtraction

\[
f_s \left[ C_{\text{ground}} \right] + f_d \left[ C_{\text{trunk-ground}} \right] = \langle [C] \rangle - \frac{f_s}{15} \begin{pmatrix} 3 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 8 \end{pmatrix} - \frac{f_d}{4} \begin{pmatrix} 1 & \pm j\sqrt{2} & -1 \\ \mp j\sqrt{2} & 2 & \pm j\sqrt{2} \\ -1 & \mp j\sqrt{2} & 1 \end{pmatrix}
\]  

(3.52)

The Yamaguchi et al. (2005) algorithm can be summarized as follows:

1. Estimate the helix scattering component using equation (3.45).
2. Depending on the ratio of VV to HH, use equations (3.46) (-2 dB < VV/HH < 2 dB) or (3.50) (VV/HH < -2 dB or VV/HH > +2 dB) to estimate the volume component.
3. Subtract the helix and volume components from the observation using equations (3.47) when \(-2 \text{ dB} < \text{VV/HH} < 2 \text{ dB}\), (3.51) when \(\text{VV/HH} < -2 \text{ dB}\) or (3.52) when \(\text{VV/HH} > +2 \text{ dB}\).

4. Use equation (3.39) and the process described by Freeman and Durden to estimate the strength of the surface and double reflection scattering terms.

**Figure 3-14.** Two color overlays displaying the results of the Yamaguchi *et al.* decomposition applied to the L-band image of the Black Forest. The image on the left displays the surface contribution in blue, the double reflection contribution in red, and the vegetation contribution in green. The image on the right shows only those pixels with non-negative eigenvalues once the vegetation contribution has been subtracted. See the text for more discussion.

Figure 3-14 shows the results of applying the Yamaguchi *et al.* (2005) decomposition to the L-band image of the Black Forest. The image on the left shows the relative strength of the volume, double bounce and surface scattering with the same color scheme that we used for the Freeman and Durden decomposition in Figure 3-13. Also shown in the figure on the right are the pixels with negative eigenvalues after the helix and volume components have been subtracted blanked out. Comparing Figures 3-14 and 3-13, we note that the qualitative results of the decompositions are very similar. This is not surprising, since the helix components are typically much smaller than the others. The major difference between the two decomposition methods lies in the number of pixels with negative eigenvalues. The Yamaguchi *et al.* (2005) decomposition results show
significantly fewer pixels with negative eigenvalues. It should be pointed out that Yamaguchi et al. (2005) recognized that areas where the HV returns exceed approximately half that of HH or VV could lead to negative powers in the decomposition. Their solution was to apply a slightly different algorithm to pixels that show this high relative value of HV. In the next Section we shall describe a systematic way of determining the level of the volume scattering, while still ensuring that no negative powers will result.

Figure 3-15. The image on the left displays the relative strength of the helix term in the Yamaguchi et al. decomposition as a fraction of the total power. While the helix term is slightly stronger in the vegetated areas, it is relatively small compared to the overall scattering. The image on the right shows the strength of the helix term when it is not normalized. The arrows indicate areas where the strength of the helix term is modulated by the topographic slopes in the along-track direction. Images were calculated at L-band.

Figure 3-15 shows the relative strength of the helix term in the Yamaguchi et al. decomposition. Notice that the helix terms are stronger in the vegetated areas, but overall is still relatively small, rarely exceeding 10% of the scattering. The image in the right shows the strength of the helix term when it is not normalized by the total power. It is interesting to note that the areas indicated by the arrows show modulations that appear to be consistent with the along-track slopes caused by the local topography. This image was analyzed by van Zyl (1993) who showed that the scattering mechanisms at P-band are
strongly modulated by the topographic slopes in the range direction. The fact that the strength of the helix term might be influenced by the along-track slopes should not be a surprise, because it is well known that along-track tilts lead to non-zero correlations between co- and cross-polarized components of the scattering matrix (Shuler et al., 1996; Lee et al., 2000), even for terrains that otherwise would exhibit reflection symmetry.

To better understand how azimuth slopes affect the co- and cross-polarized correlation, consider the analysis shown in Appendix A for a tilted surface. We show that the scattering matrix of the tilted surface $S(\theta)$ can be written as a transformation of the scattering matrix of the surface without tilts $S_i(\theta_i)$ as follows

$$S(\theta) = \mathbf{T}S_i(\theta_i)\mathbf{T}^{-1}$$

(3.53)

The angles refer to the angle of incidence in either the global coordinate system or the local coordinate system for the surface. For details, please see the appendix. The transformation matrix takes the form of a coordinate rotation

$$\mathbf{T} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}; \quad \tan \varphi = \frac{h_x}{u}; \quad u = \sin \theta - h_z \cos \theta$$

(3.54)

Where $h_x$ and $h_z$ are the surface slopes in the range (cross-track) and azimuth (along-track) directions, respectively. Note that these expressions are completely general; no assumptions are made about the form of the scattering matrix. In the special case where the range slopes are small, the rotation is directly proportional to the azimuth slope.

Performing the transformation, we find that

$$\langle S_{hh} (\varphi) S_{hv}^* (\varphi) \rangle + \langle S_{hv} (\varphi) S_{vv}^* (\varphi) \rangle = \frac{1}{2} \sin 2\varphi \left( \langle S_{hh} S_{hh}^* \rangle - \langle S_{vv} S_{vv}^* \rangle \right)$$

$$+ \frac{1}{2} (1 + \cos 2\varphi) \left( \langle S_{hh} S_{hv}^* \rangle - \langle S_{hv} S_{hv}^* \rangle \right)$$

$$- \frac{1}{2} (1 - \cos 2\varphi) \left( \langle S_{hv} S_{hv}^* \rangle - \langle S_{vv} S_{hv}^* \rangle \right)$$

(3.55)

If the terrain exhibits reflection symmetry, the second and third terms are zero and this quantity becomes a real number. In that case, the estimate of the helix component as given by equation (3.45) will be zero. If the terrain does not exhibit reflection symmetry, or if we only average over a relatively small number of pixels (recall that reflection symmetry only says that in the average the co-cross product will be zero) the estimate of the helix component will be

$$f_c = 2 \left| \text{Im} \left\{ \langle S_{hh} (\varphi) S_{hv}^* (\varphi) \rangle + \langle S_{hv} (\varphi) S_{vv}^* (\varphi) \rangle \right\} \right|$$

$$= \left| \text{Im} \left\{ (1 + \cos 2\varphi) \left( \langle S_{hh} S_{hv}^* \rangle - \langle S_{hv} S_{hv}^* \rangle \right) - (1 - \cos 2\varphi) \left( \langle S_{hv} S_{hv}^* \rangle - \langle S_{vv} S_{hv}^* \rangle \right) \right\} \right|$$

(3.56)

This quantity is clearly modulated by the along-track slopes, so the results in the image on the right in Figure 3-15 are not surprising.
3.3.2.3 The Non-Negative Eigenvalue Decomposition (NNED)

The results in the previous two Sections show a significant flaw in these decompositions; some negative powers may result after subtraction of the volume components from the observation. This is clearly a non-physical result. If the hypothesis is that the observed radar cross-section is the linear sum of radar cross-sections representing different types of scattering mechanisms, a crucial requirement is that each scattering mechanism must represent a physically realizable scatterer or collection of scatterers. This must also mean that the radar cross-section representing each scattering mechanism must be zero or positive for all polarization combinations. As shown in equation (3.5), this implies that all eigenvalues of the matrices representing the individual scattering processes must be non-negative. A decomposition method that takes this property into account was proposed by van Zyl et al. (2008), which we shall call the non-negative eigenvalue decomposition (NNED).

To explain the results on the right in Figure 3-11 and to introduce the NNED technique, let us take a more generic decomposition where we want to express the scattering as follows

\[ \langle [C] \rangle = a[C_{\text{model}}] + [C_{\text{remainder}}] \]  \hspace{1cm} (3.57)

Here, the first term on the right represents the covariance matrix predicted by some model such as randomly oriented branches. Recognizing that the form of this covariance matrix may be different from the measured matrix, we add the second term, which will contain whatever is in the measured matrix that is not consistent with the model matrix. The question now is what value of \( a \) to use in equation (3.57). To answer this question, we need to recognize that all matrices in equation (3.57) must represent physically realizable covariance matrices. That is, if we look at each matrix by itself, it should satisfy all the restrictions that we expect for a measured covariance matrix. In particular, we need to insist that the eigenvalues for each matrix are real and greater than or equal to zero. Let us rewrite equation (3.57) in the form

\[ [C_{\text{remainder}}] = \langle [C] \rangle - a[C_{\text{model}}] \]  \hspace{1cm} (3.58)

The matrix on the left must have eigenvalues that are real and are larger than or equal to zero. This requirement allows us to derive a limit on the values of \( a \). The largest value of \( a \) that still ensures that all three eigenvalues of the matrix on the left would be greater than or equal to zero, is the maximum value of \( a \) that we could use in equation (3.57).

To derive the general expressions limiting the values of \( a \), we start with the average covariance matrix for terrain with reflection symmetry as given in (3.27), and write the model covariance matrix as

\[ [C_{\text{model}}] = \begin{pmatrix} \xi_a & 0 & \rho_a \\ 0 & \eta_a & 0 \\ \rho_a^* & 0 & \xi_a \end{pmatrix} \]  \hspace{1cm} (3.59)
Then equation (3.58) becomes

\[
[C_{\text{rem}}] = \begin{pmatrix} \xi & 0 & \rho \\ 0 & \eta & 0 \\ \rho^* & 0 & \zeta \end{pmatrix} - a \begin{pmatrix} \xi_a & 0 & \rho_a \\ 0 & \eta_a & 0 \\ \rho^*_a & 0 & \zeta_a \end{pmatrix}
\]  (3.60)

The eigenvalues for this matrix are the roots of the following equation

\[
(\eta - a\eta_a - \lambda) \left\{ \lambda^2 - (\xi + \zeta - a\xi_a - a\zeta_a)\lambda + (\xi - a\xi_a)(\zeta - a\zeta_a) - |\rho - a\rho_a|^2 \right\} = 0  (3.61)
\]

which are

\[
\lambda_1 = \frac{1}{2} \left\{ \xi + \zeta - a\xi_a - a\zeta_a + \sqrt{(\xi + \zeta - a\xi_a - a\zeta_a)^2 - 4(\xi - a\xi_a)(\zeta - a\zeta_a) + 4|\rho - a\rho_a|^2} \right\}
\]

\[
\lambda_2 = \frac{1}{2} \left\{ \xi + \zeta - a\xi_a - a\zeta_a - \sqrt{(\xi + \zeta - a\xi_a - a\zeta_a)^2 - 4(\xi - a\xi_a)(\zeta - a\zeta_a) + 4|\rho - a\rho_a|^2} \right\}  (3.62)
\]

\[
\lambda_3 = \eta - a\eta_a
\]

Since \( \lambda_1 \geq \lambda_2 \), the maximum value of \( a \) is found when either \( \lambda_2 = 0 \) or when \( \lambda_3 = 0 \). To find the value of \( a \) that would make \( \lambda_2 = 0 \), we need to solve the equation

\[
(\xi - a\xi_a)(\zeta - a\zeta_a) = |\rho - a\rho_a|^2
\]

\[
\Rightarrow a^2 \left( \xi_a\zeta_a - |\rho_a|^2 \right) - \left\{ (\xi_a\xi_a + \xi_a\zeta_a) - \rho\rho^*_a - \rho^*\rho_a \right\} a + \xi\zeta - |\rho|^2 = 0  (3.63)
\]

This quadratic has two roots that are both positive. The smaller of the two is

\[
a = \frac{1}{2(\xi_a\zeta_a - |\rho_a|^2)^2} \left\{ \left( \xi_a\xi_a + \xi_a\zeta_a \right) - \rho\rho^*_a - \rho^*\rho_a \right\} - \sqrt{\left\{ \left( \xi_a\xi_a + \xi_a\zeta_a \right) - \rho\rho^*_a - \rho^*\rho_a \right\}^2 - 4\left( \xi_a\zeta_a - |\rho_a|^2 \right)(\xi\zeta - |\rho|^2)} \} \right)  (3.64)
\]

The case of \( \lambda_3 = 0 \) is straightforward. The resulting maximum \( a \) is therefore

\[
a_{\text{max}} = \min \left\{ \eta/\eta_a, \frac{1}{2(\xi_a\zeta_a - |\rho_a|^2)^2} \left\{ \left( \xi_a\xi_a + \xi_a\zeta_a \right) - \rho\rho^*_a - \rho^*\rho_a \right\} - \sqrt{\left\{ \left( \xi_a\xi_a + \xi_a\zeta_a \right) - \rho\rho^*_a - \rho^*\rho_a \right\}^2 - 4\left( \xi_a\zeta_a - |\rho_a|^2 \right)(\xi\zeta - |\rho|^2)} \} \right)  (3.65)
\]

It should be pointed out that in the general form for the model matrix we assumed in (3.59), we have explicitly assumed that the model matrix represents an average scattering
process for terrain with reflection symmetry. It also includes cases of a single scatterer with no cross-polarized component, such as a pure dihedral or first order scattering from a slightly rough surface. It does not, however include scattering from terrain that does not exhibit reflection symmetry, or single scatterers with cross-polarized components. For those cases, the expressions become significantly more complicated, with the eigenvalues being the roots to a cubic polynomial. However, most observed scattering seems to approximate reflections symmetry well as evidenced by the relatively small values of the helix component in the Yamaguchi et al. (2005) decomposition discussed in the previous Section, so we do not believe the present analysis to be too severely limited by these assumptions. In any case, the fundamental approach remains the same. We always insist that no eigenvalue be negative.

We note that if the model represents a single scatterer with no cross-polarized return, \( \xi_a \zeta_a = |\rho_a|^2 \), and we can simplify the expressions in (3.65) to

\[
a_{\text{max}} = \min \left( \frac{\eta/\eta_a}{\xi_a \zeta_a - |\rho|^2} \frac{(\xi \zeta + \zeta \xi) - \rho \rho^* - \rho^* \rho_a}{(\xi \zeta + \zeta \xi - \rho \rho^* - \rho^* \rho_a)} \right)
\]

(3.66)

Let’s illustrate this process with an example. For the model we shall assume a uniformly random oriented cloud of thin cylinders. The expression for the average covariance matrix for this model was derived in Chapter 2, and for the backscatter case, is given by

\[
\begin{pmatrix} 0.472 & 0.008 + i0.010 & 0.056 - i0.029 \\ 0.008 - i0.010 & 0.235 & 0.003 - i0.002 \\ 0.056 + i0.029 & 0.003 + i0.002 & 0.293 \end{pmatrix}
\]

(3.69)

Putting these together in equation (3.65), we find the resulting maximum \( a \) is therefore

\[
a_{\text{max}} = \min \left( \frac{4\eta}{2 \left\{ 3(\xi + \zeta) - \rho - \rho^* - \sqrt{3(\xi + \zeta) - \rho - \rho^*} - 32(\xi \zeta - |\rho|^2) \right\} \right)
\]

(3.68)

To illustrate this process with a numerical example, consider the case of a covariance matrix extracted from the Black Forest image at L-Band. The normalized matrix is

\[
\begin{pmatrix} 0.472 & 0.008 + i0.010 & 0.056 - i0.029 \\ 0.008 - i0.010 & 0.235 & 0.003 - i0.002 \\ 0.056 + i0.029 & 0.003 + i0.002 & 0.293 \end{pmatrix}
\]

(3.69)

Note that this matrix is not the same as the one we assumed in (3.27), but the terms assumed to be zero in (3.27) are indeed much smaller than the others. We shall therefore ignore those terms and effectively set them equal to zero, consistent with what Freeman and Durden assumed. Using equation (3.68) we find that

\[
a_{\text{max}} = \min (0.940, 0.752) = 0.752
\]

(3.70)
Simply setting the strength of the returns from the randomly oriented cylinders based on the cross-polarized return will force us to use the value 0.940 for $a$. Clearly this is much larger than the allowable value of 0.752. Using this maximum value of $a$, we find that the decomposition would be

\[
\begin{pmatrix}
0.472 & 0.008 + i0.010 & 0.056 - i0.029 \\
0.008 - i0.010 & 0.235 & 0.003 - i0.002 \\
0.056 + i0.029 & 0.003 + i0.002 & 0.293
\end{pmatrix}
\approx
\begin{pmatrix}
0.281 & 0 & 0.094 \\
0 & 0.188 & 0 \\
0.094 & 0 & 0.281
\end{pmatrix}
+ \begin{pmatrix}
0.191 & 0 & -0.038 - i0.029 \\
0 & 0.047 & 0 \\
-0.038 + i0.029 & 0 & 0.012
\end{pmatrix}
\]

(3.71)

**Figure 3-16.** These images display the fraction of the observed total power that can be attributed to scattering from a uniformly randomly oriented layer of cylinders. The image on the left follows the calculation proposed by Freeman and Durden (1998), while the image on the right is calculated by requiring that the remaining eigenvalues not be negative. On the average the image on the right shows about 10-20% lower values for the forested areas than the one on the left.
Figure 3-16 shows a comparison of the fraction of the scattering assigned to the randomly oriented canopy by using the approach suggested by Freeman and Durden and that using the maximum value of $a$ as we have described above. The fraction is expressed as the total power of the scattering from the randomly oriented cloud of cylinders as compared to the total power in the measured covariance matrix. In the case of the Freeman and Durden decomposition, this ratio is simply the thin vegetation index as defined in (3.11). We note from this figure that the fraction of scattering that should be assigned to the vegetation is generally quite a bit lower than if we simply assume that all the cross-polarized scattering comes from the randomly oriented branches. In fact, on the average, the maximum amount of scattering from the branches is between 10 and 20% lower if we insist that the eigenvalues remain positive. Nevertheless, the random scattering clearly still dominates in the vegetated areas, and this correction would not change the qualitative interpretation of the scattering for this scene significantly.

It is important to remember that the image on the right in Figure 3-16 represents the maximum amount of scattering we can assign to the randomly oriented cylinders. We are, of course, free to assign less than that, and we would not violate our requirement that the eigenvalues of the remainder not be negative. This result points to one of the difficulties with model-based decompositions. How are we to decide how much of the scattering to assign to the cylinders? Unfortunately, there is no simple answer to this question. A reasonable suggestion is to use the amount of power in the left-over matrix, i.e. the one labeled “remainder” in (3.57), as a guide. One could make the argument that the best value of $a$ to use would be that value of $a$ that results in the minimum amount of power in this left-over matrix. We can then simply compare the total power in the remainder matrix to that in the original measurement and select that value of $a$ that minimizes this ratio. For the single model case discussed so far, this simply leads to picking the maximum value of $a$.

The question now is if there is a way to pick the “best” solution for a generic model-based decomposition for vegetation scattering. We shall assume, like Freeman and Durden, that the scattering from vegetated terrain consists of three major components: scattering from the canopy, double reflection scattering and single reflections from the underlying ground surface. We then propose a hybrid approach that combines model-based decomposition and eigenvalue decomposition. In the case of the Freeman and Durden decomposition, they artificially have to decide whether to explicitly solve for the double reflection or the single reflection parameters by examining the phase difference between HH and VV. But as we showed before (van Zyl, 1992), the eigenvalue decomposition itself can be interpreted in terms of single (or odd numbers of) reflections, double (or even numbers of) reflections and a diffuse part. Our hybrid approach then is the following. We first subtract a covariance matrix representing a model for canopy scattering from the observed covariance matrix. We pick that multiplicative parameter that minimizes the power in the remainder matrix. We then perform an eigenvalue decomposition on the remainder matrix. Mathematically this translates to

$$
\langle [C] \rangle = f_c [C_{\text{canopy}}] + \lambda_d [C_{\text{double}}] + \lambda_s [C_{\text{single}}] + \lambda_r [C_{\text{remainder}}]
$$

(3.72)
Here, $\lambda_r$ is the “diffuse” part of the covariance matrix that is left once we subtracted the canopy contribution to the observed scattering. It is clear that if we want to minimize the power in the last matrix on the right, we need to maximize the value of $f_c$. This is done using the expressions in (3.65), where the subscript $a$ in that expression is substituted for the canopy model covariance matrix.

This decomposition needs only one assumption; that of the model to use to represent the canopy scattering. We no longer have to artificially make assumptions about the ratio of the HH to VV scattering strength for the double reflection or single reflection terms. These are determined by the eigenvectors of the matrix that is left over once the canopy contribution is subtracted.

Figure 3-17 shows a comparison of the Freeman and Durden decomposition and NNED using uniformly randomly oriented thin cylinders as the model for the canopy scattering like Freeman and Durden. In our display of the NNED, we use the green color to show the canopy strength, blue to show the single scattering term and red to show the double reflection term. We do not include the last term on the right in equation (3.72) in our display. At first look, these images are qualitatively very similar. But a closer examination shows that the details in the forested area are quite different. For example, there are significantly more double reflections in much of the forest in the modified three-component analysis than in the original Freeman and Durden decomposition. This is to be expected, since we have now assigned less of the scattered power to the randomly oriented cylinders. So, while the qualitative results are the same, quantitatively, these images are quite different. This could be quite significant if one were to use these decomposed signals in further analysis to infer geophysical parameters.
Figure 3-17. These L-band images shows a comparison between a three-component decomposition as proposed by Freeman and Durden (left image) and a modified version that ensures non-negative powers after the vegetation scattering has been subtracted (right image). Note the increase double reflections in the forested areas in the image on the right.

Figure 3-18 shows a similar result calculated using the C-band images. The differences in the forest is now even more evident between the standard Freeman-Durden decomposition and the modified version that ensures that all eigenvalues remain non-negative. In this case, there is significantly more scattering assigned to the single reflection model, especially in the older, higher biomass parts of the forest. This should not, however, be interpreted that there is more direct scattering observed from the ground surface under the forest. In fact, this increase is due to the fact that the thin cylinder model is less appropriate at C-band, because most of the branches are no longer much thinner than the radar wavelength. When this happens, the random cylinder model matrix becomes more like scattering from a flat plate or a sphere. In the limiting case where the branches become much thicker than the radar wavelength, the scattering from the randomly oriented thick branches would be indistinguishable from the scattering from a sphere or a flat plate.
Figure 3-18. Comparison between a three-component decomposition as proposed by Freeman and Durden (left image) and the modified version that ensures non-negative powers after the vegetation scattering has been subtracted (right image) at C-band. Note the increase single reflections in the forested areas in the image on the right. See the text for a discussion.

Figure 3-19 shows the comparison at P-band. The modified decomposition shows significantly less random scattering and increased double reflections when compared to the Freeman-Durden decomposition. This shows again how assigning all the cross-polarized return to the canopy scattering tends to over-estimate the contribution from the canopy scattering. Also note that the scattering near the top of the image (the portion where the angle of incidence is quite steep) has a more bluish tint in the image on the right, implying more single scattering from the underlying ground surface. This is consistent with the expectation that at steeper angles of incidence the attenuation loss through the canopy will be lower increasing the strength of the reflection from the underlying ground surface.

In all the discussion thus far we have implicitly assumed that the covariance matrix terms that involve cross-products of co-polarized returns and cross-polarized returns are negligibly small. The usual argument in favor of this assumption is that natural terrain tends to exhibit reflection symmetry over large scales. Yamaguchi et al. (2005) point out that this is not always the case. In fact they show examples of urban areas where these terms are not negligible. Their solution is to add a fourth model to the decomposition that includes scattering by helices to account for the observed correlation between the co-polarized and cross-polarized returns. They then proceed to first remove this helix component that is calculated based on the observed correlation. Note that our
proposed hybrid decomposition as shown in (3.72) explicitly includes any potential observed cross-correlation between the co-polarized and cross-polarized components. The matrices that are calculated using the eigenvalue decomposition are formed from the eigenvectors of the covariance matrix that results from the subtraction of the assumed canopy model from the observed covariance matrix. Any correlation between the cross-polarized and co-polarized returns will result in these eigenvectors not being “pure” single, double or diffuse scattering covariance matrices. The advantage of our approach is that we do not have to assume a specific scattering mechanism that accounts for these observed correlations.

![Figure 3-19](image.png)

**Figure 3-19.** Comparison between a three-component decomposition as proposed by Freeman and Durden (left image) and the modified version that ensures non-negative powers after the vegetation scattering has been subtracted (right image) at P-band. Note the increase double reflections (red color) in the forested areas in the image on the right.

### 3.3.2.3 Adaptive Model-Based Decomposition

In the previous Section we laid out a method for performing a model-based decomposition that ensures that all the individual matrices have non-negative eigenvalues. We shall now show how this same idea can be used to determine which canopy model is the “best” fit to the observations.

As we pointed out before, the idea is to minimize the amount of power in the matrix that is left once we subtract the canopy scattering. This provides a simple way to compare different canopy models. We simply calculate the amount of power that would
be in the remainder matrix once we subtract each canopy model contribution. The model with the smallest amount of power left over is then assumed to be the best fit to the observation. In fact, Yamaguchi et al. (2005) pointed out that the uniformly randomly oriented cloud of thin cylinders may not always be the appropriate model for canopy scattering. As part of their four-component scattering decomposition they included a test to apply different canopy models.

As an illustration of the idea, let us first look at an example where we will use a limited number of different canopy models to find the best fit for the L-band data of the Black Forest image we have been analyzing so far. In doing the test, we assume three different models: a uniformly randomly oriented cloud of thin cylinders, a cloud of thin cylinders oriented around the horizontal direction with a cosine squared distribution, and a cloud of thin cylinders oriented around the vertical direction with a cosine squared distribution. For each pixel in the image, we then calculate the relative amount of power in the remainder matrix if we subtracted the maximum contribution assuming each of these models. Figure 3-20 shows the result for the three cases at L-band. At the global scale of the image, it appears that the best fit model at L-band might be the cosine squared distribution around the horizontal orientation. Most of the image is in fact coniferous trees with branches that are closer to horizontal orientation than uniformly randomly oriented, so this result is not unexpected.

![Figure 3-20](image.png)

**Figure 3-20.** Comparison of different canopy models at L-band. The images show the relative amount of power in the remainder matrix after the canopy contribution has been subtracted from the observation. The results show that the cosine squared distribution around the horizontal orientation provides the best fit over the majority of the image.

It should be pointed out that this result cannot be taken as conclusive proof that the canopy contains branches that have orientations that are closer to horizontal. In drawing such conclusions, one has to remember that this result shows the best fit of a canopy model to the observation by first subtracting the canopy contribution. In some
sense this means we assume that the scattering is dominated by the canopy contribution. If, for example, we had the case of a pure double reflection (for which HH is large than VV) with a small contribution from a uniformly oriented canopy, the resulting covariance matrix would have HH larger than VV. In that case, we might find that the best fit canopy model is a cosine squared around the horizontal orientation, because that model predicts that HH will be larger than VV.

We can now extend this concept by adaptively performing the decomposition on each pixel to find the best canopy model for that pixel. The process is the same as we described above, except it is performed for each pixel separately. For each pixel we now compare the remainders for the set of models and pick than model that leaves the smallest relative amount of power in the remainder after we subtract the canopy contribution. We shall first introduce this concept with just the three models described above (the same three used by Yamagichi et al. (2005)) and later generalize the concept. Figure 3-21 shows the results for the Black Forest image at C-band, L-band and P-band. At C-band the uniformly randomly oriented model is the best fit, consistent with scattering from the needles in the coniferous trees. At L-band and P-band, we see that most of the pixels are better represented by a model with a cosine squared distribution around the horizontal orientation. At L-band this result is likely because of the primary branch orientation. At P-band, however, this is probably more indicative that double bounce scattering from the ground surface and tree trunks likely dominate.

Figure 3-21. Adaptive non-negative eigenvalue decompositions at different frequencies with three canopy scattering models. At the longer wavelengths the scattering is best modeled by a cosine squared distribution around the horizontal orientation, while at C-band a uniformly random orientation provides the best fit. See the text for more discussion.
Figure 3-22 shows the L-band comparison of the adaptive decomposition using three models compared to the Yamaguchi et al. (2005) choice of model based on their simplified selection criteria involving the ratio of HH to VV power. We have blanked out pixels for which either the single reflections or the double reflections are stronger than the canopy component, since for those pixels our original assumption that the canopy scattering dominates clearly does not hold. First, we notice a difference in number of pixels that are blanked out between the adaptive decomposition and the Yamaguchi et al. decomposition. This is primarily related to the amount of power assigned to the canopy component, and the fact that for our adaptive decomposition we insist that all eigenvalues be non-negative. This will typically assign less power to the canopy component. Apart from this difference, however, it is obvious that the two adaptive techniques give very similar results.

![Figure 3-22](image)

**Figure 3-22.** This figure shows a comparison of the optimum model indicated by the color schemes for the adaptive NNED and Yamaguchi et al. models using the L-band image. We have blanked out those pixels for which the canopy scattering is not the strongest component. Note the great similarity between the NNED and Yamaguchi results in terms of which models best represents the canopy scattering.

Building on these initial results, Arii (2009) and Arii et al. (2009) extended the adaptive decomposition technique to include a generalized canopy component that selects the best fit to the canopy scattering over different amounts of randomness and different mean orientations of the canopy. Their characterization of the canopy scattering starts with the scattering matrix of a vertically oriented thin cylinder, which apart from a multiplicative constant that depends on the cylinder length and dielectric constant, can be written as
They then allow this cylinder to be rotated by an angle $\theta$ with respect to the vertical direction about the line of sight, with positive values of $\theta$ indicating a clockwise rotation. The resulting covariance matrix is found to be

$$
\begin{align*}
\left[ C_{cyl} (\theta) \right] &= \frac{1}{4} \begin{pmatrix}
(1 - \cos 2\theta)^2 & \sqrt{2} \sin 2\theta (1 - \cos 2\theta) & \sin^2 2\theta \\
\sqrt{2} \sin 2\theta (1 - \cos 2\theta) & 2 \sin^2 2\theta & \sqrt{2} \sin 2\theta (1 + \cos 2\theta) \\
\sin^2 2\theta & \sqrt{2} \sin 2\theta (1 + \cos 2\theta) & (1 + \cos 2\theta)^2
\end{pmatrix} \\
&= \begin{pmatrix}
3 & 0 & 1 \\
0 & 2 & 0 \\
1 & 0 & 3
\end{pmatrix} + \frac{2}{8} \begin{pmatrix}
-2 \cos 2\theta & \sqrt{2} \sin 2\theta & 0 \\
\sqrt{2} \sin 2\theta & 0 & \sqrt{2} \sin 2\theta (1 + \cos 2\theta) \\
0 & \sqrt{2} \sin 2\theta & 2 \cos 2\theta
\end{pmatrix} \\
&\quad + \frac{1}{8} \begin{pmatrix}
\cos 4\theta & -\sqrt{2} \sin 4\theta & -\cos 4\theta \\
-\sqrt{2} \sin 4\theta & -2 \cos 4\theta & \sqrt{2} \sin 4\theta \\
-\cos 4\theta & \sqrt{2} \sin 4\theta & \cos 4\theta
\end{pmatrix}
\end{align*}
$$

We can write this as the sum of three matrices:

$$
\left[ C_{cyl} (\theta) \right] = \frac{1}{8} \left[ C_{a} \right] + \frac{2}{8} \left[ C_{\beta} (2\theta) \right] + \frac{1}{8} \left[ C_{\gamma} (4\theta) \right]
$$

The next step is then to calculate the average covariance matrix for a given probability distribution of the cylinder orientation angles. The resulting average covariance matrix is

$$
\langle \left[ C_{cyl} \right] \rangle = \int_{0}^{2\pi} \left[ C_{cyl} (\theta) \right] p(\theta) \, d\theta.
$$

Where $p(\theta)$ describes the probability density function of the cylinder orientation angles. In the discussion above, we have used three cases: uniform distribution for which

$$
p_{\text{uniform}} = \frac{1}{2\pi}
$$

and cosine squared distributions around the vertical and horizontal directions,

$$
p_{\cos_{-sq}} = \frac{\cos^2 (\theta - \theta_0)}{\int_{0}^{2\pi} \cos^2 (\theta - \theta_0) \, d\theta}
$$
For the distribution peaking around vertical, $\theta_0 = 0$, while for the horizontal case, $\theta_0 = \pi/2$. The uniform distribution represents the most random orientation. In the other extreme, the delta function around a fixed angle

$$p_{\delta}(\theta - \theta_0) = \frac{1}{2} \left\{ \delta(\theta - \theta_0) + \delta(\theta - \theta_0 - \pi) \right\}$$

represents the least amount of randomness. Arii (2009) and Arii et al. (2009) suggested that all these cases can be approximated by a generalized $n$-th power cosine squared distribution

$$p_\nu(\theta, \theta_0, n) = \frac{\left\{ \cos^2(\theta - \theta_0) \right\}^n}{\int_0^{2\pi} \left\{ \cos^2(\theta - \theta_0) \right\}^n d\theta}$$

When $n = 0$ this distribution becomes exactly same as the uniform distribution, whereas it approximates the delta function distribution with infinitely large $n$.

The average covariance matrix for the generalized probability density function can be written as

$$\mathbb{E}\left[ C_{vol}(\theta_0, n) \right] = \frac{1}{A_n} \int_0^{2\pi} \left[ C_{cyl}(\theta) \right] \cos^{2n}(\theta - \theta_0) d\theta$$

With

$$A_n = \int_0^{2\pi} \cos^{2n}(\theta - \theta_0) d\theta$$

To derive expressions for these, we note that we can write

$$\cos^{2n} x = \frac{1}{2^{2n}} \binom{2n}{n} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos(2(n-k)x)$$

Using the series expansion in equation (3.83), we find

$$A_n = \frac{\pi}{2^{2n-1}} \binom{2n}{n}$$

Also, note that

$$\int_0^{2\pi} \left[ C_{cyl}(\theta) \right] \cos^{2n}(\theta - \theta_0) d\theta = \frac{1}{2^{2n}} \binom{2n}{n} \int_0^{2\pi} \left[ C_{cyl}(\theta) \right] d\theta$$

$$+ \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \int_0^{2\pi} \left[ C_{cyl}(\theta) \right] \cos(2(n-k)(\theta - \theta_0)) d\theta$$

It is easily shown that
\[
\frac{1}{2^{2n}} \int_0^{2\pi} \left[ C_{\text{cyl}}(\theta) \right] d\theta = \frac{\pi}{2^{2n}} \left( \begin{array}{c} 2n \end{array} \right) [C_a]
\]  

(3.86)

Next, we note that
\[
\int_0^{2\pi} \left[ C_{\text{cyl}}(\theta) \right] \cos\{2(n-k)(\theta-\theta_0)\} d\theta
= \cos\{2(n-k)\theta_0\} \int_0^{2\pi} \left[ C_{\text{cyl}}(\theta) \right] \cos\{2(n-k)\theta\} d\theta
\]
\[
+ \sin\{2(n-k)\theta_0\} \int_0^{2\pi} \left[ C_{\text{cyl}}(\theta) \right] \sin\{2(n-k)\theta\} d\theta
\]  

(3.87)

We easily find that
\[
\int_0^{2\pi} [C_a] \cos\{2(n-k)\theta\} d\theta = \int_0^{2\pi} [C_a] \sin\{2(n-k)\theta\} d\theta = 0
\]  

(3.88)

After considerable algebra, we find
\[
\int_0^{2\pi} \left[ C_\beta(2\theta) \right] \cos\{2(n-k)(\theta-\theta_0)\} d\theta = \begin{cases} 
2\pi \left[ C_\beta(2\theta_0) \right] & \text{for} \quad k = n-1 \\
0 & \text{otherwise}
\end{cases}
\]  

(3.89)

\[
\int_0^{2\pi} \left[ C_\gamma(4\theta) \right] \cos\{2(n-k)(\theta-\theta_0)\} d\theta = \begin{cases} 
\pi \left[ C_\gamma(4\theta_0) \right] & \text{for} \quad k = n-2 \\
0 & \text{otherwise}
\end{cases}
\]

Using equations (3.89), (3.88), (3.86), and (3.84) in (3.81), we find
\[
\left\langle \left[ C_{\text{vol}}(\theta_0, n) \right] \right\rangle = [C_a] + \frac{2n}{n+1} [C_\beta(2\theta_0)] + \frac{n(n-1)}{(n+1)(n+2)} [C_\gamma(4\theta_0)]
\]  

(3.90)

Note that we have derived this expression assuming that \( n \) is an integer. We have also verified this expression numerically for values of \( n \) that are not integers (Arii et al., 2009). For the three cases we have used so far, we find
\[
\left\langle \left[ C_{\text{vol}}^{\text{uniform}} \right] \right\rangle = [C_a]
\]
\[
\left\langle \left[ C_{\text{vol}}^{\text{cos}}(\theta_0) \right] \right\rangle = [C_a] + [C_\beta]
\]  

(3.91)
\[
\left\langle \left[ C_{\text{vol}}^{\text{delta}}(\theta_0) \right] \right\rangle = [C_a] + 2[C_\beta] + [C_\gamma].
\]

The eigenvalues of the covariance matrix in (3.90) only depend on the value of the power of the cosine squared function, and not on the mean orientation angle. The eigenvalues are:
\[
\lambda_1 = \frac{1}{4(n+1)(n+2)} \left\{ 2n^2 + 4n + 3 + \sqrt{4n^2 (n+2)^2 + (2n+1)^2} \right\}
\]
\[
\lambda_2 = \frac{1}{4(n+1)(n+2)} \left\{ 2n^2 + 4n + 3 - \sqrt{4n^2 (n+2)^2 + (2n+1)^2} \right\}
\]
\[
\lambda_3 = \frac{2n+1}{2(n+1)(n+2)}
\]

Equation (3.92)

Figure 3-23 shows the three eigenvalues as a function of the power of the cosine squared function. Note that for all values of \( n \), \( \lambda_2 \leq \lambda_3 \leq \lambda_1 \).

![Figure 3-23](image)

Figure 3-23. This figure shows the relative strength of the eigenvalues of the covariance matrix representing the generalized volume scattering model. For all values of \( n \), we find that \( \lambda_2 \leq \lambda_3 \leq \lambda_1 \).

This figure shows that there is little practical difference for distributions for values of \( n > 20 \). To illustrate this, we show the entropy and the RVI of the resulting average covariance matrix as a function of \( n \) in Figure 3-24. The results confirm that especially the RVI varies very little for values of \( n > 20 \).

With this theoretical framework in hand, we can now describe the proposed adaptive model-based decomposition. In fact, it differs from the previously described NNED only in the first step of the decomposition. Instead of choosing a specific model for the canopy scattering, the adaptive decomposition calculates the best fit over all powers \( n \), and all mean orientation angles \( \theta_0 \). We decide which pair of parameters represent the best fit by selecting that pair that results in the least amount of power in the “remainder” matrix:
\[
\langle [C^\prime_{\text{remainder}}] \rangle = \langle [C] \rangle - f_v \left\{ \left[ C_\alpha \right] + \frac{2n}{n+1} \left[ C_\beta \right] + \frac{n(n-1)}{(n+1)(n+2)} \left[ C_\gamma \right] \right\}.
\] (3.93)

Note that the canopy distribution function as used here does not assume reflection symmetry. In fact, reflection symmetry only results for special mean orientation angles. As such, equation (3.93) describes a general decomposition; we do not need to resort to adding special scatterers in order to generate non-zero correlations between the cross- and co-polarized terms in the covariance matrix.

**Figure 3-24.** This figure shows the relative strength of the eigenvalues of the covariance matrix representing the generalized volume scattering model.

We first illustrate our results with a numerical example. For our comparison here we extracted the average covariance matrix in the forest portion of the Black Forest image at all three frequencies. The resulting matrices are

\[
\langle C \rangle_L = \begin{pmatrix}
0.36 & -0.07 + i0.0 & -0.18 - i0.03 \\
-0.07 - i0.0 & 0.20 & -0.08 + i0.0 \\
-0.18 + i0.03 & -0.08 - i0.0 & 0.44
\end{pmatrix}
\]

\[
\langle C \rangle_L = \begin{pmatrix}
0.52 & -0.09 - i0.03 & -0.09 + i0.08 \\
-0.09 + i0.03 & 0.22 & -0.06 + i0.01 \\
-0.09 - i0.08 & -0.06 - i0.01 & 0.26
\end{pmatrix}
\]

\[
\langle C \rangle_p = \begin{pmatrix}
0.67 & -0.07 + i0.0 & -0.03 + i0.13 \\
-0.07 - i0.0 & 0.13 & -0.04 + i0.01 \\
-0.03 - i0.13 & -0.04 - i0.01 & 0.20
\end{pmatrix}
\] (3.94)

We shall illustrate the use of our model by assuming the canopy scattering dominates, and that we can consider the canopy as a collection of randomly oriented dipoles. This is equivalent to assuming that the scattering is from a collection of randomly oriented cylinders, but that the cylinders are much thinner than the radar wavelength.
In fitting our observations to the model, we use the methodology described above. The results are:

\[
\begin{align*}
C - Band & : n = 0.92 \quad \theta_0 = 143.4^\circ \\
L - Band & : n = 1.66 \quad \theta_0 = 107.7^\circ \\
P - Band & : n = 3.47 \quad \theta_0 = 99.1^\circ 
\end{align*}
\]

(3.95)

These results show a decrease in randomness with increase in wavelength. Also, the longer wavelengths sense a mean orientation angle that is close to horizontal (recall the angles are measured with respect to the orientation angle of a vertically oriented dipole). Since the longer wavelengths interact preferentially with the larger branches, which for the type of trees present in this data set are oriented closer to the horizontal direction, the results appear reasonable. The C-band results indicate more randomness, consistent with interactions with more vertically oriented needles that are also more randomly oriented than the larger branches.

Next we show the results of applying the adaptive decomposition to the Black Forest image. First we show the randomness results in Figure 3-25.

![Randomness maps derived from the C-band (left), L-band (center), and P-band (right) AIRSAR images in the Black Forest. As in the numerical example, the randomness decreases for the longer wavelengths.](image)

**Figure 3-25.** Randomness maps derived from the C-band (left), L-band (center), and P-band (right) AIRSAR images in the Black Forest. As in the numerical example, the randomness decreases for the longer wavelengths.

A significant difference is observed in the forested area as a function of wavelength, consistent with the numerical example as reported in van Zyl et al. (2009). Randomness values close to that of the uniform distribution is found over the C-band image, whereas the P-band image shows values with less randomness than cosine squared distribution. From a physical point of view, this is reasonable considering the orientation
distributions of needles, branches and trunks. The shorter wavelength mainly interacts with the needles which have higher variance than trunks and branches. And the P-band can penetrate needles and branches so that trunks having much lower variance become dominant. Some of the areas of increased randomness visible at P-band have previously been shown to be areas where the trunk scattering is reduced due to topographic effects (van Zyl, 1993). The L-band result lies in between these two cases, and this corresponds mainly to the branch distribution.

![Mean orientation angle maps derived from the C-band (left), L-band (center), and P-band (right) AIRSAR images in the Black Forest.](image)

**Figure 3-26.** Mean orientation angle maps derived from the C-band (left), L-band (center), and P-band (right) AIRSAR images in the Black Forest.

Next we show the mean orientation angle for all three frequencies in Figure 3-26. Pixels with horizontal orientation are widely distributed in the forested area of the L- and P-band images. Before continuing with the interpretation, it is necessary to look at the decomposition results of all three scattering mechanisms as in Figure 3-27. Here we display each scattering mechanism separately, and the value for each pixel is normalized by the total power. Since the scattering in the forested area at the L-band is mostly contributed by the volume component, the inferred horizontal orientation in the L-band image in Figure 3-26 should be indicative of scattering by branches. However, the horizontal orientation in the P-band image is not related to the physical orientation of scatterers in the volume layer. It is well-known that the double-bounce scattering raises the HH contribution due to the Fresnel reflections at the trunk and ground surface. Since the algorithm subtracts the volume component first and then infers the other scattering mechanisms, strong double-bounce contribution misleads us to interpret a horizontally oriented volume component. One must pay particular attention to interpret the mean orientation angle map when double-bounce scattering contribution is dominant. The C-band mean orientation angle map indicates scattering from more vertically oriented scatterers. Given that the shorter C-band wavelength mostly interact with the needles, and mostly with those needles near the top of the canopy, (scattering lower down in the canopy is expected to be attenuated) this result is reasonable.
Figure 3-27. Volume (left), double-bounce (center), and surface(right) components of the adaptive algorithm for the C-band (top), L-band (middle), and P-band (bottom) Black Forest images. Each pixel is normalized by total power. Dotted lines in red indicate river (upper) and road (lower), respectively.
A clear contrast exists along the river and road in the P-band in Figure 3-27. The scattering from the forested area is basically contributed by both of volume and double-bounce terms. However the volume component is significant along the river and road whereas the double-bounce is considerably smaller. This result is due to the local topography, as pointed out by van Zyl (2009), where the double bounce contribution is drastically reduced because the Fresnel reflections no longer dominate when the forest floor is tilted.

3.4. Polarimetric Interferometry

SAR interferometry refers to a class of techniques where additional information is extracted from SAR images that are acquired from different vantage points, or at different times. Various implementations allow different types of information to be extracted. For example, if two SAR images are acquired from slightly different viewing geometries, information about the topography of the surface can be inferred. On the other hand, if images are taken at slightly different times, a map of surface velocities can be produced. Finally, if sets of interferometric images are combined, subtle changes in the scene can be measured with extremely high accuracy. These techniques are summarized in more detail in Chapter 6 of Elachi and van Zyl (2006), from which the following description is taken.

SAR interferometry was first demonstrated by Graham (1974), who demonstrated a pattern of nulls or interference fringes by vectorally adding the signals received from two SAR antennas; one physically situated above the other. Later, Zebker and Goldstein (1986) demonstrated that these interference fringes can be formed after SAR processing of the individual images if both the amplitude and the phase of the radar images are preserved during the processing.

![Figure 3-28. Basic interferometric radar geometry. The path length difference between the signals measured at each of the two antennas is a function of the elevation of the scatterer. (From van Zyl and Elachi, 2006)](image-url)
The basic principles of interferometry can be explained using the geometry shown in Figure 3-28. Using the law of cosines on the triangle formed by the two antennas and the point being imaged, it follows that

\[(R + \delta R)^2 = R^2 + B^2 - 2BR \cos \left(\frac{\pi}{2} - \theta + \alpha\right)\]  

(3.96)

where \( R \) is the slant range to the point being imaged from the reference antenna, \( \delta R \) is the path length difference between the two antennas, \( B \) is the physical interferometric baseline length, \( \theta \) is the look angle to the point being imaged, and \( \alpha \) is the baseline tilt angle with respect to the horizontal.

From Eq. (3.96) it follows that we can solve for the path length difference \( \delta R \). If we assume that \( R \gg B \), (a very good assumption for most interferometers) one finds that

\[\delta R \approx -B \sin(\theta - \alpha)\]  

(3.97)

The radar system does not measure the path length difference explicitly, however. Instead, what is measured, is an interferometric phase difference that is related to the path length difference through

\[\phi = \frac{a2\pi}{\lambda} \delta R = -\frac{a2\pi}{\lambda} B \sin(\theta - \alpha)\]  

(3.98)

where \( a = 1 \) for the case where signals are transmitted out of one antenna and received through both at the same time, and \( a = 2 \) for the case where the signal is alternately transmitted and received through one of the two antennas only. The radar wavelength is denoted by \( \lambda \).

From Figure 3-28 it also follows that the elevation of the point being imaged is given by

\[z(y) = h - R \cos \theta\]  

(3.99)

with \( h \) denoting the height of the reference antenna above the reference plane with respect to which elevations are quoted. From Eq. (3.99) one can infer the actual radar look angle from the measured interferometric phase as

\[\theta = \alpha - \sin^{-1}\left(\frac{\lambda \phi}{a2\pi B}\right)\]  

(3.100)

Using Eq. (6-139) and Eq. (6-138), one can now express the inferred elevation in terms of system parameters and measurables as:

\[z(y) = h - R \cos \left(\alpha - \sin^{-1}\left(\frac{\lambda \phi}{a2\pi B}\right)\right)\]  

(3.101)

This expression is the fundamental interferometric SAR equation for broadside imaging geometry.

To understand better how the information measured by an interferometer is used, consider again the expression for the measured phase difference given in (3.98). This
expression shows that even if there is no relief on the surface being imaged, the measured phase will still vary across the radar swath as shown in Figure 3-29a for a radar system with parameters similar to that of the Shuttle Radar Topography Mission (SRTM) (Elachi and van Zyl, 2006). To show how this measurement is sensitive to topographical relief, consider now the case where there is indeed topographical relief in the scene. For illustration purposes, we used a digital elevation model of Mount Shasta in California, shown in Figure 3-29b, and calculated the expected interferometric phase of the SRTM system for this area. The result, shown in Figure 3-29c shows that the parallel lines of phase difference for a scene without relief are distorted by the presence of the relief. If we now subtract the expected “smooth” earth interference pattern shown in Figure 3-29a from the distorted pattern, the resulting interference pattern is known as the flattened interferogram. For the Mount Shasta example, the result is shown in Figure 3-29d. This figure shows that the flattened interferogram resembles a contour map of the topography.

Figure 3-29. This figure shows how the topography of a scene is expressed in the interferometric phase. If there is no topography, all interferometric fringes will be parallel to the radar flight path as shown in (a). Using the topography of Mount Shasta, California, shown in perspective view in (b), the expected fringes for the SRTM system are shown in (c). Once the contribution from a smooth earth as shown in (a) is subtracted from (c), the resulting flattened interferogram resembles a contour map of the topography. (From van Zyl and Elachi, 2006)
The distortion of the “smooth” earth interferogram by the topographical relief is a consequence of the fact that the presence of topography slightly modifies the radar look angle from the value in the absence of topography (see Fig. 3-28). If we denote the look angle in the absence of relief for a given range by \( \theta_0 \), and \( z \) is the elevation of the pixel including the topography at the same range, it follows from the geometry in Figure 3-28 that the change in the look angle introduced by the relief is

\[
\delta \theta \approx \frac{z}{R \sin \theta_0}
\]  

(3.102)

From (3.98), we can write the phase of the pixel as

\[
\phi = - \frac{a2\pi}{\lambda} B \sin(\theta_0 + \delta \theta - \alpha) \approx - \frac{a2\pi}{\lambda} B \sin(\theta_0 - \alpha) - \frac{a2\pi}{\lambda} B \cos(\theta_0 - \alpha) \delta \theta
\]  

(3.103)

The first term on the right in (3.103) is simply the phase one would measure in the absence of relief, i.e. the phase shown in Figure 3-29a in the example of the Mount Shasta scene. If this phase field due to a smooth earth is subtracted from the actual interferometric phase, the resulting phase difference is the flattened interferogram

\[
\phi_{flat} \approx - \frac{a2\pi}{\lambda} B \cos(\theta_0 - \alpha) \frac{z}{R \sin \theta_0}
\]  

(3.104)

where we have combined (3.102) and the second term on the right in (3.103). This expression now clearly shows the sensitivity of the measured phase in the flattened interferogram to the topographical relief.

Besides allowing us to measure the topography of a scene being imaged, SAR polarimetry has the great added advantage over traditional SAR sensors that the interferogram contains all the information needed to geometrically rectify the SAR image. This aspect of the interferometric SAR technique is better understood by briefly reviewing the difference between traditional and interferometric SAR processing. In traditional (non-interferometric) SAR processing, it is assumed that the imaged pixel is located at the intersection of the Doppler cone (centered on the velocity vector), the range sphere (centered at the antenna) and an assumed reference plane (see Elachi and van Zyl, 2006 for a complete discussion). Since the Doppler cone has its apex at the center of the range sphere, and its axis of symmetry is aligned with the velocity vector, it follows that all points on the intersection of the cone and the plane orthogonal to the velocity vector lie in a plane orthogonal to the velocity vector. However, we do not know exactly where on this circle that forms the intersection of the cone and the plane orthogonal to the velocity vector the actual scatterer is. Therefore, the traditional SAR processor assumes that all scatterers lie in some reference plane, and places their images at the intersection of this reference plane and the circle forming the intersection of the Doppler cone and the plane orthogonal to the velocity vector, resulting in the well-known geometrical distortions in traditional SAR images.

The additional information provided by cross-track interferometry is that the imaged point also has to lie on the cone described by a constant phase, which means that one no longer has to assume an arbitrary reference plane. This cone of equal phase has its axis of symmetry aligned with the interferometer baseline and also has its apex at the
center of the range sphere. It then follows that the imaged point lies at the intersection of the Doppler cone, the range sphere and the equal phase cone. It should be pointed out that in actual interferometric SAR processors, the two images acquired by the two interferometric antennas are actually processed individually using the traditional SAR processing assumptions. The resulting interferometric phase then represents the elevation with respect to the reference plane assumed during the SAR processing. This phase is then used to find the actual intersection of the range sphere, the Doppler cone and the phase cone in three dimensions.

Once the images are processed and combined, the measured phase in the interferogram must be unwrapped before the topography can be reconstructed. During this procedure, the measured phase, which only varies between 0 and 360 degrees, must be unwrapped to retrieve the original phase by adding or subtracting multiples of 360 degrees. The earliest phase unwrapping routine was published by Goldstein et al. (1988). In this algorithm, areas where the phase will be discontinuous due to layover or poor signal-to-noise ratios are identified by branch cuts, and the phase unwrapping routine is implemented such that branch cuts are not crossed when unwrapping the phases. A second class of phase unwrapping algorithms is based on a least-squares fitting of the unwrapped solution to the gradients of the wrapped phase. This solution was first introduced by Ghiglia and Romero (1989). The major difference between these two classes of phase unwrapping algorithms lies in how errors are distributed in the image after phase unwrapping. Branch cut algorithms localize errors in the sense that areas with low correlation (resulting in high so-called residue counts) are fenced off an the phase is not unwrapped in these areas, leaving holes in the resulting topographic map. In the case of least-squares algorithms, the unwrapping is the result of a global fit, resulting in unwrapping even in areas with low correlation. The errors, however, are no longer localized, but instead are distributed through the image. Phase unwrapping remains one of the most active areas of research, and many algorithms remain under development. Detailed discussions and additional references on phase unwrapping can be found in van Zyl and Hjelmstadt (1999) and Rosen et al. (2000).

Even after the phases have been unwrapped, the absolute phase is still not known. This absolute phase is required to produce a height map that is calibrated in the absolute sense. One way to estimate this absolute phase is to use ground control points with known elevations in the scene. However, this human intervention severely limits the ease with which interferometry can be used operationally. Madsen et al. (1993) reported a method by which the radar data itself is used to estimate this absolute phase. The method breaks the radar bandwidth up into upper and lower halves, and then uses the differential interferogram formed by subtracting the upper half spectrum interferogram from the lower half spectrum interferogram to form an equivalent low frequency interferometer to estimate the absolute phase. Unfortunately, this algorithm is not robust enough in practice to fully automate interferometric processing. This is one area where significant research is needed if the full potential of automated SAR interferometry is to be realized.

Absolute phase determination is followed by height reconstruction. Once the elevations in the scene are known, the entire digital elevation map can be geometrically rectified. Madsen et al. (1993) reported accuracies ranging between 2.2 m r.m.s. for flat
terrain and 5.5 m r.m.s. for terrain with significant relief for the NASA/JPL TOPSAR interferometer.

An alternative way to form the interferometric baseline is to use a single channel radar to image the same scene from slightly different viewing geometries. This technique, known as repeat-track interferometry, has been mostly applied to spaceborne data starting with data collected with the L-band SEASAT SAR. Other investigators used data from the L-band SIR-B, SIR-C and JERS and the C-band ERS-1/2, Radarsat and Envisat ASAR radars. Repeat-track interferometry has also been demonstrated using airborne SAR systems (Gray and Farris-Manning, 1993).

Two main problems limit the usefulness of repeat-track interferometry. The first is due to the fact that, unlike the case of single-pass interferometry, the baseline of the repeat-track interferometer is not known accurately enough to infer accurate elevation information form the interferogram. Zebker et al. (1994) show how the baseline can be estimated using ground control points in the image. The second problem is due to differences in scattering and propagation that results from the fact that the two images forming the interferogram are acquired at different times. The radar signal from each pixel is the coherent sum (i.e. amplitude and phase) of all the voltages from the individual scatterers contained in the radar pixel. If there are many such individual scatterers in each pixel, and they move relative to each other between observations, the observed radar signal amplitude and phase will also change between observations. If we consider an extended area with uniform average scattering properties, consisting of several radar pixels, and the relative movement of scatterers inside each pixel exceeds the radar wavelength, then the observed radar interferometric phase will appear quite noisy from pixel to pixel. The result is that the correlation between interferometric phases of neighboring pixels is lost, and the phase cannot be unwrapped reliably. This is known as temporal decorrelation, which is worst at the higher frequencies (Zebker and Villasenor, 1992). For example, C-band images of most vegetated areas decorrelate significantly over as short a time as one day. This is not surprising, since most components of the vegetation can actually move more than the approximately 6 cm of the C-band radar wavelength, even in the presence of gentle breezes. This problem more than any other limits the use of the current operational spaceborne single-channel SARs for topographic mapping, and is the main reason why the SRTM mission was implemented as a fixed baseline interferometer.

**Radar interferometry for measuring surface velocity**

The previous discussion assumed that the surface imaged by the radar is stationary. Now consider the case where the radar images the surface from the same observation point, but at two different times. This implementation is known as along-track interferometry, and was first described by Goldstein and Zebker (1987). In their experiment, they used two SAR antennas mounted on the body of the NASA DC-8 aircraft such that one antennas was some distance forward of the other. The radar transmitted signals out of one antenna, and the returns were measured through both antennas simultaneously. In this configuration, the one antenna would image the scene as if the phase center is at the transmitting antenna, while the phase center for the receive-
only antenna lies at the center of the baseline connecting the two antennas. Since the aircraft is moving at a velocity \( v \), this means that two images are acquired, separated in time by

\[
T_{\text{obs}} = \frac{B}{2v} \quad (6-157)
\]

Note that if the system is implemented by alternately transmitting and receiving out of each antenna, the observation time difference would be doubled. Now assume that a scatterer moves with a velocity vector \( \mathbf{v}_s \) on a horizontal surface as shown in Figure 6-50. This would be the case, for example, if we were imaging an ocean current, or a flowing river. During the time \( T_{\text{obs}} \) the position of the scatterer is displaced by

\[
\mathbf{D} = \mathbf{v}_s T_{\text{obs}} \quad (6-158)
\]

The change in range of the scatterer relative to the radar between these observations is then

\[
\delta R = \hat{n} \cdot \mathbf{D} = \hat{n} \cdot \mathbf{v}_s T_{\text{obs}} \quad (6-159)
\]

where \( \hat{n} \) is a unit vector pointing from the radar to the original position of the scatterer. This change in range will be recorded as a change in phase of the radar signal from the first image to the second. This change in phase is

\[
\phi = \frac{2\pi}{\lambda} \delta R = \frac{2\pi B}{2\lambda v} \hat{n} \cdot \mathbf{v}_s \quad (6-160)
\]

As long as the velocity of the projected velocity if the scatterer is low enough such that the phase change between observations is less than \( 2\pi \), the measured phase field can be unambiguously inverted for the velocity of the scatterer. Note however, that with a single baseline, we would only measure the velocity of the scatterer projected onto the radar line-of-sight. If the scatterer moves in a direction orthogonal to the radar line-of-sight, the phase will not change. Figure 6-51b shows an example of along-track interferometric phases measured over the Juan de Fuca Straits in the northwestern part of the Pacific Ocean with the along-track interferometry mode of the NASA/JPL AIRSAR system. This image clearly shows the interferometric phase change associated with the tidal outflows from the bays into the straits.
Figure 6-50. (a) Along-track interferometry imaging geometry. (b) Interferogram acquired over the Straits of Juan de Fuca with the NASA/JPL AIRSAR system in 1999. The interferogram shown was acquired with the L-Band system, with a baseline of 20 m in the along-track direction. Given the normal flight parameters of the NASA DC-8 aircraft, this image has a velocity ambiguity of 2.4 m/s. (Image courtesy of Dr. David Imel, Jet Propulsion Laboratory. (Get permission))

So far we have discussed the change in phase as if we were observing a single scatterer. In reality each radar pixel contains many scatterers, and they may be moving relative to each other during the time between observations, as discussed in the previous Section. This temporal decorrelation limits the largest separation, or baseline, we can use between the antennas. This concept is analogous to the critical baseline discussed before. The difference here is that the maximum baseline for along-track interferometry is a function of the relative movement of scatterers inside each pixel. This is an expression of the way in which the surface is changing over time, and can be characterized by a surface coherence time.

**Differential interferometry for surface deformation studies**

One of the most exciting applications of radar interferometry is implemented by subtracting two interferometric pairs separated in time from each other to form a so-called *differential interferogram*. In this way surface deformation can be measured with unprecedented accuracy. This technique was first demonstrated by Gabriel *et al.* (1989) using data from SEASAT data to measure cm-scale ground motion in agricultural fields. Since then this technique has been applied to measure cm scale co-seismic displacements and to measure cm-scale volcanic deflation. The added information provided by high spatial resolution co-seismic deformation maps was shown to provide insight into the slip mechanism that would not be attainable from the seismic record alone. See the
references at the end of this Chapter for many examples. A summary of the results can be found in Rosen et al. (2000).

Differential interferometry is implemented using repeat pass interferometric measurements made at different times. If the ground surface moved between observations by an amount $\Delta r$ towards the radar, the phase difference in the repeat pass interferogram will include this ground displacement in addition to the topography. In that case, the phase difference in the flattened interferogram becomes

$$
\Delta \phi_{\text{flat}} \approx -\frac{4\pi}{\lambda} B_z \frac{z}{r_i \sin \theta_0} + \frac{4\pi}{\lambda} \Delta r.
$$

(6-161)

This expression shows that the phase difference is far more sensitive to changes in topography, i.e. surface displacement or deformation, than to the topography itself. The elevation has to change by one ambiguity height to cause one cycle in phase difference, whereas a surface displacement of $\lambda/2$ would cause the same amount of phase change. This is why surface deformations of a few centimeters can be measured from orbital altitudes using SAR systems.

To extract the surface deformation signal from the interferometric phase, one has to separate the effects of the topography and the surface deformation. Two methods are commonly used to do this. If a good DEM is available, one can use that to form synthetic fringes, and subtract the topography signal from the measured phase difference. The remaining signal then is due only to surface deformation. In this case, the relative lack of sensitivity to topography works to our benefit. Therefore, DEM errors on the order of several meters usually translate to deformation errors of only centimeters in the worst case.

The second way in which the deformation signal is isolated is to use images acquired during three overpasses as shown in Figure 6-51. This method generates a DEM from one pair of images, and then uses that as a reference to subtract the effects of topography from the second pair. For this case, the differential phase can be written as

$$
\Delta \phi_{3\text{pass}} = \Delta \phi_{\text{flat}} - \frac{B_1}{B_2} \cos(\theta - \alpha_1) \Delta \phi_{\text{flat}}
$$

(6-162)

where the subscripts refer to the first pair and second pair. If the motion only occurred between the first and second passes, one interferogram will include the deformation signal, while the other will not. The resulting phase difference in (6-162) will then contain only the deformation signal.
Figure 6-51. Three-pass differential interferometry imaging geometry. The surface deformation occurs between the second and third data acquisitions.

This process is illustrated in Figure 6-52. Here we artificially added a deformation signal to the digital elevation model of Mount Shasta previously discussed in Figure 6-46. The deformation signal has an amplitude of 30 cm, and the center of the deformation is situated on the side of Mount Shasta. Figure 6-52 shows the two individual flattened interferograms. The one on the left was calculated without the deformation signal, while the one in the middle includes the deformation signal. These two interferograms would be acquired if the three-pass scheme shown in Figure 6-51 is used. If the two flattened interferograms are subtracted from each other, the deformation signal is clearly visible in the image on the right in Figure 6-52.

Figure 6-52. An illustration of differential interferometry. A deformation signal of 30 cm amplitude was added to the digital elevation model of Mount Shasta shown in Figure 6-46, and the resulting interferograms were calculated. The surface deformation occurs between the second and third data acquisitions.
Polarimetric interferometry

In the preceding discussion on radar interferometry, it was assumed that only one polarization was transmitted, and only one polarization measured upon reception of the radar waves. However, as mentioned before, electromagnetic wave propagation is by nature a vector phenomenon. Therefore, in order to capture the complete information about the scattering process, interferometric measurements should really be made in the full polarimetric implementation of a radar system. In this case, there are really three different measurements being made at the same time. First, there are the two polarimetric radar measurements at each end of the baseline, represented below by the two covariance matrices \( C_{11} \) and \( C_{22} \). Since the baseline is generally short compared to the distance to the scene, these two measurements can be expected to be nearly identical, except for the very small change in the angle of incidence from one end of the baseline to the other. (The exception, of course, is if the two measurements are made in the repeat-track implementation. In that case, temporal changes could cause the two covariance matrices to be quite different.) The third measurement, of course, is the full vector interferogram as opposed to the scalar implementation described earlier.

The vector interferogram, which is the complex cross-correlation of the signal from one end of the baseline with that from the other end of the baseline, can be described as

\[
V_1 V_2^* = \tilde{A}_1 \tilde{T}_1 \tilde{T}_2^* A_2^* = A_1 \cdot [C_{12}] A_2^* \tag{6-163}
\]

The correlation of the two signals after averaging is

\[
\mu = \frac{\langle V_1 V_2^* \rangle}{\sqrt{\langle V_1 V_1^* \rangle \langle V_2 V_2^* \rangle}} = \frac{A_1 \cdot [C_{12}] A_2^*}{\sqrt{\langle A_1 \cdot [C_{11}] A_1^* \rangle \langle A_2 \cdot [C_{22}] A_2^* \rangle}} \tag{6-164}
\]

where the angular brackets \( \langle \rangle \) denote averaging. The interferometric phase is the phase angle of the numerator of (6-164)

\[
\phi_{\text{int}} = \text{arg}(A_1 \cdot [C_{12}] A_2^*) \tag{6-165}
\]

Using this formulation, Cloude and Papathanassiou (1999) showed, using repeat-track SIR-C interferometric data, that polarization diversity can be used successfully to optimize the correlation between images. They also showed significant differences in the measured elevation in forested areas when using polarization optimization. At present, polarimetric interferometry is a very active research area. Unfortunately, progress is hampered severely by lack of availability of well-calibrated data, as only a hand-full of radar systems have been upgraded to full polarimetric interferometry capability.

3.5. Summary

In this Chapter we introduced several more advanced polarimetric concepts. The vector-matrix duality allows us to understand decomposition algorithms in more detail.
We also discussed several parameters based on the eigenvalues of the covariance matrix. These parameters provide useful information about the amount of randomness in the observed scatter, as well as potentially what the actual scattering mechanisms might be.

Finally, we examined different approaches for interpreting the scattering mechanisms in an image. Using orthogonal bases, we can derive unique decompositions. These decompositions, while unique, are not straightforward to interpret. In particular, the decomposition based on the eigenvectors of the covariance matrix, suffers from the fact that the basis in which the decomposition is done varies from pixel to pixel in the image. This further complicates the interpretation. At the other end of the spectrum are model based decompositions. These are not unique and picking the appropriate solution is not obvious, but their interpretation is more straightforward. We showed that some of the popular model-based decompositions suffer from serious limitations imposed by the assumptions of how the observations are to be decomposed that lead to results that are non-physical in the sense that negative powers could be generated. We showed that this limitation can easily be removed using a simple check based on the eigenvalues of the covariance matrix. A simple hybrid approach can be implemented that corrects these limitations. This approach is then easily extended to show a simple way to find the best model to fit the observed canopy scattering.

It is important to remember that all these decomposition techniques are simply tools to make interpretation of the observed scattering easier. The strength of the pure eigenvalue and eigenvector approaches is that the answers are unique, and no assumptions are required to perform the decomposition. The interpretation of the results, however, requires an interpretation of the basis vectors in terms of scattering mechanisms, which may not be unique. The model-based decompositions provide an easy interpretation. But this easy interpretation assumes that the models are indeed applicable to the observations, which may not be the case. Furthermore, the results are not unique in the sense that many different model combinations could be used. In the final analysis, which tool is used depends on personal preference.

**References and Further Reading**


