Separable subgroups of mapping class groups

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Abstract

We investigate separability questions for the mapping class group of a surface. While this group is not subgroup separable in general, we prove a large family of interesting subgroups are separable. This includes many classically studied subgroups such as solvable subgroups, Heegaard and Handlebody groups, geometric subgroups, and all the terms in the Johnson filtration.

1 Introduction and main results

A subgroup H of G is said to be *separable* in G if it can be expressed as an intersection of finite index subgroups of G. If the trivial subgroup $\{1\}$ is separable, G is *residually finite*. More generally, if every finitely generated subgroup of G is separable, G is *subgroup separable* (or *LERF*).

Subgroup separability has been an important tool in geometry; for example it often permits the lifting of a π_1–injective immersion to an embedding in a finite cover [34] (see in addition [6], [20, 21], [17, 18], and [26]). Algebraically, it can also be viewed as an indication of an abundance of finite index subgroups and a rich interaction of these subgroups with the finitely generated ones. This powerful property is generally difficult to establish and the class of groups known to be subgroup separable is small. It is a theorem of M. Hall [14] that free groups are subgroup separable. P. Scott reproved this and the subgroup separability of surface groups [34]. More recently, I. Agol, D. Long, and A. Reid [2] proved the Bianchi groups are subgroup separable (see [1], [19], or [21]). In contrast, the mapping class group $\mathrm{Mod}(S)$ of a finite type surface S is known not to be subgroup separable except in a few very special cases (see the appendix). Nevertheless, it is well-stocked with subgroups of finite index and many interesting subgroups of $\mathrm{Mod}(S)$ are separable.

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We briefly mention the main results of the paper, referring the reader to §2 for definitions. Our first theorem is the separability of solvable subgroups of $\mathrm{Mod}(S)$—the proof we give is a variation on the main idea of [31].

Theorem 1.1 *Solvable subgroups of* $\mathrm{Mod}(S)$ *are separable.*

The next theorem gives a potentially large class of subgroups that are separable. For the statement, set $\Gamma_S = \pi_1(S)$, $\mathfrak{X}_n(\Gamma_S)$, the $\mathrm{SL}(n;\mathbf{C})$–character variety, and for a subvariety V, $\mathrm{Stab}(V)$ and $\mathrm{Triv}(V)$ the set and pointwise stabilizers of V under the action of $\mathrm{Mod}(S)$, respectively.

Theorem 1.2 *For any number field k and proper k–algebraic subvariety V of $\mathfrak{X}_n(\Gamma_S)$, the subgroups* $\mathrm{Stab}(V)$ *and* $\mathrm{Triv}(V)$ *are separable in* $\mathrm{Mod}(S)$.

This result is analogous to the linear setting, where for a finitely generated subgroup Λ of a k–algebraic linear group $\mathbf{G}$, the subgroup $\Lambda \cap \mathbf{H}$, for any algebraic subgroup $\mathbf{H}$ defined over any number field, is separable in Λ (see [24], [18], and [6]).

One application of Theorem 1.2 implies the separability of the handlebody groups and the Heegaard groups.

Corollary 1.3 *The two handlebody groups* $\mathrm{Mod}(S,H)$, $\mathrm{Mod}_0(S,H)$ *and any Heegaard group* $\mathrm{Mod}(S,M^3)$ *are separable.*

Finally, a generalization of the proof of the residual finiteness of $\mathrm{Mod}(S)$ [13] bears our final result.

Theorem 1.4 *The stabilizers of multi-curves, or more generally geometric subgroups of* $\mathrm{Mod}(S)$ *are separable.*

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2 Definitions and background

In this article, we denote a closed orientable surface with genus g and n marked points by S, the surface minus the marked points by $\dot{S}$, and $\Gamma_S = \pi_1(\dot{S})$. We define the mapping class group to be the quotient

$$\mathrm{Mod}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S),$$

where $\mathrm{Diff}^+(S)$ is the group of orientation preserving diffeomorphisms of S leaving marked points invariant and $\mathrm{Diff}_0(S)$ is the component containing the identity. It is often useful to note the natural map from $\mathrm{Mod}(S)$ to $\mathrm{Out}(\Gamma_S)$ is injective. Indeed, when $n = 0$ and $g > 0$, the Dehn–Nielsen Theorem says this is onto a subgroup of index 2.

On occasion we will want to consider surfaces-with-boundary and denote these by Σ to distinguish them from closed surfaces. In this case we modify the definition of $\mathrm{Diff}^+(\Sigma)$ and demand boundary components be fixed point-wise.

2.1 The Johnson filtration

From the lower central series C_j on Γ_S, we obtain a descending series N_j on $\mathrm{Mod}(S)$ from the induced a family of homomorphisms

$$\rho_j\colon \mathrm{Mod}(S) \longrightarrow \mathrm{Out}(\Gamma_S/C_j).$$

The series N_j is given by $\ker \rho_j$, and following B. Farb, we refer to $\{N_j\}$ as the *Johnson filtration*:

$$\mathrm{Mod}(S) \rhd N_1 \rhd N_2 \rhd N_3 \rhd \cdots \rhd N_j \rhd \ldots.$$

The first non-trivial term $N_1 = \mathcal{T}$ is usually referred to as the *Torelli subgroup* and the second $N_2 = \mathcal{K}$ as the *Johnson Kernel*.

Theorem 2.1 (Bass–Lubotzky; [4]) *The Johnson filtration* $\{N_j\}_{j>0}$ *on the group* $\mathrm{Mod}(S)$ *satisfies:*

(a)
$$\bigcap_j N_j \subset Z(\mathrm{Mod}(S)).$$

(b) N_j/N_{j+1} *is torsion free for* $j > 0$.

2.2 Solvable subgroups

Using ideas from the Nielsen–Thurston classification for elements of $\mathrm{Mod}(S)$, J. Birman, A. Lubotzky, and J. McCarthy [9] show virtually solvable subgroups are virtually abelian (with bounded rank). N. Ivanov [15] strengthened this to the following form.

Theorem 2.2 (Birman–Lubotzky–McCarthy, Ivanov) *There is a finite index subgroup* $\mathrm{Mod}'(S)$ *of* $\mathrm{Mod}(S)$ *so that for any virtually solvable subgroup* G *of* $\mathrm{Mod}(S)$, $G \cap \mathrm{Mod}'(S)$ *is free abelian with rank at most* $3g-3$.

An element ϕ of $\mathrm{Mod}(S)$ is called *pure* if for any ϕ–invariant finite set of conjugacy classes in Γ_S, ϕ leaves each conjugacy class invariant, and a subgroup is *pure* if it is comprised of pure elements. Ivanov [17] has shown the existence of finite index pure subgroups $\mathrm{Mod}'(S)$ of $\mathrm{Mod}(S)$, and moreover, Theorem 2.2 holds for any finite index pure subgroup $\mathrm{Mod}'(S)$.

2.3 Multi-curve stabilizers and geometric subgroups

By a *multi-curve*, we mean the isotopy class of a closed embedded 1–manifold in $\dot{S}$ for which each component is non-peripheral and homotopically essential. $\mathrm{Mod}(S)$ acts on the set of multi-curves, and we denote the stabilizer of a multi-curve A by $\mathrm{Stab}(A)$.

Given a proper subsurface Σ of S, if each component of Σ is π_1–injective, we say Σ is *incompressible* (which is equivalent to saying each boundary component is homotopically non-trivial and non-peripheral). We will often consider subsurfaces as well defined up to isotopy without comment.

If Σ is an incompressible subsurface of S, L. Paris and D. Rolfsen [28] have proven the inclusion induces a homomorphism $\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Mod}(S)$ which is injective, unless two components of the boundary of Σ are isotopic in $\dot{S}$. For a general incompressible surface Σ, the kernel of this homomorphism is contained in the center of $\mathrm{Mod}(\Sigma)$. Following [28], when Σ is incompressible, even if the groups only inject modulo centers, we call the image of $\mathrm{Mod}(\Sigma)$ in $\mathrm{Mod}(S)$ a *geometric subgroup* and denote it by $G(\Sigma)$.

2.4 Handlebody and Heegaard groups

A pair of subgroups of $\mathrm{Mod}(S)$ arising in connection with 3–manifolds are the handlebody groups. These groups have been studied from diverse points of view; see Birman [8] and H. Masur [25]. Setting H be a handlebody and selecting a diffeomorphism $S \longrightarrow \partial H$, the *first handlebody group* $\mathrm{Mod}(S,H)$ is the group consisting of those automorphisms of S which extend over H, while the *second handlebody group* $\mathrm{Mod}_0(S,H)$ of $\mathrm{Mod}(S,H)$ is comprised of those elements which induce the identity outer automorphism on $\pi_1(H)$. A similar subgroup studied by L. Goeritz [12], J. Powell [29], and recently M. Scharlemann [33] is the *Heegaard group* $\mathrm{Mod}(S,S^3)$. This is defined by choosing a diffeomorphism from S to a Heegaard surface in S^3, then taking the subgroup of automorphisms of S that extend to S^3. There is nothing special about S^3 in this construction, and so for any 3–manifold M and diffeomorphism from S to a Heegaard surface in M, we define the Heegaard group of (S,M), denoted by $\mathrm{Mod}(S,M)$, in an analogous fashion.

We remark in the present notation, we suppress the identification of S with the boundary of the handlebody or Heegaard surface.

2.5 Representation and character varieties

Given a finitely generated group Λ and natural number n, the set $\mathrm{Hom}(\Lambda;\mathrm{SL}(n;\mathbf{C}))$ is called the $\mathrm{SL}(n;\mathbf{C})$–*representation variety*. This can be equipped with a natural analytic structure (see [30]); in fact, the $\mathbf{Z}$–algebraic structure on $\mathrm{SL}(n;\mathbf{C})$ provides $\mathrm{Hom}(\Lambda;\mathrm{SL}(n;\mathbf{C}))$ with a $\mathbf{Z}$–algebraic structure. $\mathrm{SL}(n;\mathbf{C})$ acts on $\mathrm{Hom}(\Lambda;\mathrm{SL}(n;\mathbf{C}))$ by conjugation and the quotient (in the sense of geometric invariant theory) is a set $\mathcal{X}_n(\Lambda)$ called the $\mathrm{SL}(n;\mathbf{C})$–*character variety* which is a $\mathbf{Z}$–defined affine variety (see [27]).

The inner automorphism group action of Λ on $\mathrm{Hom}(\Lambda;\mathrm{SL}(n;\mathbf{C}))$ is absorbed by the action of $\mathrm{SL}(n;\mathbf{C})$ and thus affords an action of the outer automorphism group $\mathrm{Out}(\Lambda)$ on $\mathcal{X}_n(\Lambda)$ by $\mathbf{Z}$–algebraic automorphisms—see [23]. For any subvariety V of $\mathcal{X}_n(\Lambda)$ we define

$$\begin{aligned}\mathrm{Stab}(V) &= \{\gamma \in \mathrm{Out}(\Lambda) : \gamma(V) \subset V\}\\ \mathrm{Triv}(V) &= \{\gamma \in \mathrm{Out}(\Lambda) : \gamma|_V = \mathrm{id}_V\}.\end{aligned}$$

If k is a number field, $\mathcal{O}_k$ its ring of integers, and V is a k–defined variety, for any finite extension ring $R/\mathcal{O}_K$, we denote the set of R–points of V by $V(R)$. For an ideal $\mathfrak{p}$ of R, reducing the coordinates of the R–points modulo $\mathfrak{p}$ gives birth to a set we denote by $V(R/\mathfrak{p})$. The reduction map (viewed as sets)

$$r_{\mathfrak{p}} \colon V(R) \longrightarrow V(R/\mathfrak{p})$$

induces a homomorphism

$$(r_{\mathfrak{p}})_* \colon \mathrm{Aut}(V(R)) \longrightarrow \mathrm{Sym}(V(R/\mathfrak{p})),$$

where $\mathrm{Sym}(V(R/\mathfrak{p}))$ is the symmetric group on $V(R/\mathfrak{p})$. Note as $R/\mathfrak{p}$ is finite, $\mathrm{Sym}(V(R/\mathfrak{p}))$ is a finite group.

3 Subgroup separability

3.1 Separability results

For convenience we collect the requisite material on separability needed in this article here, referring the reader to the listed references for proofs. For a general reference, the reader can consult [22].

Lemma 3.1 (Long–Reid; [20]) *If $H < K < G$, $[K : H] < \infty$, and H separable in G, then K is separable in G.*

Lemma 3.2 (Scott; [34]) *If H, G_0 are subgroups of G with $[G : G_0] < \infty$, then H is separable in G if and only if $H \cap G_0$ is separable in G_0.*

Lemma 3.3 *If $\rho : G \longrightarrow H$ is a surjective homomorphism and $K < H$ is separable in H, then $\rho^{-1}(K)$ is separable in G.*

Proof. This is a consequence of the fact finite index subgroups of H pull back to finite index subgroups of G under ρ. □

Theorem 3.4 (Grossman; [13]) $\mathrm{Mod}(S)$ *is residually finite.*

One proof of this theorem uses the fact Γ_S satisfies a strong type of residual finiteness: A group G is said to be *conjugacy separable* if for any two non-conjugate elements $x, y \in G$, there is a homomorphism from G to a finite group for which the images of x and y remain non-conjugate.

Theorem 3.5 (Stebe; [36]) Γ_S *is conjugacy separable.*

3.2 Lattices and nilpotent groups

The reduction of solvable subgroup separability in $\mathrm{Mod}(S)$ to the quotients of $\mathrm{Mod}(S)$ by the terms in the Johnson filtration requires solvable subgroup separability in the image $\mathrm{Out}(\Gamma_S/C_j)$.

Theorem 3.6 *Solvable subgroups of* $\mathrm{Out}(\Gamma_S/C_j)$ *are separable.*

Theorem 3.6 follows from a pair of results, the first due to D. Segal [35] (see [5]), the latter due to the second author [26].

Theorem 3.7 (Segal) *The outer automorphism group of a finitely generated nilpotent group is a finite extension of an arithmetic lattice.*

Theorem 3.8 *Solvable subgroups of arithmetic lattices are separable.*

Using the residual finiteness of linear groups, Theorem 3.7, and Lemma 3.2, one can see the separability of the terms in the Johnson filtration.

4 Solvable subgroup separability in $\mathrm{Mod}(S)$

In this section, we prove Theorem 1.1. Our approach is a variation of the one taken in [31] for $\mathrm{Aut}(\Gamma_S)$. By Theorem 2.2, it suffices to separate abelian subgroups. To this end, we prove the following proposition.

Proposition 4.1 *For every torsion free abelian subgroup A of $\mathrm{Mod}(S)$, there exists $j(A) = j$ such that the quotient homomorphism $\rho_j \colon \mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(S)/N_j$ restricted to A is an injection.*

Proof. To begin, set

$$A_i = \ker \rho_i \cap A = N_i \cap A.$$

The inclusions $N_{i+1} \leq N_i$ produce inclusions $A_{i+1} \leq A_i$ and so $\mathrm{rank}_{\mathbf{Z}} A_i \geq \mathrm{rank}_{\mathbf{Z}} A_{i+1}$. As $\{\mathrm{rank}_{\mathbf{Z}} A_i\}$ is a non-increasing sequence of non-negative integers, there exists $j = j(A) > 0$ such that for all $i, k \geq j$, $\mathrm{rank}_{\mathbf{Z}} A_i = \mathrm{rank}_{\mathbf{Z}} A_k$. By Theorem 2.1 (a),

$$\bigcap_i A_i = \{1\},$$

since $Z(\mathrm{Mod}(S))$ is finite and A is torsion free. Consequently, it suffices to show the sequence of groups $\{A_i\}$ is eventually constant. Assuming otherwise, let $i \geq j$ be such that $A_i \neq A_{i+1}$. By our selection of i,

$$\mathrm{rank}_{\mathbf{Z}} A_i = \mathrm{rank}_{\mathbf{Z}} A_{i+1},$$

and so

$$A_i / A_{i+1} < N_i / N_{i+1}$$

is a non-trivial finite group. However, this is in opposition with Theorem 2.1 (b).□

Proof of Theorem 1.1. Fix a pure subgroup $\mathrm{Mod}'(S)$ of $\mathrm{Mod}(S)$ with finite index. If A_0 is any solvable subgroup of $\mathrm{Mod}(S)$, then according to Theorem 2.2, $A = A_0 \cap \mathrm{Mod}'(S)$ is a torsion free abelian group. By Lemma 3.2 it suffices to separate A in $\mathrm{Mod}'(S)$.

If we let $N'_i = N_i \cap \mathrm{Mod}'(S)$, then

$$\bigcap_i N'_i = \{1\}; \tag{1}$$

$Z(\mathrm{Mod}(S))$ is finite and so intersects $\mathrm{Mod}'(S)$ trivially. In an abuse of notation, we write ρ_i for the homomorphism from $\mathrm{Mod}'(S)$ to $\mathrm{Mod}'(S)/N'_i$, and note Proposition 4.1 remains valid for $j = j(A)$.

For any $i \geq j$,

$$\rho_i^{-1}(\rho_i(A)) = AN_i' = A \ltimes N_i'$$

since ρ_i restricted to A is injective. The inclusion of

$$A \ltimes N_k' \hookrightarrow A \ltimes N_i'$$

for $k > i$ respects the semidirect product structure. This in combination with (1) yields

$$\bigcap_{k \geq i} \rho_k^{-1}(\rho_k(A)) = \bigcap_{k \geq i} A \ltimes N_k' = A \ltimes \bigcap_{k \geq i} N_k' = A.$$

An application of Theorem 3.6 implies $\rho_i(A)$ is separable. By Lemma 3.3, $\rho_i^{-1}(\rho_i(A))$ is separable, and hence so is A, being the intersection of these separable subgroups. □

This argument, along with the theorems for $\mathrm{Out}(F_n)$ analogous to Theorem 2.1 (see [4]) and Theorem 2.2 (see [7]), yields

Theorem 4.2 *Solvable subgroups of* $\mathrm{Out}(F_n)$ *are separable.*

5 Subvariety stabilizer separability

To prove Theorem 1.2 we would like to use the reduction maps from Section 2.5 to construct finite index subgroups of $\mathrm{Mod}(S)$. This, in turn, requires the existence of sufficiently many algebraic points on the variety V. The starting point for our proof of Theorem 1.2 is thus the following consequence of Hilbert's Nullstellensatz.

Proposition 5.1 *If k is a number field and V is a k–algebraic variety, then*

$$V(\bar{k}) = \bigcup_{\substack{k < K \\ |K:k| < \infty}} V(K)$$

is Zariski dense in V, where $\bar{k}$ is the algebraic closure of k.

Given a finite extension K/k and q in $V(K)$, it follows q is in $V(R)$, for some finite extension ring $R/\mathcal{O}_K$. For example, if m is the product of the denominators occurring in the coordinates of q, one can take $R = \mathcal{O}_K[1/m]$. Coupled with Proposition 5.1, this produces a plethora of points algebraically defined in the variety V.

Proof of Theorem 1.2. Let V be a k–algebraic subvariety of $\mathcal{X}_n(\Gamma_S)$, and $\mathfrak{a}, \mathfrak{a}_V \subset \mathbf{C}[\mathbf{T}]$ the associated ideals for $\mathcal{X}_n(\Gamma_S)$ and V, respectively. For $\gamma \in \mathrm{Mod}(S) \setminus$

Stab(V), there exists $q_0 \in V$ such that $\gamma(q_0) \notin V$. Since $V(\bar{k})$ is Zariski dense in V, we can assume $q_0 \in V(K)$, for some finite extension K/k. Indeed, from the discussion above, q_0 is in $V(R)$ for some finite extension ring $R/\mathcal{O}_K$. Next, select a generating set $f_1, \dots, f_r \in R[\mathbf{T}]$ for $\mathfrak{a}_V$. It follows both $f_j(q_0), f_j(\gamma(q_0))$ are in R for all j. By assumption, $\gamma(q_0) \notin V$, and so $f_j(\gamma(q_0)) \neq 0$ for some generator f_j. Thus, we can select an ideal $\mathfrak{p}$ of R such that

$$f_j(\gamma(q_0)) \neq 0 \mod \mathfrak{p}.$$

Since the points of $V(R/\mathfrak{p})$ are precisely those of the form $r_\mathfrak{p}(q)$ with

$$f_i(q) = 0 \mod \mathfrak{p}$$

for each $1 \leq i \leq r$, it follows the point $r_\mathfrak{p}(\gamma(q_0))$ is not in $V(R/\mathfrak{p})$. As

$$(r_\mathfrak{p})_*(\gamma)(r_\mathfrak{p}(q_0)) = r_\mathfrak{p}(\gamma(q_0)),$$

the transformation $(r_\mathfrak{p})_*(\gamma)$ does not stabilize $V(R/\mathfrak{p})$. On the other hand, because $r_\mathfrak{p}(V(R)) = V(R/\mathfrak{p})$, we see $(r_\mathfrak{p})_*(\mathrm{Stab}(V))$ is contained in Stab$(V(R/\mathfrak{p}))$. Therefore, $(r_\mathfrak{p})_*^{-1}(\mathrm{Stab}(V(R/\mathfrak{p})))$ separates γ from Stab(V).

Next let $\gamma \in \mathrm{Mod}(S) \setminus \mathrm{Triv}(V)$. As above, we may select q_0 in $V(R)$, for some finite extension $R/\mathcal{O}_K$, such that $\gamma(q_0) \neq q_0$. The separation of Triv(V) from γ is obtained in a similar fashion to the above by selecting an ideal $\mathfrak{p}$ of R such that

$$\gamma(q_0) \neq q_0 \mod \mathfrak{p}.$$

Specifically, with such an ideal, $(r_\mathfrak{p})_*^{-1}(\mathrm{Triv}(V(R/\mathfrak{p})))$ separates γ and Triv(V). □

As mentioned in the introduction, separability of the handlebody groups and Heegaard groups follows from Theorem 1.2.

Proof of Corollary 1.3. A Heegaard group is the intersection of two conjugates of the handlebody group Mod(S,H), so it suffice to prove the corollary for Mod(S,H) and Mod$_0(S,H)$.

We note the embedding $i \colon S \longrightarrow H$ induces an inclusion

$$i^* \colon \mathcal{X}_2(\pi_1(H)) \longrightarrow \mathcal{X}_2(\Gamma_S).$$

The image under i^*, denoted by V_H, is $\mathbf{Z}$–defined. Specifically, it is obtained by declaring certain elements in Γ_S—those bounding a disk in H—to be trivial. Clearly, Mod(S,H) is contained in Stab(V_H). If $\phi \in \mathrm{Mod}(S) \setminus \mathrm{Mod}(S,H)$, it takes a simple closed curve that bounds a disk in H to one which does not. Therefore, by Dehn's Lemma [32] it is not homotopically trivial in H. Since there exist faithful representations of $\pi_1(H)$ to SL$(2;\mathbf{C})$, it follows ϕ is not in Stab(V_H). Hence

$\mathrm{Stab}(V_H) = \mathrm{Mod}(S,H)$, and thus by Theorem 1.2, $\mathrm{Mod}(S,H)$ is separable. Similarly, $\mathrm{Mod}_0(S,H) = \mathrm{Triv}(V_H)$ and so is separable by Theorem 1.2. □

The idea of using algebraic actions of groups on varieties to deduce residual properties is not new. H. Bass and A. Lubotzky [3] used this to produce another proof of the residual finiteness of $\mathrm{Mod}(S)$. In addition, though difficult to identify, the subgroups separated by Theorem 1.2 are natural generalizations of totally geodesic stabilizer of arithmetic groups acting on symmetric or homogenous spaces. Little seems to be known about the structure of these subgroups aside from the special ones considered in Corollary 1.3.

6 Geometric subgroups

The next theorem easily implies Theorem 1.4.

Theorem 6.1 *If Δ is any finite set of conjugacy classes in Γ_S, then the group* $\mathrm{Stab}(\Delta)$ *is separable.*

Proof. Write the elements of Δ as $\Delta = \{a_1,\ldots,a_n\}$. Let $\phi \in \mathrm{Mod}(S)\setminus \mathrm{Stab}(\Delta)$ and a_i be such that $\phi(a_i) = x$ and $x \neq a_j$ for any j. By Theorem 3.5, there exists a finite group F and epimorphism

$$\psi\colon \Gamma_S \longrightarrow F$$

for which $\psi(x)$ is not conjugate to $\psi(a_i)$ for any i. Without loss of generality, we may assume $\ker(\psi)$ is characteristic (if not, replace F by the quotient of Γ_S by the characteristic core of $\ker(\psi)$). The homomorphism ψ induces a homomorphism

$$\psi_*\colon \mathrm{Out}(\Gamma_S) \longrightarrow \mathrm{Out}(F).$$

The group $\mathrm{Out}(F)$ acts on the set of conjugacy classes of F and we can consider the subgroup $\mathrm{Stab}(\psi(\Delta))$. Visibly, $\mathrm{Stab}(\Delta) \subset \psi^{-1}(\mathrm{Stab}(\psi(\Delta)))$ and does not contain $\psi_*(\phi)$ since $\psi(x)$ is not in any of the conjugacy classes of $\psi(\Delta)$. □

Proof of Theorem 1.4. The proof for multi-curve stabilizers is immediate from Theorem 6.1. Suppose Σ is an incompressible subsurface of S. To separate $G(\Sigma)$ we show for any finite index pure subgroup $\mathrm{Mod}'(S)$ of $\mathrm{Mod}(S)$

$$G(\Sigma)\cap \mathrm{Mod}'(S) = \mathrm{Stab}(\Delta)\cap \mathrm{Mod}'(S)$$

for some multi-curve Δ. Given this, the confirmation of the separability of $G(\Sigma)$ follows at once from applications of Lemma 3.2 and Theorem 6.1.

It remains to construct Δ, a task achieved via the following line of attack. Let $\partial\Sigma$ denote the union of the conjugacy classes in Γ_S representing the boundary components of Σ and $R_1,\ldots,R_k$ denote those components of the complementary subsurface

$\overline{S\setminus\Sigma}$ that are not homeomorphic to annuli or pairs of pants. For each R_i, let α_i, β_i be a pair of simple closed curves which bind R_i (meaning every other non-peripheral essential closed curve on R_i intersects one of α_i or β_i). Any automorphism of R_i fixing both α_i and β_i must have finite order (up to Dehn twisting in curves parallel to the boundary)—compare [16]. Finally, set $\Delta = \partial\Sigma \cup \{\alpha_1, \beta_1, ..., \alpha_k, \beta_k\}$.

If ϕ is any pure automorphism which preserves Δ, it must be the identity on $\overline{S\setminus\Sigma}$. This is true for each R_i by the discussion in the previous paragraph, the definition of pure automorphism, and the fact Dehn twisting in the boundary of $\overline{S\setminus\Sigma}$ can be absorbed into Σ. On the other hand, since the automorphism group of an annulus or pair of pants is finite, modulo Dehn twisting in the boundary, the purity assumption shows that it holds for these components as well. Thus $\mathrm{Stab}(\Delta) \cap \mathrm{Mod}'(S) < G(\Sigma) \cap \mathrm{Mod}'(S)$. Since being in $\mathrm{Stab}(\Delta)$ puts absolutely no constraint on the restriction to Σ, the other inclusion evidently holds. ☐

7 The profinite topology and continuous actions

Each finite index characteristic subgroup K of Γ_S endows $\mathrm{Mod}(S)$ with a finite index normal subgroup via the kernel of the induced homomorphism

$$\rho\colon \mathrm{Out}(\Gamma_S) \longrightarrow \mathrm{Out}(\Gamma_S/K).$$

Any such normal subgroup of $\mathrm{Mod}(S)$ is called a *principal congruence subgroup* and any subgroup containing a principal congruence subgroup is called a *congruence subgroup*. This name is justified by considering the nature of the subgroups when S is a 2–torus. In this case, $\mathrm{Mod}(S)$ is the modular group $\mathrm{PSL}(2;\mathbf{Z})$ and the congruence subgroups are precisely the congruence subgroups of $\mathrm{PSL}(2;\mathbf{Z})$; namely the subgroups which contain a kernel of a reduction

$$r_m\colon \mathrm{PSL}(2;\mathbf{Z}) \longrightarrow \mathrm{PSL}(2;\mathbf{Z}/m\mathbf{Z}).$$

For any group G with automorphism group $\mathrm{Aut}(G)$, we have a map

$$\Phi\colon G \times \mathrm{Aut}(G) \longrightarrow G$$

given by $\Phi(g,\varphi) = \varphi(g)$. If G is equipped with the profinite topology, the coarsest group topology on $\mathrm{Aut}(G)$ for which the mapping Φ is continuous is the topology generated by the congruence subgroups—this is called the *congruence topology*. For $\mathrm{Out}(G)$, we have a natural action on the orbit space $G/\mathrm{Inn}(G)$ equipped with the quotient topology induced by the surjective map

$$G \longrightarrow G/\mathrm{Inn}(G),$$

and with this and the set map

$$\Phi\colon G/\mathrm{Inn}(G) \times \mathrm{Out}(G) \longrightarrow G/\mathrm{Inn}(G),$$

the congruence topology on $\mathrm{Out}(G)$ is the coarsest topology for which this map is continuous.

Proposition 7.1 (Closed stabilizer lemma) *For a group G, a subgroup H is closed in the profinite topology on G if and only if there exists a topological space X with a continuous G–action where G equipped with the profinite topology and a closed subset S of X such that $H = \mathrm{Stab}_G(S)$.*

Proof. The direct implication follows by setting $X = G$, $S = H$, with G–action given by left (right) translation. For the reverse implication, we must show if $H = \mathrm{Stab}_G(S)$ for some closed subset S of a G–space X, then H is closed. Consider a convergent net h_α in H with limit h. By the net characterization of closed sets, it suffices to show $h \in H$, which in turn requires for all $s \in S$, we verify $h(s) \in S$. We associate to the net h_α, the net $h_\alpha(s) = s_\alpha$ in X. Indeed, $h_\alpha \in H$ and $H = \mathrm{Stab}_G(S)$, and so s_α is a net in S. By assumption, the G–action on X is continuous, and since convergent nets are preserved under continuous maps, s_α is convergent. As S is closed, the limit of s_α, say t, is a point of S, and by continuity,

$$t = \lim_\alpha s_\alpha = \lim_\alpha h_\alpha(s) = (\lim_\alpha h_\alpha)(s) = h(s).$$

Thus, $h(s) \in S$, as required. □

The following is an immediate consequence of Proposition 7.1.

Corollary 7.2 *The stabilizer in* $\mathrm{Mod}(S)$ *of any closed set of conjugacy classes in* Γ_S *is separable.*

We note that since conjugacy separability for Γ_S is equivalent to the statement that the quotient topology on the set of conjugacy classes induced by the profinite topology on Γ_S is Hausdorff. Thus Theorem 6.1 is a special case this corollary.

Appendix: $\mathrm{Mod}(S)$ is not subgroup separable

For completeness, in this section we prove $\mathrm{Mod}(S)$ is not subgroup separable—Dasbach and Mangum [11] established this for $\mathrm{Out}(F_n)$. The idea is to show $\mathrm{Mod}(S)$ contains a subgroup which is not subgroup separable. To begin, whenever S contains two disjoint incompressible subsurfaces more complicated than a pair of pants, Mod(S) contains an isomorphic copy of $F_2 \times F_2$, and so cannot be subgroup separable. In the remaining cases, Mod(S) is virtually free, and hence obviously separable, or else S is a torus with two marked points or a sphere with 5 marked points. In these cases, Mod(S) contains the fundamental group of every 3–manifold fibered over the circle with fiber a once punctured torus or 4–punctured sphere, respectively. It is well known that there are such 3–manifold groups (with reducible monodromy) which are not subgroup separable (see [10]). Therefore, the mapping class groups in these special cases are not subgroup separable.

One cannot overemphasize the importance of this in regards to a venture in separating subgroups of Mod(S). The nature of this result prescribes that only finitely generated subgroups of a special nature can be separated. In some sense, without introducing new methods, results like Theorem 1.2 or Corollary 7.2 are the most general one could hope for.

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