A generalization of Long's verbal lemma

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1 Separability

For a group G, subgroup $H < G$, and $g \in G \setminus H$, we say that H and g are *separated* in G if there exists a finite index subgroup $K < G$ such that $H < K$ and $g \notin K$. If for every $g \in G \setminus H$, g and H are separated, we say that H is *separable* in G. G is residually finite if $\{1\}$ is separable in G.

Whether a subgroup H is separable in G is intimately tied to the profinite topology on G. Recall that the topology on a topological group is completely determined by a neighborhood basis at any $g \in G$ since left (respectively, right) translation is a homeomorphism. Consequently, one need only declare a neighborhood basis at any g; indeed, this always endows G with a topology for which G is a topological group. The *profinite topology* on G is the topology given by declaring the finite index subgroups of G as a neighborhood basis for the identity. Equivalently, this is the weak topology on the collection of all surjective homomorphisms of G onto finite groups where the latter are equipped with the discrete topology.

The following is easily verified.

Proposition 1.1. *H is separable in G if and only if H is closed in the profinite topology on G.*

2 Abstract functions and Long's lemma

For a non-abelian free group F_r with free generating set $x_1, \dots, x_r$, a reduced word $w \in F_r$, and group G, we define the *abstract function* $\Phi_w \colon G^r \longrightarrow G$ by

$$\Phi_w(g_1, \dots, g_r) = w(g_1, \dots, g_r).$$

Example 2.1.

For $r = 2$ and $w = x_1 x_2^2 x_1^{-2} x_2$, the function Φ_w is given by

$$\Phi_w(g, h) = gh^2 g^{-2} h.$$

The following appears in [1].

Proposition 2.1 (Long's verbal lemma). *Let G be a residually finite group, Φ_w an abstract function, and $H < G$ be maximal with respect to vanishing on Φ_w. Then H is separable in G.*

Proof. For $g \in G \setminus H$, there exists $h_1, \dots, h_{r-1} \in H$ such that

$$\Phi_w(h_1, \dots, h_{j-1}, g, h_j, \dots, h_{r-1}) = g'$$

and $g' \neq 1$. Since G is residually finite, there exists a finite group L and a surjective homomorphism $\rho\colon G \longrightarrow L$ such that $\rho(g) \neq 1$. Let $L' < L$ be subgroup maximal with respect to vanishing on Φ_w, viewed as an abstract function on L. In addition, we insist that $\rho(H) < L'$. By construction, $\rho(g) \notin L'$. Thus, setting $K = \rho^{-1}(L')$, we obtain the needed finite index subgroup which separates H and g. $\square$

The following is a straightforward generalization of Proposition 2.1.

Proposition 2.2. *Let G be a residually finite group, $\mathcal{F}$ a family of abstract functions on G, and H a subgroup maximal with respect to vanishing on $\mathcal{F}$. Then H is separable in G.*

Proof. For each function $\Phi \in \mathcal{F}$, let K_Φ be maximal with respect to vanishing on Φ and $H < K_\Phi$. By Proposition 2.1, K_Φ is separable in G. The proof is completed by noting that

$$H = \bigcap_{\Phi \in \mathcal{F}} K_\Phi$$

since H is maximal with respect to vanishing on $\mathcal{F}$ and the intersection of separable subgroups is separable. $\square$

3 Continuity of abstract functions

The following is our first generalization of Proposition 2.1.

Theorem 3.1 (Continuity of abstract functions). *Let G be a group and $\Phi_w\colon G^r \longrightarrow G$, an abstract function. Then Φ_w is continuous with respect to the profinite topology.*

Proof. It suffices to show that for a closed set $C \subset G$, $\Phi_w^{-1}(C)$ is closed. In turn, it suffices to show for each $(g_1, \dots, g_r) \notin \Phi_w^{-1}(C)$, there exists a finite group L and a surjective homomorphism

$$\rho'\colon G^r \longrightarrow L'$$

such that $\rho((g_1, \dots, g_r)) \notin \rho(\Phi_w^{-1}(c))$. By assumption $w(g_1, \dots, g_r) \notin C$. Since C is closed, there exists a finite group L and a surjective homomorphism $\rho\colon G \longrightarrow L$ such that $w(g_1, \dots, g_r) \notin \rho(C)$. Altogether, we have the diagram

$$\begin{array}{ccc} G^r & \xrightarrow{\rho \times \cdots \times \rho} & L^r \\ \Phi_w \downarrow & & \downarrow \Phi_w \\ G & \xrightarrow[\rho]{} & L \end{array}$$

Setting $L' = L^r$ and $\rho' = \rho \times \cdots \times \rho$ yields the desired pair. To see this, note that by construction $\rho'(g_1, \ldots, g_r) \notin \Phi_w^{-1}(\rho(C))$. However,

$$\rho'(\Phi_w^{-1}(C)) \subset \Phi_w^{-1}(\rho(C)).$$

Thus, $(\rho')^{-1}(\Phi_w^{-1}(\rho(C)))$ is a closed set which contains $\Phi_w^{-1}(C)$ but not $(g_1, \ldots, g_r)$. □

For a pair of subgroups $H, K < G$ and an abstract function Φ_w, we say that K *evaluates in* H under Φ_w if $\Phi_w(K^r) \subset H$.

Corollary 3.2. *Let G be a group, $H < G$ a closed subgroup, and K a subgroup of G maximal with respect to evaluating in H under Φ_w. Then K is separable in G.*

The corollary follows at once from Theorem 3.1 in combination with the following proposition (see [2]).

Proposition 3.3. *Let G be a topology group and $H < G$ a subgroup. Then the closure of H in G is the intersection of closed subgroups which contain G.*

4 Generalized abstract functions

For a finite collection $\Phi_{w_1}, \ldots, \Phi_{w_\ell}$ of abstract functions with $w_1, \ldots, w_r \in F_r$ and a finite number of elements $g_0, \ldots, g_{\ell+1}$, we define the function

$$\Phi_{w_1,\ldots,w_\ell,g_0,\ldots,g_{\ell+1}} \colon G^r \longrightarrow G$$

by

$$\Phi_{w_1,\ldots,w_\ell,g_0,\ldots,g_{\ell+1}}(h_1, \ldots, h_r) = g_0\Phi_{w_1}(h_1, \ldots, h_r)g_1 \ldots g_\ell\Phi_{w_\ell}(h_1, \ldots, h_r)g_{\ell+1}.$$

We call such functions *generalized abstract functions*

Theorem 4.1. *If G is a residually finite group, Φ is a generalized abstract function, and H a subgroup maximal with respect to vanishing on Φ, then H is separable in G.*

Proof. The proof is identical to the proof of Proposition 2.1 □

Corollary 4.2. *If G is a residually finite group, $\mathcal{F}$ is a collection of abstract functions, and H is maximal with respect to vanishing on $\mathcal{F}$, then H is separable in G.*

Corollary 4.3. *If G is residually finite and H is a maximal k–step solvable subgroup of G, then H is separable.*

Proof. Set $H^j = [H, H^{j-1}]$, $\quad H^0 = H$. Since H is k–step solvable, $H^k = 0$. Define the following functions:

$$\begin{aligned} {}^1\Phi_{h_1}(g) &= [h_1, g], \quad h_1 \in H \\ {}^2\Phi_{h_1,h_2}(g) &= [h_2,\ {}^1\Phi_{h_1}(g)], \quad h_1, h_2 \in H \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \\ {}^k\Phi_{h_1,\ldots,h_k}(g) &= [h_k,\ {}^{k-1}\Phi_{h_1,\ldots,h_{k-1}}(g)]. \end{aligned}$$

Let $\mathcal{F}_H$ denote the collection of all ${}^k\Phi_{h_1,\ldots,h_k}$. By assumption, for every $\Phi \in \mathcal{F}_H$, $\Phi(H) = 0$. If there exists $g \in G \setminus H$ such that $\Phi(g) = 0$ for all $\Phi \in \mathcal{F}_H$, then $\langle g, H \rangle$ is a solvable subgroup of step size k. However, the existence of such a g is in direct opposition with the maximality of H. Thus, H is maximal with respect to vanishing on $\mathcal{F}_H$. Therefore, by Corollary 4.2, H is separable. $\square$

Corollary 4.4. *If G is a residually finite group and H is a maximal solvable subgroup, then H is separable in G.*

Corollary 4.5. *Let G be a residually finite group, $H < G$ a subgroup. Then the centralizer of $C_G(H)$ is separable in G.*

Proof. For each $h \in H$, define $\Phi_h(g) = [h, g]$ and set $\mathcal{F} = \{\Phi_h\}_{h \in H}$. Then $C_G(H)$ is maximal with respect to vanishing on $\mathcal{F}$. Thus by Corollary 4.2, $C_G(H)$ is separable. $\square$

Theorem 4.6. *Let G be a group, $\mathcal{F}$ a collection of generalized functions, H a separable subgroup, and K maximal with respect to evaluating in H on the family $\mathcal{F}$. Then K is separable in G.*

Corollary 4.7. *Let G be a group, H a separable subgroup, and $K = H_G(H)$ the normalizer of H in G. Then K is separable in G. More generally, let $\overline{H}$ be the profinite closure of H and K be maximal with respect to conjugating H into $\overline{H}$. Then K is separable in G.*

References

[1] D. D. Long, *Immersions and embeddings of totally geodesic surfaces*, Bull. London Math. Soc. **19** (1987), no. 5, 481–484.

[2] Albert Wilansky, *Topology for analysis*, Robert E. Krieger Publishing Co. Inc., Melbourne, FL, 1983.