All nil 3-manifolds are cusps of complex hyperbolic 2-orbifolds

D. B. McReynolds

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Abstract

In this paper, we prove that every closed nil 3-manifold is diffeomorphic to a cusp cross-section of a finite volume complex hyperbolic 2-orbifold.

1 Introduction

A classical question in topology is whether a compact manifold bounds. Hamrick and Royster [9] showed that every flat manifold bounds, and it was conjectured in [6] that any almost flat manifold bounds (for some progress on this see [18]). In the same paper, Farrell and Zdravkovska made a stronger geometric conjecture [6]:

Conjecture 1.

(a) If M^n is a flat Riemannian manifold, then $M^n = \partial W^{n+1}$ where $W \setminus \partial W$ supports a complete hyperbolic structure with finite volume.

(b) If M^n supports an almost flat structure, then $M^n = \partial W^{n+1}$, where $W \setminus \partial W$ supports a complete Riemannian metric with finite volume of whose sectional curvatures are negative.

We say that a flat manifold M^n *geometrically bounds* if we have (a) in Conjecture 1 with M^n and W^{n+1}. Long and Reid [11] showed that (a) is false by proving that for a flat 3-manifold to geometrically bound the η-invariant is an integer. Furthermore, flat 3-manifolds with nonintegral η-invariant are easily constructed using [13]. An equivalent formulation of the result of Long and Reid is that it shows that some flat manifolds cannot be diffeomorphic to a cusp cross section of a 1-cusped, finite volume real hyperbolic 4-manifold. On the other hand, Long and Reid showed [12] that every flat n-manifold is diffeomorphic to a cusp cross section of a finite volume real hyperbolic $(n+1)$-orbifold.

Our main result is the complex hyperbolic plane analogue of [12] (see §2 for definitions). Namely,

Theorem 1.1. *Every closed nil 3-manifold is diffeomorphic to a cusp cross section of a finite volume, complex hyperbolic 2-orbifold.*

The proof is similar in spirit to that of [12]. First, we show that the fundamental group of a closed nil 3-manifold injects into a discrete arithmetic group of isometries of the complex hyperbolic plane. The proof is completed by a separation argument which parallels that made in [12].

2 Background material and notation

In this section we give the necessary background material for the proof of Theorem 1.1.

2.1 Models of complex hyperbolic space

See [1] or [7, Ch. 3-4] for a reference on this material.

Let

$$H : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$$

be defined by

$$H(z,w) = z_1\overline{w_1} + \cdots + z_n\overline{w_n} - z_{n+1}\overline{w_{n+1}}.$$

Define

$$V_- = \left\{ z \in \mathbb{C}^{n+1} \ : \ H(z,z) < 0 \right\},$$

the negative vectors associated with H and

$$q : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow P\mathbb{C}^{n+1}$$

be the projection map of the nonzero vectors onto complex projective space. We define *complex hyperbolic n-space* to be the image $q(V_-)$ and denote this by $\mathbf{H}^n_{\mathbb{C}}$. As a subset of $P\mathbb{C}^{n+1}$, $\partial\mathbf{H}^n_{\mathbb{C}}$ is the set $q(V_0)$, where

$$V_0 = \left\{ z \in \mathbb{C}^{n+1} \setminus \{0\} \ : \ H(z,z) = 0 \right\}.$$

The metric on $\mathbf{H}^n_{\mathbb{C}}$ is given by the equation

$$\cosh^2\left(\frac{d([z],[w])}{2}\right) = \frac{H(z,w)H(w,z)}{H(z,z)H(w,w)},$$

where $q(z) = [z]$ and $q(w) = [w]$. Letting

$$\mathrm{U}(n,1) \overset{\mathrm{def}}{=} \{A \in \mathrm{GL}(n+1,\mathbb{C}) \ : \ H(Az,Aw) = H(z,w)\}$$

then

$$\mathrm{Isom}(\mathbf{H}^n_{\mathbb{C}}) = \langle \mathrm{PU}(n,1), \tau \rangle$$

where τ is the isometry induced by complex conjugation. The subgroup consisting of those isometries which are holomorphic is simply $\mathrm{PU}(n,1)$.

We need another model of $\mathbf{H}^n_{\mathbb{C}}$ analogous to the upper half space model. Following [7, p. 111–119], we identify $\mathbf{H}^n_{\mathbb{C}}$ with an unbounded subset of $\mathbb{C}^n$, namely the set

$$\left\{(z, z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \ : \ 2 \operatorname{Re} z_n - \langle z, z \rangle > 0\right\}.$$

With this,

$$\partial \mathbf{H}^n_{\mathbb{C}} = \left\{(z, z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \ : \ 2 \operatorname{Re} z_n - \langle z, z \rangle = 0\right\} \cup \{p_\infty\}.$$

This is the *paraboloid model.* Next, we identify this model with the set

$$\left\{(z, u, v) \in \mathbb{C}^{n-1} \times (\operatorname{Re} \mathbb{C})^+ \times \operatorname{Im} \mathbb{C} \ : \ z = z, \ u + iv = 2\overline{z_n} - \langle z, z \rangle\right\}.$$

This is called the *Siegel model* with *horospherical coordinates.*

We define *horospheres* at p_∞ to be sets of the form

$$\mathcal{H}_c = \{(z, u, v) \in \mathbf{H}^n_{\mathbb{C}} \ : \ u = c\}.$$

Horoballs are sets where we have an inequality $u \geq c$. As in real hyperbolic space, we need to establish which isometries stabilize p_∞. Indeed our only interest are in those isometries which also leave invariant each horosphere centered at p_∞. Following [7, p. 116–120] (see also [4]), we take a generator of the 1-parameter family of isometries which translates along the geodesic connecting the origin and p_∞. If η is an infinitesimal generator of such an isometry, then the negative eigenspace decomposition of $\operatorname{ad} \eta$ gives the associated Lie algebras of $\operatorname{Stab}_{\mathrm{PU}(n,1)}(p_\infty)$. In particular, it yields

$$\operatorname{Stab}_{\mathrm{PU}(n,1)}(p_\infty) = \mathfrak{N}_{2n-1} \rtimes (\mathrm{U}(n-1) \cdot \mathbb{R}^\times).$$

Here $\mathfrak{N}_{2n-1}$ is the set $\mathbb{C}^{n-1} \times \mathbb{R}$ with group operation

$$(z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2 \operatorname{Im} \langle z_1, z_2 \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian product on $\mathbb{C}^{n-1}$. This group is the $(2n-1)$-dimensional *Heisenberg group.* The last factor, $\mathbb{R}^\times$, is the 1-parameter family generated by $\exp(\eta)$. This isometries are loxodromic, and in particular, those holomorphic isometries which are parabolic and elliptic are $\mathfrak{N}_{2n-1} \rtimes \mathrm{U}(n-1)$. Explicitly by exponentiating the associated Lie algebra, we get

$$\mathfrak{N}_{2n-1} \rtimes \mathrm{U}(n-1) \longrightarrow \mathrm{U}(n,1)$$

given by

$$(\xi, t, U) \longmapsto \begin{pmatrix} I_{n-1} & \xi & \xi \\ -\xi^* & 1 - \frac{1}{2}(\|\xi\|^2 - it) & -\frac{1}{2}(\|\xi\|^2 - it) \\ \xi^* & \frac{1}{2}(\|\xi\|^2 - it) & 1 + \frac{1}{2}(\|\xi\|^2 - it) \end{pmatrix} \begin{pmatrix} U & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

For future reference, we refer to this map as ψ.

As the action of $\mathfrak{N}_{2n-1}$ on $\partial\mathbf{H}^n_{\mathbb{C}} \setminus \{p_\infty\}$ is a simple, transitive action, we can identify $\partial\mathbf{H}^n_{\mathbb{C}} \setminus \{p_\infty\}$ with $\mathfrak{N}_{2n-1}$ (or $\partial\mathbf{H}^n_{\mathbb{C}}$ with the 1-point compactification of $\mathfrak{N}_{2n-1}$). Moreover, each horosphere $\mathcal{H}_c$ can be identified with $\mathfrak{N}_{2n-1}$. In fact, in the Siegel model, the action of $\mathfrak{N}_{2n-1}$ is given by

$$(\xi, t)(z, u, v) = (\xi + z, u, t + v + 2\operatorname{Im}\langle \xi, z\rangle).$$

The action of $\mathrm{U}(n-1)$ is given by

$$U(z, u, v) = (Uz, u, v).$$

In all that remains, set $\mathfrak{N}(2n-1) = \mathfrak{N}_{2n-1} \rtimes \mathrm{U}(n-1)$. Then the elements of $\mathrm{Stab}(p_\infty)$ which are elliptic and parabolic are given by $\mathfrak{N}(2n-1)$ together with τ.

If $\Pi < \mathrm{Isom}(\mathbf{H}^n_{\mathbb{C}})$ is a discrete subgroup acting properly discontinuously, then $\mathbf{H}^n_{\mathbb{C}}/\Pi$ is a complex hyperbolic orbifold. When the action is free, the resulting space is a manifold.

Let Π be a finite covolume, discrete subgroup of $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{C}})$ and let v be a cusp of the associated orbifold $M = \mathbf{H}^n_{\mathbb{C}}/\Pi$. By conjugating v to the ideal point, p_∞, we may assume that $v = p_\infty$. We suppress the conjugation in all that follows. The *maximal peripheral subgroup* of Π fixing p_∞ is defined to be

$$\triangle_\infty = \mathrm{Stab}(p_\infty) \cap \Pi.$$

By the Kazhdan-Margulis theorem,

$$L = \triangle_\infty \cap \mathfrak{N}_{2n-1}$$

is finite index in $\triangle_\infty$. In fact, $\mathfrak{N}_{2n-1}/L$ is compact. As $\triangle_\infty$ leaves invariant every horosphere at p_∞, we can select the horosphere $\mathcal{H}$ such that $\mathcal{H}/\triangle_\infty$ embeds in M (see [1]). In this case, we call $\mathcal{H}/\triangle_\infty$ a *cusp cross section* of the cusp at p_∞.

2.2 Finite volume complex hyperbolic orbifolds

In this subsection, for convenience, we give a family of finite volume, complex hyperbolic orbifolds. These are constructed using arithmetic methods (see [3] or [14, Ch. X]). We summarize want is needed in the sequel.

Theorem 2.1. *Let K be an imaginary, quadratic extension of $\mathbb{Q}$. Then $\mathrm{U}(n,1;\mathcal{O}_K)$ is a finite covolume, noncocompact, discrete subgroup of $\mathrm{U}(n,1)$. In particular, $\mathbf{H}^n_{\mathbb{C}}/\,\mathrm{U}(n,1;\mathcal{O}_K)$ is a noncompact, finite volume complex hyperbolic orbifold.*

2.3 Nil 3-manifolds

Since we are interested in the complex hyperbolic plane, we make some comments about nil 3-manifolds. See [15] or [16, p. 184–185] for a reference on this material.

The 3-dimensional Heisenberg group also has a definition as

$$\mathfrak{N}_3 = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \ : \ x, y, t \in \mathbb{R} \right\}.$$

We will identify S^1 with the rotations in the xy-plane. An *orientable nil 3-manifold* is a manifold of the form $\mathfrak{N}_3/\Gamma$, where Γ is a discrete subgroup of $\mathfrak{N}_3 \rtimes S^1$ which acts freely and properly discontinuously. As we will have need for this in the sequel, we must also consider an orientation reversing involution given by

$$\widetilde{\tau} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{\tau} \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As automorphisms of lattices of $\mathfrak{N}_3$ uniquely determine automorphisms of $\mathfrak{N}_3$ (see [5, Thrm 1.2.3]), this determines a continuous isomorphism of $\mathfrak{N}_3$. A *nil 3-manifold* is a manifold of the form $\mathfrak{N}_3/\Gamma$, where Γ is a discrete subgroup of $\left\langle \mathfrak{N}_3 \rtimes S^1, \widetilde{\tau} \right\rangle$ which acts freely and properly discontinuously.

In the above subsection, we gave a general definition of the Heisenberg group. That these two definitions coincide follows from the fact that both groups are connected, simply connected 2-step nilpotent Lie groups of dimension three and there is a unique such Lie group (up to Lie isomorphism) that has these properties. More generally, any simply connected, connected, 2-step nilpotent group with 1-dimensional center is uniquely determined (see [17]) and is Lie isomorphic to the the Heisenberg group which we defined in the above subsection.

In this form, we identify S^1 with U(1), where U(1) acts (as above) by

$$U(z,t) = (Uz, t).$$

We also identify τ and $\widetilde{\tau}$, where τ is the isometry induced by complex conjugation. Moreover, when the nil 3-manifold is closed, $\mathfrak{N}_3 \cap \Gamma$ must be finite index in Γ and $\mathfrak{N}_3/(\mathfrak{N}_3 \cap \Gamma)$ must be compact. Such manifolds are *almost flat* (see [5, p. 15] or [8]). From our analysis of cusp cross-sections, if M is a finite volume complex hyperbolic 2-orbifold with a cusp at v, the cusp cross sections at v are seen to be closed nil 3-orbifolds.

It is known (see [5, p. 144–154] or [15]) that every closed nil 3-manifold is a large Seifert fibered space. Consequently, we have the following:

Proposition 2.2. *A closed nil 3-manifold is determined (up to diffeomorphism) by its fundamental group.*

Since this will be important in what follows, we give the complete list of the fundamental groups of closed nil 3-manifolds (see [5, p. 159–167]).

(1)
$$< a, b, c \ : \ [b,a] = c^k, \ [c,a] = [c,b] = 1 > .$$
with $k \in \mathbb{N}$.

(2)

$$< a, b, c, \alpha \ : \ [b,a] = c^k, \ [c,a] = [c,b] = [\alpha, c] = 1, \ \alpha a = a^{-1}\alpha \\ \alpha b = b^{-1}\alpha, \ \alpha^2 = c >,$$

with $k \in 2\mathbb{N}$.

(3)

$$< a, b, c, \alpha \ : \ [b,a] = c^{2k}, \ [c,a] = [c,b] = [a, \alpha] = 1, \ \alpha c = c^{-1}\alpha, \\ \alpha b = b^{-1}\alpha c^{-k}, \ \alpha^2 = a >$$

with $k \in 2\mathbb{N}$.

(4)

$$< a, b, c, \alpha, \beta \ : \ [b,a] = c^{2k}, \ [c,a] = [c,b] = [c,\alpha] = [a,\beta] = 1, \\ \beta c = c^{-1}\beta, \ \alpha a = a^{-1}\alpha c^k, \alpha b = b^{-1}\alpha c^{-k}, \\ \alpha^2 = c, \ \beta^2 = a, \ \beta b = b^{-1}\beta c^{-k}, \ \alpha\beta = a^{-1}b^{-1}\beta\alpha c^{-k-1} >,$$

with $k \in 4\mathbb{N}$.

(5)

$$< a, b, c, \alpha \ : \ [b,a] = c^k, \ [c,a] = [c,b] = [c,\alpha] = 1, \ \alpha a = b\alpha, \\ \alpha b = a^{-1}\alpha, \ \alpha^4 = c^p >,$$

with $k \in 2\mathbb{N}$ and $p = 1$ or $k \in 4\mathbb{N}$ and $p = 3$.

(6)

$$< a, b, c, \alpha \ : \ [b,a] = c^k, \ [c,a] = [c,b] = [c,\alpha] = 1, \ \alpha a = b\alpha c^{k_2}, \\ \alpha b = a^{-1}b^{-1}\alpha, \ \alpha^3 = c^{k_4} >$$

with $k > 0$ and

$$k \equiv 0 \mod 3, \quad k_2 = 0, \ k_4 = 1, \ \text{or } k \equiv 0 \mod 3, \quad k_2 = 0, \ k_4 = 2,$$

or

$$k \equiv 1, 2 \mod 3, \quad k_2 = 1, \ k_4 = 1.$$

(7)

$$< a, b, c, \alpha \ : \ [b,a] = c^k, \ [c,a] = [c,b] = [c,\alpha] = 1, \ \alpha a = ab\alpha, \\ \alpha b = a^{-1}\alpha, \ \alpha^6 = c^{k_4} >,$$

with $k > 0$ and

$$k \equiv 0 \mod 6, \quad k_4 = 1, \ \text{or } k \equiv 4 \mod 6, \quad k_4 = 1,$$

or

$$k \equiv 0 \mod 6, \quad k_4 = 5, \ \text{or } k \equiv 2 \mod 6, \quad k_4 = 5.$$

2.4 Subgroup separability results

We review the essential definitions and results needed for the separation argument at the end of §3.

Let G be a group. We say that $g \in G \setminus H$ and $H < G$ are *separated* if there exists a subgroup K of finite index in G which contains H but not g. We say that $H < G$ is *separable* in G or G is *H-separable*, if every $g \in G \setminus H$ and H can be separated.

Lemma 2.3. *Let G be a residually finite group and N a maximal nilpotent subgroup of nilpotent class k. Then G is N-separable.*

Lemma 2.3 will follow from a more general result (see [10] for example).

Lemma 2.4. *Let G be a residually finite group and $f(x_1, \dots, x_n)$ an abstract polynomial. Let $H < G$ be maximal with respect to*

$$f(h_1, \dots, h_n) = 1.$$

Then G is H-separable.

Proof of Lemma 2.3. Define a family of abstract polynomials for a group G by

$$\begin{aligned} f_1(x_1, x_2) &= x_1 x_2 x_1^{-1} x_2^{-1} \\ f_2(x_1, x_2, x_3) &= f_1(x_1, x_2) x_3 (f_1(x_1, x_2))^{-1} x_3^{-1} \\ &\vdots \\ f_i(x_1, \dots, x_{i+1}) &= f_{i-1}(x_1, \dots, x_i) x_{i+1} (f_{i-1}(x_1, \dots, x_i))^{-1} x_{i+1}^{-1}. \end{aligned}$$

Notice that N is nilpotent of step k if and only if for every $n_1, \dots, n_{k+1} \in N$, we have $f_k(n_1, \dots, n_{k+1}) = 1$ and this k is the smallest such positive integer for which this holds. Therefore, by Lemma 2.4, a maximal k-step nilpotent group N is separable in G. $\square$

We will make use of the following lemma (see [12] for a proof).

Lemma 2.5. *Let G be a group and $H < K < G$. If H is separable in G and $[K : H] < \infty$, then K is separable in G.*

We need one final result on subgroup separability.

Lemma 2.6. *Let G be a group, G_0 a subgroup of finite index, and H a subgroup. G is H-separable if and only if G_0 is $(G_0 \cap H)$-separable.*

Proof. Assume that G is H-separable and $g \in G_0 \setminus H$. Then there exists $K < G$ of finite index containing H but not g. Take $K_0 = G_0 \cap K$, which is a finite index subgroup of G_0 containing H but not g, as desired.

To show that H is separable in G, it suffices by Lemma 2.5 to show that $G_0 \cap H$ is separable in G. As G_0 is a finite index subgroup of G, G_0 is certainly separable in G. We are now reduced to separating the elements in $G_0 \setminus (G_0 \cap H)$. However, by assumption this can be done. Therefore H is separable in G. $\square$

3 Proof of Theorem 1.1

Let M be a closed nil 3-manifold with $\pi_1(M) = \Gamma$. By Proposition 2.2, in order to prove Theorem 1.1, it suffices to show that $\Gamma \cong \triangle_\infty(W)$, where W is a finite volume complex hyperbolic 2-orbifold.

Recall that $\mathfrak{N}(3) = \mathfrak{N}_3 \rtimes \mathrm{U}(1)$ and τ the isometry of $\mathbf{H}^2_\mathbb{C}$ induced by conjugation. For a subring $R \subset \mathbb{C}$, we define

$$\mathfrak{N}_3(R) = R \times \operatorname{Im} R \subset \mathfrak{N}_3$$

with the induced group operation. Then

$$\mathfrak{N}(3, R) = \mathfrak{N}_3(R) \rtimes \mathrm{U}(1; R).$$

We begin by establishing injectivity. For the statement, let ζ_3 be a primitive third root of unity, say $\zeta_3 = -1/2 + \sqrt{-3}/2$.

Theorem 3.1. *Let M be a closed nil 3-manifold M and $\Gamma = \pi_1(M)$. Then there exists a faithful representation*

$$\varphi : \Gamma \longrightarrow \langle \mathfrak{N}(3, \mathcal{O}_K), \tau \rangle$$

with $K = \mathbb{Q}(i)$ or $\mathbb{Q}(\zeta_3)$ and $\mathcal{O}_K$ the ring of integers of K.

Proof. We begin by summarizing the strategy of the proof, which depends heavily on the list in §2 of presentations for the fundamental group of a closed nil 3-manifold. The idea is to show that an injective homomorphism on the lattice subgroup $\langle a, b, c\rangle$ (recall §2) can be promoted to the full 3-manifold group (see [5, Thrm 3.1.3]). To get a representation with the coefficients in $\mathbb{Z}[i]$ or $\mathbb{Z}[\zeta_3]$, we are reduced to solving some simple equations. The details are as follows.

Let

$$p_1 : \mathfrak{N}_3 \longrightarrow \mathbb{C}, \quad p_2 : \mathfrak{N}_3 \longrightarrow \mathbb{R}$$

be projection onto the first factor and second factor, respectively.

Lemma 3.2.

(a) Let a and b be as above and

$$\rho : \langle a, b, c\rangle \longrightarrow \mathfrak{N}_3$$

be a homomorphism. If $p_1(\rho(a))$ and $p_1(\rho(b))$ are $\mathbb{Z}$-linearly independent and $c \notin \ker \rho$, then ρ is injective.

(b) Let

$$\rho : \Gamma \longrightarrow \langle \mathfrak{N}(3), \tau \rangle$$

be a homomorphism. Assume that $\rho(\alpha), \rho(\beta) \notin \mathfrak{N}_3$ and $\rho(\langle a, b, c\rangle) = \rho(\Gamma) \cap \mathfrak{N}_3$. If $\rho_{|\langle a,b,c\rangle}$ is injective, and

$$\langle \alpha, \beta \rangle \cap \ker \rho = \{1\},$$

then ρ is injective.

Proof. For (a), let $w \in \ker \rho$ and write

$$w = a^{n_1} b^{n_2} c^{n_3}.$$

Also, let $\rho(a) = (v_1, t_a)$, $\rho(b) = (v_2, t_b)$, and $\rho(c) = (0, s)$. Then

$$\rho(w) = (n_1 v_1 + n_2 v_2, n_1 t_a + n_2 t_b + 2\,\mathrm{Im}\,\langle n_1 v_1, n_2, v_2 \rangle + n_3 s).$$

Since $w \in \ker \rho$,

$$n_1 v_1 + n_2 v_2 = 0.$$

By $\mathbb{Z}$-linear independence of v_1 and v_2, $n_1 = n_2 = 0$. Therefore $n_3 s = 0$, and so $n_3 = 0$, as $s \neq 0$.

For (b), let $w \in \ker \rho$ and write

$$w = \ell \alpha^{n_1} \beta^{n_2},$$

where $\ell \in \langle a, b, c \rangle$. Since $\rho(\alpha)$ and $\rho(\beta)$ are not in $\mathfrak{N}_3$, we must have

$$q_2 \rho(\alpha^{n_1}) \rho(\beta^{n_2}) = 0,$$

where

$$q_2 : \mathfrak{N}(3) \longrightarrow \langle \mathrm{U}(1), \tau \rangle$$

is projection onto the second factor. This can happen if and only if

$$\rho(\alpha^{n_1}) \rho(\beta^{n_2}) \in \mathfrak{N}_3.$$

This implies that $w \in \langle a, b, c \rangle$ and thus $w = 0$. ☐

Let $L = \langle a, b, c \rangle$ and define two homomorphisms

$$\varphi_1, \varphi_3 : L \longrightarrow \mathfrak{N}_3$$

by

$$\begin{aligned} \varphi_1(a) &= (1, 0), & \varphi_3(a) &= (1, 0) \\ \varphi_1(b) &= (i, 0), & \varphi_3(b) &= (\zeta_3, 0). \end{aligned}$$

This determines c, since some power of c is a commutator of a and b. By Lemma 3.2 (a), both maps are injective.

We extend this to Γ be declaring

$$\varphi_j(\alpha) = (z_1, t_1, \eta_1), \quad \varphi_j(\beta) = (z_2, t_2, \eta_2), \tag{2}$$

where $z_1, z_2 \in \mathbb{C}$, $t_1, t_2 \in \mathbb{R}$, and $\eta_1, \eta_2 \in \langle \mathrm{U}(1), \tau \rangle$.

To solve Equation (2), we simply use the presentations in §2 to ensure that this yields a homomorphism. By applying Lemma 3.2 (b), one can see that these are injective. For clarity, we solve the equations in the second family (2) and give a list of the equations and solutions for the seventh family (7). We relegate the rest of the solutions to the appendix.

From §2, the second family has presentation

$$\left\langle a, b, c, \alpha \ : \ [b,a] = c^k, \ [c,a] = [c,b] = [c,\alpha] = 1, \ \alpha a = a^{-1}\alpha, \alpha b = b^{-1}\alpha, \alpha^2 = c \right\rangle$$

with $k \in 2\mathbb{N}$. We take the map φ_1 for this family. First, consider the relation $[b,a] = c^k$. Then

$$\begin{aligned} [\varphi_1(b), \varphi_1(a)] &= [(i,0,1),(1,0,1)] \\ &= (i+1, 2\operatorname{Im}\langle i,1\rangle, 1)(-i-1, 2\operatorname{Im}\langle -i,-1\rangle, 1) \\ &= (0,4,1). \end{aligned}$$

Since $[\varphi_1(b), \varphi_1(b)] = \varphi_1(c)^k$, it follows that $\varphi_1(c) = (0, 4/k, 1)$.

Next, consider the relation $\alpha^2 = c$. We have

$$\begin{aligned} (z_1, t_1, \eta_1)(z_1, t_1, \eta_1) &= (z_1 + \eta_1 z_1, 2t_1 + 2\operatorname{Im}\langle z_1, \eta_1 z_1\rangle, \eta_1^2) \\ &= (0, 4/k, 1). \end{aligned}$$

In particular, $\eta_1^2 = 1$. If $\eta = 1$, then the above injection would yield an isomorphism between a group in the first family with a group in the second family. This is impossible, therefore $\eta = -1$. By considering the first coordinate equation with $\eta = -1$, we get no information. The second coordinate equation is $2t_1 = 4/k$, therefore $t_1 = 2/k$. One can now check that $[c,\alpha] = 1$, regardless of z_1.

Now, we take the relation $\alpha a = a^{-1}\alpha$. We have

$$(z_1, 2/k, -1)(1,0,1) = (z_1 - 1, 2/k - 2\operatorname{Im} z_1, -1).$$

On the other hand,

$$(-1,0,1)(z_1, 2/k, -1) = (-1 + z_1, 2/k + 2\operatorname{Im} z_1, -1).$$

The first and last coordinate equations yield no information, while the second coordinate equation yields $4\operatorname{Im} z_1 = 0$, hence $\operatorname{Im} z_1 = 0$.

Lastly, we have the relation $\alpha b = b^{-1}\alpha$. We have

$$(z_1, 2/k, -1)(i,0,1) = (z_1 - i, 2/k + 2\operatorname{Re} z_1, -1).$$

On the other hand,

$$(-i,0,1)(z_1, 2/k, -1) = (-i + z_1, 2/k - 2\operatorname{Re} z_1, -1).$$

As above, the first and second coordinates yields no information, while the second coordinate implies that $\operatorname{Re} z_1 = 0$.

Hence, we deduce from the above computations,

$$\begin{aligned} \varphi_1(a) &= (1,0,1), & \varphi_1(b) &= (i,0,1) \\ \varphi_1(c) &= (0,4/k,1) & \varphi_1(\alpha) &= (0,2/k,-1). \end{aligned}$$

Notice that these solutions are in $\mathbb{Q}(i)$ and not $\mathbb{Z}[i]$. This is rectified by conjugating the above representation by a dilation of $2k$. This dilation is linear on the first factor, quadratic on the second factor, and trivial on the third factor. The resulting faithful representation is

$$\begin{aligned} \varphi_1(a) &= (2k, 0, 1), & \varphi_1(b) &= (2ki, 0, 1) \\ \varphi_1(c) &= (0, 16k, 1) & \varphi_1(\alpha) &= (0, 8k, -1). \end{aligned}$$

For the seventh family (7), we have the presentation

$$\left\langle a, b, c, \alpha \ : \ [b,a] = c^k, \ [c,a] = [c,b] = [c,\alpha] = 1, \ \alpha a = ab\alpha, \ \alpha b = a^{-1}\alpha, \ \alpha^6 = c^{k_4} \right\rangle$$

with

$$\begin{aligned} &k \equiv 0 \mod 6, \ k_4 = 1, \text{ or } k \equiv 4 \mod 6, \ k_4 = 1, \text{ or} \\ &k \equiv 0 \mod 6, \ k_4 = 5, \text{ or } k \equiv 2 \mod 6, \ k_4 = 5. \end{aligned}$$

We take φ_3 in this case. By considering all the relations, we get

$$\begin{aligned} (0, 4/k, 1) &= (0, s, 1) \\ (z_1 + \eta_1, t_1 + 2\,\mathrm{Im}\,\langle z, \eta_1 \rangle, \eta_1) &= (1 + \zeta_3 + z_1, t_1 + \\ &\qquad 2\,\mathrm{Im}\,\langle 1 + \zeta_3, z_1 \rangle + 2\,\mathrm{Im}\,\langle 1, \zeta_3 \rangle + t_1, \eta_1) \\ (z_1 + \eta_1\zeta_3, t_1 + 2\,\mathrm{Im}\,\langle z_1, \eta_1\zeta_3 \rangle, \eta_1) &= (-1 + z_1, t_1 + 2\,\mathrm{Im}\,\langle -1, z_1 \rangle, \eta_1). \end{aligned}$$

The last relation, $\alpha^6 = c^{k_4}$ is a bit long to write out in its general form. Of course, the commutator relations (aside from $[b,a] = c^k$) are all trivially satisfied. Solving these equations and conjugating by a dilation of $12k$ to get the coefficients in $\mathbb{Z}[\zeta_3]$, we have

$$\begin{aligned} \varphi_3(a) &= (12k, 0, 1), & \varphi_3(b) &= (12k\zeta_3, 0, 1), \\ \varphi_3(c) &= (0, 288k\sqrt{3}, 1), & \varphi_3(\alpha) &= (-6k, 12k(4k_4 + 3k)\sqrt{3}, \zeta_6). \end{aligned}$$

The other remaining families are handled similarly. □

To complete the proof of Theorem 1.1, we need some extra control. Let us establish the following notation. Let

$$\rho : \Gamma \longrightarrow \langle U(2,1; \mathcal{O}_K), \tau \rangle$$

be the faithful representation obtain from Theorem 3.1 and Equation (1), and let $G_K = \langle \mathrm{U}(2,1; \mathcal{O}_K), \tau \rangle$. Finally, recall the map ψ given in Equation (1) of §2. Then

Proposition 3.3. *For ρ and L as above, we have:*

(a) $\rho(\Gamma) \subset \triangle_\infty(G_K)$, and

(b) there exists an ideal $\mathfrak{p} \subset \mathcal{O}_K$ *such that*

$$\psi(\mathfrak{N}_3(\mathfrak{p})) \subset \rho(L) \subset \psi(\mathfrak{N}_3(\mathcal{O}_K)) \subset \triangle_\infty(G_K)$$

with

$$[\rho(L) : \psi(\mathfrak{N}_3(\mathfrak{p}))],\ [\psi(\mathfrak{N}_3(\mathcal{O}_K)) : \rho(L)] < \infty.$$

Proof. That (a) holds is a consequence of the fact that Equation (1) identifies $\mathfrak{N}(3)$ with a subgroup of $\mathrm{Stab}(p_\infty)$. Moreover, the elements in this subgroup consist of only parabolic and elliptic elements. To see that (b) holds, we simply take an ideal contained in the intersection of each principal ideal generated by the complex coefficients of a and b under the map φ_j into $\mathfrak{N}_3$. Then take a generator for this ideal and take the least common multiple of this element with the coefficient of c under the mapping φ_j. Finally let $\mathfrak{p}$ be the ideal generated by this least common multiple.

For example, in the family (1),

$$a \longmapsto (2k, 0), \quad b \longmapsto (2ki, 0), \quad c \longmapsto (0, 16k).$$

As $(2k) = (2ki)$ as ideals in $\mathbb{Z}[i]$, we can take either principal ideal. We then take the generator, say $2k$, and $16k$. The least common multiple is $16k$, and so $\mathfrak{p} = 16k\mathbb{Z}[i]$. To see that $\mathfrak{N}_3(\mathfrak{p}) < \varphi_1(L)$, we must verify that $(16k, 0)$, $(16ki, 0)$, $(0, 16) \in \varphi_1(L)$, as these certainly generator $\mathfrak{N}_3(\mathfrak{p})$. For this notice that

$$\varphi_1(a)^8 = (16k, 0), \quad \varphi_1(b)^8 = (16ki, 0), \quad \varphi_1(c) = (0, 16k).$$

We refer the reader to the appendix for the remaining ideals $\mathfrak{p}$ for each of the seven families. ☐

In the construction of the faithful representation ρ in Theorem 3.1, we cannot ensure that $\Gamma = \triangle_\infty(W)$. A separation argument is needed to guarantee that we can achieve this by lifting to a finite cover.

Proposition 3.4. *G_K is $\rho(\Gamma)$-separable.*

Proof. By Lemma 2.6 if suffices to show that $\mathrm{U}(2,1;\mathcal{O}_K) \cap \rho(\Gamma)$ is separable in $\mathrm{U}(2,1;\mathcal{O}_K)$. Thus there is no loss in generality in assuming that $\rho(\Gamma) \subset \mathrm{U}(2,1;\mathcal{O}_K)$. Let $H_K = \mathrm{U}(2,1;\mathcal{O}_K)$.

To show that H_K is $\rho(\Gamma)$-separable, it suffices by Lemma 2.5 to show that H_K is $\rho(L)$-separable. In fact, we need only show that H_K is $\psi(\mathfrak{N}_3(\mathfrak{p}))$-separable, where $\mathfrak{p}$ is given in Proposition 3.3. Since $\psi(\mathfrak{N}_3(\mathcal{O}_K))$ is a maximal 2-step nilpotent subgroup of H_K and H_K is residually finite, by Lemma 2.3, it suffices to separate the points in $\psi(\mathfrak{N}_3(\mathcal{O}_K)) \setminus \psi(\mathfrak{N}_3(\mathfrak{p}))$ from $\psi(\mathfrak{N}_3(\mathfrak{p}))$. To this end, we can use the reduction homomorphism induced by the ring homomorphism

$$r_\mathfrak{p} : \mathcal{O}_K \longrightarrow \mathcal{O}_K/\mathfrak{p}.$$

Note that the latter is a finite ring. Let

$$r_\mathfrak{p} : H_K \longrightarrow \mathrm{U}(2,1;\mathcal{O}_K/\mathfrak{p})$$

denote the induced homomorphism and $K_{\mathfrak{p}} = \ker r_{\mathfrak{p}}$. Let $\gamma \in \mathfrak{N}_3(\mathcal{O}_K) \setminus \mathfrak{N}_3(\mathfrak{p})$. We claim that $K_{\mathfrak{p}}$ separates $\mathfrak{N}_3(\mathfrak{p})$ and γ. To see this, simply notice that $\mathfrak{N}_3(\mathcal{O}_K) \cap K_{\mathfrak{p}} = \mathfrak{N}_3(\mathfrak{p})$. Therefore, $\gamma \notin K_{\mathfrak{p}}$. That $K_{\mathfrak{p}}$ is finite index is immediate from the fact that $\mathcal{O}_K/\mathfrak{p}$ is a finite ring. $\square$

We now prove Theorem 1.1.

Proof of Theorem 1.1. Using Proposition 3.3, we have a faithful representation

$$\rho : \Gamma \longrightarrow G_K$$

with $\rho(\Gamma) \subset \triangle_\infty(G_K)$. By Proposition 3.4, there exists $\Pi < G_K$ of finite index such that

$$\Pi \cap \triangle_\infty(G_K) = \rho(\Gamma).$$

By Theorem 2.1, we know that G_K (and consequently Π) is a discrete, finite covolume subgroup. It now follows that Γ is the maximal peripheral subgroup of a finite covolume, discrete subgroup of $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$. By Proposition 2.2, we have that $\mathfrak{N}_3/\Gamma = M$ is diffeomorphic to a cusp cross-section of $\mathbf{H}^2_{\mathbb{C}}/\Pi$, as desired. $\square$

4 A final comment

By unpublished work of W. Neumann and A. Reid, one cannot expect to obtain manifolds of finite volume with just a single cusp in the above construction. However, a natural question is whether or not one can guarantee that $\mathbf{H}^2_{\mathbb{C}}/\Pi$ be constructed to be a manifold. For instance, in the second family (2), the only congruence subgroup which contains Γ is the level two congruence subgroup. This is seen to have torsion though, as

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an element of this subgroup. However by results of Borel [2], we can guarantee that the first and third families map into torsion free groups. We can simply conjugate by a dilation to ensure that the representation maps into a congruence subgroup which is torsion free. [2] is needed in showing that the congruence subgroup is torsion free.

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5 Appendix

5.1 The solutions to Equation (2)

Here, we include the maps found by solving Equation (2) in §3. The final representation is obtained by applying Equation (1). In each of these, we have conjugated by a dilation as to ensure that the coefficients are in $\mathcal{O}_K$. For all the families except (6) and (7), $K = \mathbb{Q}(i)$. For (6) and (7), $K = \mathbb{Q}(\zeta_3)$. Finally, these maps are by no means unique (even up to conjugation by a dilation).

(1) We take φ_1.

$$a \longmapsto (2k, 0, 1), \qquad b \longmapsto (2ki, 0, 1), \qquad c \longmapsto (0, 16k, 1).$$

For $\mathfrak{p}$, take

$$\mathfrak{p} = 16k\mathbb{Z}[i].$$

By applying ψ, we get:

$$\begin{aligned} a &= \begin{pmatrix} 1 & 2k & 2k \\ -2k & 1-2k^2 & -2k^2 \\ 2k & 2k^2 & 1+2k^2 \end{pmatrix}, \\ b &= \begin{pmatrix} 1 & 2ki & 2ki \\ 2ki & 1-2k^2 & 2k^2 \\ -2ki & 2k^2 & 1+2k^2 \end{pmatrix}, \\ c &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+8ki & 8ki \\ 0 & -8ki & 1-8ki \end{pmatrix}. \end{aligned}$$

(2) We take φ_1.

$$\begin{aligned} a &\longmapsto (2k, 0, 1), & b &\longmapsto (2ki, 0, 1), \\ c &\longmapsto (0, 16k, 1), & \alpha &\longmapsto (0, 2k, -1). \end{aligned}$$

For $\mathfrak{p}$, take

$$\mathfrak{p} = 16k\mathbb{Z}[i].$$

By applying ψ, we get:

$$a = \begin{pmatrix} 1 & 2k & 2k \\ -2k & 1-2k^2 & -2k^2 \\ 2k & 2k^2 & 1+2k^2 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 & 2ki & 2ki \\ 2ki & 1-2k^2 & 2k^2 \\ -2ki & 2k^2 & 1+2k^2 \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+8ki & 8ki \\ 0 & -8ki & 1-8ki \end{pmatrix},$$

$$\alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1+ki & ki \\ 0 & -ki & 1-ki \end{pmatrix}.$$

(3) We take φ_1.

$$a \longmapsto (4k, 0, 1), \qquad b \longmapsto (4ki, 0, 1),$$
$$c \longmapsto (0, 32k, 1), \qquad \alpha \longmapsto (2k, 0, \tau).$$

For $\mathfrak{p}$, take

$$\mathfrak{p} = 32k\mathbb{Z}[i].$$

By applying ψ, we get:

$$a = \begin{pmatrix} 1 & 4k & 4k \\ -4k & 1-8k^2 & -8k^2 \\ 4k & 8k^2 & 1+8k^2 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 & 4ki & 4ki \\ 4ki & 1-8k^2 & 8k^2 \\ -4ki & 8k^2 & 1+8k^2 \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+16ki & 16ki \\ 0 & -16ki & 1-16ki \end{pmatrix},$$

$$\alpha = \begin{pmatrix} 1 & 2k & 2k \\ 2k & 1-2k^2 & -2k^2 \\ -2k & 2k^2 & 1+2k^2 \end{pmatrix} \tau.$$

By the last element, we mean first apply τ, then the given matrix.

(4) We take φ_1.

$$a \longmapsto (4k, 0, 1), \qquad b \longmapsto (4ki, 0, 1), \qquad c \longmapsto (0, 32k, 1),$$
$$\alpha \longmapsto (2k + 2ki, 0, -1), \qquad \beta \longmapsto (2k, 0, \tau).$$

For $\mathfrak{p}$, take

$$\mathfrak{p} = 32k\mathbb{Z}[i].$$

By applying ψ, we get:

$$a = \begin{pmatrix} 1 & 4k & 4k \\ -4k & 1-8k^2 & -8k^2 \\ 4k & 8k^2 & 1+8k^2 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 & 4ki & 4ki \\ 4ki & 1-8k^2 & 8k^2 \\ -4ki & 8k^2 & 1+8k^2 \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+16ki & 16ki \\ 0 & -16ki & 1-16ki \end{pmatrix},$$

$$\alpha = \begin{pmatrix} -1 & 2k+2ki & 2k+2ki \\ -2k+2ki & 1-4k^2 & -4k^2 \\ 2k-2ki & 4k^2 & 1+4k^2 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 2k & 2k \\ -2k & 1-4k^2 & -4k^2 \\ 2k & 4k^2 & 1+4k^2 \end{pmatrix}\tau.$$

The meaning of the last element is as in (3) above.

(5) We take φ_1.

$$a \longmapsto (2k, 0, 1), \qquad b \longmapsto (2ki, 0, 1),$$
$$c \longmapsto (0, 16k, 1), \qquad \alpha \longmapsto (0, pk, i).$$

For $\mathfrak{p}$, take

$$\mathfrak{p} = 16k\mathbb{Z}[i].$$

By applying ψ, we get:

$$a = \begin{pmatrix} 1 & 2k & 2k \\ -2k & 1-2k^2 & -2k^2 \\ 2k & 2k^2 & 1+2k^2 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 & 2ki & 2ki \\ 2ki & 1-2k^2 & 2k^2 \\ -2ki & 2k^2 & 1+2k^2 \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+8ki & 8ki \\ 0 & -8ki & 1-8ki \end{pmatrix},$$

$$\alpha = \begin{pmatrix} i & 0 & 0 \\ 0 & 1+\frac{pk}{2}i & \frac{pk}{2}i \\ 0 & -\frac{pk}{2}i & 1-\frac{pk}{2}i \end{pmatrix}.$$

(6) We take φ_3.

$$\begin{aligned}
a &\longmapsto (24k, 0, 1),\\
b &\longmapsto (24k\zeta_3, 0, 1),\\
c &\longmapsto (0, 1152k\sqrt{3}, 1),\\
\alpha &\longmapsto (-6[k+2k_2] + 6\sqrt{3}[k-2k_2]i, 192k\sqrt{3}(k\left\|z\right\|^2 + 2k_4), \zeta_3),
\end{aligned}$$

where

$$z = -6[k+2k_2] + 6\sqrt{3}[k-2k_2]i.$$

For $\mathfrak{p}$, take

$$\mathfrak{p} = 1152k\mathbb{Z}[\zeta_3].$$

By applying ψ, we get:

$$\begin{aligned}
a &= \begin{pmatrix} 1 & 24k & 24k \\ -24k & 1-288k^2 & -288k^2 \\ 24k & 288k^2 & 1+288k^2 \end{pmatrix},\\
b &= \begin{pmatrix} 1 & -12+12\sqrt{3}i & -12+12\sqrt{3}i \\ 12+12\sqrt{3}i & 1-288k^2 & -288k^2 \\ -12-12\sqrt{3}i & 288k^2 & 1+288k^2 \end{pmatrix},\\
c &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+144k\sqrt{3}i & 144k\sqrt{3}i \\ 0 & -144k\sqrt{3}i & 1-144k\sqrt{3}i \end{pmatrix},\\
\alpha &= \begin{pmatrix} 1 & \mu & \mu \\ 6[k+2k_2] + 6\sqrt{3}[k-2k_2]i & 1-\sigma & -\sigma \\ -6[k+2k_2] - 6\sqrt{3}[k-2k_2]i & \sigma & 1+\sigma \end{pmatrix} \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

where

$$\sigma = \frac{1}{2}\left[36(k+2k_2)^2 + 108(k-2k_2)^2 - 192k\sqrt{3}(k\left\|z\right\|^2 + 2k_4)i\right]$$

and

$$\mu = -6[k+2k_2] + 6\sqrt{3}[k-2k_2]i.$$

(7) We take φ_3.

$$\begin{aligned}
a &\longmapsto (12k, 0, 1), & b &\longmapsto (12k\zeta_3, 0, 1),\\
c &\longmapsto (0, 288k\sqrt{3}, 1), & \alpha &\longmapsto (-6k, 12k(4k_4+3k)\sqrt{3}, \zeta_6).
\end{aligned}$$

For $\mathfrak{p}$, take

$$\mathfrak{p} = 288k\mathbb{Z}[\zeta_3].$$

By applying ψ, we get:

$$a = \begin{pmatrix} 1 & 12k & 12k \\ -12k & 1-144k^2 & -144k^2 \\ 12k & 144k^2 & 1+144k^2 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 & -6k+6k\sqrt{3}i & -6k+6k\sqrt{3}i \\ 6k+6k\sqrt{3}i & 1-144k^2 & -144k^2 \\ -6k-6k\sqrt{3}i & 144k^2 & 1+144k^2 \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+288k\sqrt{3}i & 288k\sqrt{3}i \\ 0 & -288k\sqrt{3}i & 1-288\sqrt{3}i \end{pmatrix},$$

$$\alpha = \begin{pmatrix} 1 & -6k & -6k \\ 3k-3k\sqrt{3}i & 1-\chi & -\chi \\ -3k+3k\sqrt{3}i & \chi & 1+\chi \end{pmatrix} \begin{pmatrix} \zeta_6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\chi = -36k^2 + 12k(4k_4 + 3k)\sqrt{3}i.$$

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Department of Mathematics,
University of Texas, Austin, TX 78712.
e-mail: `dmcreyn@math.utexas.edu`