Arithmetic lattices in $\mathrm{SU}(n,1)$

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Contents

CHAPTER 1

Introduction

The purpose of this note is to provide an accessible introduction to arithmetic lattices in $\mathrm{SU}(n,1)$. The mathematical goal is two-fold; give the classification of arithmetic lattices in $\mathrm{SU}(n,1)$ and prove that there are infinitely many commensurability classes of lattices of first and second type (see below for the definitions of these types of lattices). In order to keep the technical aspects of this note to a minimum, we give the general construction of arithmetic lattices in $\mathrm{SU}(n,1)$ and only sketch the proof of their totality. For the construction of infinitely many commensurability classes for each type, some background is needed; for instance, at the least the Hasse norm theorem and the Hasse invariant for a simple algebra over a number field.

Acknowledgements

This note is an amalgam of a series of lectures given at the University of Texas and the University of Maryland. I would like to thank Bill Goldman, Alan Reid, David Saltman, and Rich Schwartz for conversation before, during, and after these lectures. I would also like to thank John Parker for his interest in these notes. Finally, I would like to thank Misha Belolipetsky and Matthew Stover for reading a draft of this notes and for pointing out several errors.

CHAPTER 2

Some basic terminology

In this section, we present some of the requisite material for the sequel.

1. Commensurability and wide commensurability

For a group G with subgroups $H_1, H_2 < G$, we say that H_1 and H_2 are *commensurable* in $H_1 \cap H_2$ if a finite index subgroup of H_1 and H_2. More generally, if a G–conjugate of H_1 is commensurable with H_2, we say that H_1 and H_2 are *commensurable in the wide sense*.

2. Number fields

By a *number field* k, we mean a finite extension of $\mathbf{Q}$ and denote the *ring of integer* of k by $\mathscr{O}_k$. This is a finitely generated $\mathbf{Z}$–module consisting of those elements of k which are roots of a monic polynomial in $\mathbf{Z}[t]$. For a fixed number field k, a *real embedding* σ of k is an embedding $\sigma\colon k \longrightarrow \mathbf{R}$. Similarly, a *complex embedding* τ of k is an embedding $\tau\colon k \longrightarrow \mathbf{C}$ such that $\tau(k)$ is not contained in $\mathbf{R}$. Up to post-composition by an automorphism of $\mathbf{R}$, there are only finitely many real embeddings $\sigma_1, \dots, \sigma_{r_1}$. Likewise, up to post-composition by an automorphism of $\mathbf{C}$, there are only finitely many complex embeddings $\tau_1, \dots, \tau_{r_2}$. It is a fundamental fact that (see [**37**]) $[k : \mathbf{Q}] = r_1 + 2r_2$. We say that k is *totally real* if $r_2 = 0$ and k is *totally imaginary* if $r_1 = 0$.

Throughout this note, we will be primarily concerned with totally imaginary quadratic extensions E of a totally real number field F. By this we mean that E is totally imaginary with totally real subfield F such that $[E : F]$ is two. The extension E/F is necessarily Galois and we denote the Galois group by $\mathrm{Gal}(E/F)$ and its non-trivial involution by $*$. Note that for a pair of *compatible embeddings* $\sigma\colon F \longrightarrow \mathbf{R}$ and $\tau\colon E \longrightarrow \mathbf{C}$ such that $\tau(E) \cap \mathbf{R} = \sigma(F)$, $*$ is nothing more than the restriction of complex conjugation in $\mathbf{C}$ to E.

In examples, we often work with cyclotomic fields. By an *nth root of unity* in a field k, we mean an element $\zeta \in k^\times$ such that $\zeta^n = 1$ and say that ζ is *primitive* if ζ is a generator for the finite multiplicative group of all nth roots of unity. Typically, we denote a primitive nth root of unity by ζ_n and call the extension $\mathbf{Q}(\zeta_n)$ a *cyclotomic extension*.

THEOREM 2.1 (Weak Approximation Theorem). *Let K be a totally real number field with real embeddings $\sigma_1, \dots, \sigma_s$. For any $p, q \in \mathbf{N}$ with $p+q = s$, there exists $\lambda \in K$ and embeddings $\sigma_{j_1}, \dots, \sigma_{j_p}$ such that $\sigma_k(\alpha) > 0$ if and only if $k = j_\ell$ for $\ell = 1, \dots, p$.*

3. Valuations on number fields

By a *valuation* ν on a number field k, we mean a map $\nu \colon k \longrightarrow \mathbf{R}$ such that

(a) $\nu(x) \geq 0$ for all $x \in k$,
(b) $\nu(xy) = \nu(x)\nu(y)$ for all $x, y \in k$, and
(c) $\nu(x+y) \leq \nu(x) + \nu(y)$.

If, in addition, ν satisfies the *ultra triangle inequality*

$$\nu(x+y) \leq \max\{\nu(x), \nu(y)\}$$

we say that ν is *non-archimedean* and otherwise say ν is *archimedean*.

Given a valuation ν on k, we obtain a metric d_ν by setting $d_\nu(x,y) = \nu(x-y)$. We say two valuation ν_1 and ν_2 are *equivalent* if the topologies induced by d_{ν_1} and d_{ν_2} are the same. Alternatively, ν_1 and ν_2 are equivalent if there exists a positive real number β such that $\nu_1(x) = \nu_2(x)^\beta$. Given such a β, it is a simple matter to see that the metric balls in the space (k, d_{ν_1}) contain (and are contained by) metric balls in the metric space (k, ν_2).

Aside from the archimedean valuation $|\cdot|$ given by $\mathbf{Q} \subset \mathbf{R}$, for each prime integer $p \in \mathbf{N}$, we obtain a nonarchimedean valuation on $\mathbf{Q}$ as follows. For a rational number x, write $x = p^s y$ such that neither the numerator nor the denominator of y are divisible by p. We define the *p–adic valuation* $|\cdot|_p$ on $\mathbf{Q}$ by $|x|_p = p^s$. One can verify that $|\cdot|_p$ is nonarchimedean.

THEOREM 2.2 ([**46**]). *Up to equivalence, the valuation on $\mathbf{Q}$ are $|\cdot|_p$ for each prime $p \in \mathbf{N}$ together with the archimedean valuation $|\cdot|$.*

4. Extensions of valuations

For any number field $k/\mathbf{Q}$, the valuations on $\mathbf{Q}$ extend to a family of valuations on k. The archimedean valuation are simply those that come from the embeddings $k \subset \mathbf{R}$ or $k \subset \mathbf{C}$ and the nonarchimedean valuations come from the prime ideal $\mathfrak{p} < \mathscr{O}_k$ (see [**30**]). Specifically, for each prime ideal $\mathfrak{p}$ over p, we obtain an extension of ν_p to k which we denote by $\nu_{\mathfrak{p}}$. We denote the equivalence classes of nonarchimedean valuation on k by $V_f(k)$ and the archimedean valuations by $V_\infty(k)$. For later use, we denote the archimedean valuation given by real embeddings and those given by complex embeddings by $V_{\mathbf{R}}(k)$ and $V_{\mathbf{C}}(k)$, respectively. Finally, the equivalence class of all valuations on k is denoted by $V(k)$.

5. Local fields

Given a valuation ν associated to a prime ideal $\mathfrak{p} \in \mathscr{O}_k$, we can form the metric completion (k, d_ν), obtaining a locally compact, complete field $k_\mathfrak{p}$ or k_ν. By a *local field*, we mean the metric completion of a number field for a fixed nonarchimedean valuation. Given a local field $k_\mathfrak{p}$ with valuation ν, define

$$\mathscr{O}_{k,\mathfrak{p}} = \{x \in k \,:\, \nu(x) \leq 1\}$$
$$\mathfrak{m}_{k,\mathfrak{p}} = \{x \in k \,:\, \nu(x) < 1\}.$$

PROPOSITION 2.3 ([**46**]). *$\mathscr{O}_{k,\mathfrak{p}}$ is a local, principle ideal domain with maximal ideal $\mathfrak{m}_{k,\mathfrak{p}}$.*

As a principle ideal, $\mathfrak{m}_{k,\mathfrak{p}}$ is generated by $\pi \in k$ and we call such an element π a *uniformizer*, which is unique up to multiplication by ε with $\nu(\varepsilon) = 1$.

THEOREM 2.4 (Strong Approximation Theorem). *Add.*

6. Algebraic groups

Let k be a field with $\mathrm{char}(k) = 0$ and algebraic closure $\overline{k}$. By a *linear algebraic group* $\mathbf{G}$, we mean a Zariski closed subgroup of $\mathrm{GL}(n;\overline{k})$. Associated to $\mathbf{G}$ is an ideal $\mathfrak{a} < \overline{k}[T]$, where T denotes an n^2–tuple of indeterminants, one for each matrix coefficient. The relationship between $\mathbf{G}$ and $\mathfrak{a}$ is simple; $\mathbf{G}$ is the zero set $V(\mathfrak{a})$ of $\mathfrak{a}$. Consequently, we call $\mathfrak{a}$ the *ideal of vanishing* for $\mathbf{G}$. It follows from the Hilbert Basis theorem that $\mathfrak{a}$ is finitely generated. We say that $\mathbf{G}$ is defined over a subfield $L \subset \overline{k}$ if there exists a finite set $P_1(T), \dots, P_s(T) \in L[T]$ which generates $\mathfrak{a}$. In this case, we say that $\mathbf{G}$ is an *L–algebraic group* and denote the ideal, generated over $L[T]$, by $\mathfrak{a}_L$.

Given an L–algebraic group $\mathbf{G} < \mathrm{GL}(n;\overline{k})$ and an algebraic embedding $\rho\colon \mathbf{G} \longrightarrow \mathrm{GL}(m;\overline{k})$ defined over L, the groups $\mathbf{G} \cap \mathrm{GL}(n;L)$ and $\rho(\mathbf{G}) \cap \mathrm{GL}(m;L)$ are isomorphic. In fact, for any subring $R \subset \overline{k}$ containing $\mathscr{O}_L$, the groups $\mathbf{G} \cap \mathrm{GL}(n;R)$ and $\rho(\mathbf{G}) \cap \mathrm{GL}(m;R)$ are commensurable upon viewing $\mathbf{G} \cap \mathrm{GL}(n;R)$ as a subgroup of $\mathrm{GL}(m;\overline{k})$ via ρ. Thus, up to commensurability, the subgroup $\mathbf{G}(R) = \mathbf{G} \cap \mathrm{GL}(n;R)$ is well-defined and called the *R–points* of $\mathbf{G}$.

7. Lattices in Lie groups

On a topological group H, for $h \in H$, we associate the homeomorphism $L_h \colon H \longrightarrow H$ to h, where $L_h(g) = hg$ and by R_h the homeomorphism given by right translation by h. A *left Haar measure* μ on H is a regular Borel measure (see [**16**] or [**21**]) such that L_h is μ–automorphism for all $h \in H$. It is a fundamental fact (see [**16**] or [**21**]) that every Lie group H can be equipped with a left (respectively right) Haar measure μ and its unique up to scaling. For the Lie group $\mathrm{SU}(n,1)$, this measure can selected to be bi-invariant or 2–sided.

A *lattice* $\Lambda < H$ is a discrete subgroup of H such that μ descends to a finite volume measure on the coset space H/Λ. Note that since μ is Λ–equivariant, μ always descends to a measure on the coset space H/Λ. If in addition H/Λ is compact in the quotient topology, we say that Λ is a *cocompact lattice*.

The following fundamental result is due to Borel and Harish-Chandra [**5**].

THEOREM 2.5 (Borel–Harish-Chandra; [**5**]). *Let* $\mathbf{G}$ *be a semi-simple* $\mathbf{Q}$*–algebraic group,* $\mathbf{G}(\mathbf{R})$ *and* $\mathbf{G}(\mathbf{Z})$*, its group of real and integral points, respectively. Then* $\mathbf{G}(\mathbf{Z})$ *is a lattice in* $\mathbf{G}(\mathbf{R})$.

8. Godement's compactness criterion

In the sequel, we also require a compactness criterion known as Godement's compactness criterion. This was proved by Borel and Harish-Chandra [**5**] and independently by Mostow and Tamagawa [**27**]. Before stating this theorem, we require an additional piece of terminology.

We say that $A \in \mathrm{GL}(n;\mathbf{C})$ is *unipotent* if A is conjugate in $\mathrm{GL}(n;\mathbf{C})$ into the group of upper triangular matrices with ones along the diagonal.

THEOREM 2.6 (Godement's compactness criterion;[**5**], [**27**]). *Let* $\mathbf{G}$ *be a semi-simple* $\mathbf{Q}$*–algebraic group and* Λ *a lattice in* $\mathbf{G}$. *Then* Λ *is cocompact if and only if* Λ *contains no non-trivial unipotent elements.*

A slightly stronger form of Godement's compactness criterion exists.

THEOREM 2.7 (Godement's compactness criterion; Strong form). *Let* $\Lambda < \mathbf{G}$ *be a lattice in the* $\mathbf{Q}$*–algebraic group* $\mathbf{G}$ *commensurable with* $\mathbf{G}(\mathbf{Z})$. *Then* Λ *is cocompact if and only if* $\mathbf{G}(\mathbf{Q})$ *contains no nontrivial unipotent elements.*

PROOF. If Λ is noncocompact, we simply apply Theorem 2.6 to obtain a nontrivial unipotent element $x \in \mathbf{G}(\mathbf{Z}) \subset \mathbf{G}(\mathbf{Q})$. For the converse, assume that $x \in \mathbf{G}(\mathbf{Q})$ is a nontrivial unipotent element. To demonstrate the noncocompactness of Λ, it suffices to construct a noncocompact lattice Λ_1 commensurable with Λ. To this end, let $\mathbf{Z}_x$ be the Zariski closure of the cyclic group $\langle x\rangle$ in $\mathbf{G}$. Since $x \in \mathbf{G}(\mathbf{Q})$, this is a $\mathbf{Q}$–defined subgroup of $\mathbf{G}$. In addition, $\langle x\rangle$ is a lattice in $\mathbf{Z}_x$. Note, the inclusion $\iota\colon \mathbf{Z}_x \longrightarrow \mathbf{G}$ is a $\mathbf{Q}$–defined injection. It follows that there exists a lattice Λ' in $\mathbf{G}(\mathbf{R})$, commensurable with $\mathbf{G}(\mathbf{Z})$ such that $\iota(\langle x\rangle) \subset \Lambda'$ (see [**32**, p.165–166] for a proof of this). Thus, by Theorem 2.6, Λ' is noncocompact. $\square$

9. Restriction of scalars

As an $\mathbf{R}$–algebra, $\mathbf{C}$ can be embedded in $\mathrm{M}(2;\mathbf{R})$ via the map

$$\mathrm{Res}_{\mathbf{C}/\mathbf{R}}(x+iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

This extends to $\mathrm{GL}(n;\mathbf{C})$ by performing this expansion of each coefficient and produces an embedding $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\colon \mathrm{GL}(n;\mathbf{C}) \longrightarrow \mathrm{GL}(2n;\mathbf{R})$. One way to obtain the map $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}$ is as follows. Select an $\mathbf{R}$–vector space basis for $\mathbf{C}$, say 1 and i. For $\alpha \in \mathbf{C}$, we have an $\mathbf{R}$–linear map $L_\alpha(\beta) = \alpha\beta$ given by left multiplication. This provides us with an injection (of $\mathbf{R}$–algebras) $\mathbf{C} \longrightarrow \mathrm{End}(\mathbf{R}^2)$. To see that we have constructed precisely the map $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}$ above, let $z = x+iy$ be a fixed complex number. In the basis $\{1,i\}$, we have (taking column vectors)

$$L_z(1) = \begin{pmatrix} x \\ y \end{pmatrix}, \quad L_z(i) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

as asserted. Since $\{1,i\}$ is a $\mathbf{Z}$–module basis for $\mathscr{O}_{\mathbf{Q}(i)} = \mathbf{Z}[i]$, not surprisingly,

$$\mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{Z}[i]) = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{C}) \cap \mathrm{M}(2;\mathbf{Z}).$$

For any other imaginary quadratic extension k of $\mathbf{Q}$, by selecting instead a $\mathbf{Z}$–module basis $\mathscr{B}$ for $\mathscr{O}_k$, we can arrange for $\mathscr{O}_k$ to be precisely the $\mathbf{Z}$–points under the induced $\mathbf{R}$–algebra injection $\mathbf{C} \longrightarrow \mathrm{End}(\mathbf{R}^2)$ afforded by $\mathscr{B}$. More generally, for any finite extension F of $\mathbf{Q}$, we obtain an injection $F \longrightarrow \mathrm{End}(\mathbf{R}^d)$, where $d = [F:\mathbf{Q}]$ by selecting a $\mathbf{Q}$–basis for F, viewed as a d–dimensional $\mathbf{Q}$–vector space. If, in addition, we select this basis to be a $\mathbf{Z}$–module basis for $\mathscr{O}_F$, $\mathscr{O}_F = F \cap \mathrm{M}(d;\mathbf{Z})$.

This process is known as *restriction of scalars* or *corestriction*. Given a k–algebraic group $\mathbf{G} < \mathrm{GL}(n;\mathbf{C})$ with $[k:\mathbf{Q}] = d$, this process can also be performed on $\mathbf{G}$ and is denoted by $\mathrm{Res}_{k/\mathbf{Q}}(\mathbf{G})$. Abstractly, this provides us with a functor from the category of k–algebraic groups with k–algebraic homomorphisms to the category of $\mathbf{Q}$–algebraic groups with $\mathbf{Q}$–algebraic homomorphisms.

A useful take on restriction of scalars is the following. Let k_{gal} denote the Galois closure of k over $\mathbf{Q}$ and $\mathrm{Gal}(k_{gal}/\mathbf{Q})$ its Galois group. For a k–algebraic group $\mathbf{G}$ with associated ideal $\mathfrak{a}$, select a finite generating set $P_1(T), \dots, P_r(T) \in k[T]$ for $\mathfrak{a}$. Each element $\sigma \in \mathrm{Gal}(k_{gal}/k)$ acts on $k_{gal}[T]$ yielding a new set of $\sigma(k)$–polynomials ${}^{\sigma}P_1(T), \dots, {}^{\sigma}P_r(T) \in \sigma(k)[T]$. The ideal generated by these polynomials is denoted by ${}^{\sigma}\mathfrak{a}$ and produces a new algebraic group ${}^{\sigma}\mathbf{G}$. By construction, ${}^{\sigma}\mathbf{G}$ is a $\sigma(k)$–algebraic group. For two Galois automorphisms $\sigma_1, \sigma_2 \in \mathrm{Gal}(k_{gal}/\mathbf{Q})$ equivalent modulo $\mathrm{Gal}(k_{gal}/k)$, we obtain isomorphic groups ${}^{\sigma_1}\mathbf{G}$ and ${}^{\sigma_2}\mathbf{G}$. Indeed, the images of k under the action of $\mathrm{Gal}(k_{gal}/\mathbf{Q})$ produce all of the embeddings of $k \longrightarrow \mathbf{C}$ and it is a simple matter to see that we only need to take a distinct set of coset representatives S of $\mathrm{Gal}(k_{gal}/\mathbf{Q}) / \mathrm{Gal}(k_{gal}/k)$. Define

$$\mathrm{Res}_{k/\mathbf{Q}}(\mathbf{G}) = \prod_{\sigma \in S} {}^{\sigma}\mathbf{G}.$$

This group is invariant under the action of $\mathrm{Gal}(k_{gal}/\mathbf{Q})$. Consequently, $\mathrm{Res}_{k/\mathbf{Q}}(\mathbf{G})$ is a $\mathbf{Q}$–algebraic group. By construction, the groups $\mathrm{Res}_{k/\mathbf{Q}}(\mathbf{G})(\mathbf{Z})$ and $\mathrm{Res}_{k/\mathbf{Q}}(\mathbf{G}(\mathscr{O}_k))$ are commensurable.

CHAPTER 3

A motivational example

Before embarking on our first general construction of lattices in SU$(n,1)$, we give a motivational family of lattices whose construction is simplest among the arithmetic lattices in SU$(n,1)$. These groups are in the same spirit as SL$(n;\mathbf{Z})$ and the Bianchi groups PSL$(2;\mathscr{O}_d)$.

1. The groups U$(n,1)$ and SU$(n,1)$

Let $I_{n,1}$ denote the diagonal matrix diag$(1,1,\ldots,1,-1) \in$ GL$(n+1;\mathbf{C})$. Associated to $I_{n,1}$ is a Hermitian form $\langle\cdot,\cdot\rangle_{n,1} : \mathbf{C}^{n+1} \times \mathbf{C}^{n+1} \longrightarrow \mathbf{C}$ given by $\langle x,y\rangle_{n,1} = y^* I_{n,1} x$, where $*$ denotes the complex transpose of the column vector y. The group U$(n,1)$ is the subgroup of GL$(n+1;\mathbf{C})$ of matrices A such that for all $x,y \in \mathbf{C}^{n+1}$, $\langle Ax,Ay\rangle_{n,1} = \langle x,y\rangle_{n,1}$. The subgroup of those elements A of U$(n,1)$ such that $\det(A) = 1$ is denoted by SU$(n,1)$.

The group SU$(n,1)$, though embedded in GL$(n+1;\mathbf{C})$, is not a complex algebraic group but instead a real algebraic group. In order to apply Theorem 2.5, we view SU$(n,1)$ as the $\mathbf{R}$–points of an algebraic group.

Using $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}$ described in the previous section, we obtain an embedding $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathrm{SU}(n,1)) \longrightarrow \mathrm{GL}(2n+2;\mathbf{R})$ such that there exists a $\mathbf{Q}$–algebraic group $\mathbf{G} < \mathrm{GL}(2n+2;\mathbf{C})$ for which $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathrm{SU}(n,1)) = \mathbf{G}(\mathbf{R})$ (see the proof of Theorem 14.4). Under the particular description of $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}$ in the previous section, we have $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathrm{SU}(n,1;\mathbf{Z}[i])) = \mathbf{G}(\mathbf{Z})$. In particular, by Theorem 2.5 SU$(n,1;\mathbf{Z}[i])$ is a lattice in SU$(n,1)$.

2. A family of lattices

For an imaginary quadratic number field $E/\mathbf{Q}$, a selection of a $\mathbf{Z}$–basis for $\mathscr{O}_E$ provides us with a $\mathbf{Q}$–basis for E. In turn, we are afforded an embedding $\mathrm{Res}_{E/\mathbf{Q}} \colon E \longrightarrow \mathrm{M}(2;\mathbf{Q})$ such that $\mathrm{Res}_{E/\mathbf{Q}}(\mathscr{O}_E) \subset \mathrm{M}(2;\mathbf{Z})$. This extends to a $\mathbf{Q}$–algebraic embedding of SU$(n,1)$ into GL$(2n+2;\mathbf{R})$ such that SU$(n,1;\mathscr{O}_E)$ is precisely the set of $\mathbf{Z}$–points of SU$(n,1)$ under this embedding.

THEOREM 3.1. SU$(n,1;\mathscr{O}_E)$ *is a noncocompact lattice in* SU$(n,1)$.

PROOF. That $\mathrm{SU}(n,1;\mathscr{O}_E)$ is a lattice follows from Theorem 2.5. To see that $\mathrm{SU}(n,1;\mathscr{O}_E)$ is noncocompact, note that $\mathrm{SU}(n,1;\mathscr{O}_E)$ contains a nontrivial unipotent element. For example, let $v = (0,0,\dots,0,1,1)$ and $w = (1,0,0,\dots,0)$. Then $u(\cdot) = \exp(\langle \cdot, v\rangle_{n,1}\, w - \langle \cdot, w\rangle_{n,1}\, v)$ is a unipotent (see [**13**, p. 119]) in $\mathrm{SU}(n,1;\mathscr{O}_E)$. Thus by Theorem 2.6, $\mathrm{SU}(n,1;\mathscr{O}_E)$ is noncompact. □

3. Real algebraic structure

I should probably say a little more about the real algebraic structure of $\mathrm{SU}(n,1)$ since this appears on the very last page before the appendix.

CHAPTER 4

Hermitian matrices, forms, and involutions

In this section, we introduce Hermitian matrices from three different angles. The first two are familiar to the reader acquainted with complex hyperbolic n–space or complex projective geometry. However, it is the third view that is most important for us in the construction of lattices in $\mathrm{SU}(n,1)$.

1. Hermitian and skew-Hermitian matrices

In this section, $*$ will denote complex transposition on the matrix algebra $\mathrm{M}(n+1;\mathbf{C})$. We say that $H \in \mathrm{M}(n+1;\mathbf{C})$ is *Hermitian* if $H^* = H$ and *skew-Hermitian* if $H^* = -H$. We denote the set of Hermitian matrices and skew-Hermitian matrices by $\mathscr{H}(n+1)$ and $\mathscr{S}\mathscr{H}(n+1)$. It follows from elementary properties of complex transposition that both $\mathscr{H}(n+1)$ and $\mathscr{S}\mathscr{H}(n+1)$ are $\mathbf{R}$–vector subspaces of $\mathrm{M}(n+1;\mathbf{C})$. In addition, it is easy to verify that $\mathscr{H}(n+1) \cap \mathscr{S}\mathscr{H}(n+1) = \{0\}$. Thus, as an $\mathbf{R}$–vector space,

$$\mathrm{M}(n+1;\mathbf{C}) = \mathscr{H}(n+1) \oplus \mathscr{S}\mathscr{H}(n+1).$$

In the remainder of this note, we denote $\mathscr{H}(n+1)$ simply by $\mathscr{H}$ and the subset of invertible Hermitian matrices by $\mathscr{H}^\times$.

2. Hermitian forms and unitary groups

Given $H \in \mathscr{H}^\times$, we can associate to H a non-degenerate Hermitian form $\langle \cdot,\cdot \rangle_H$ by $\langle x,y \rangle_H = y^* H x$. We call $\langle \cdot,\cdot \rangle_H$ the *associated Hermitian form* for H. Conversely, given a Hermitian form $\langle \cdot,\cdot \rangle$, we can associate to $\langle \cdot,\cdot \rangle$ a Hermitian matrix by selecting a $\mathbf{C}$–basis $e_1,\dots,e_{n+1}$ for $\mathbf{C}^{n+1}$ and define the matrix $H_{\langle \cdot,\cdot \rangle}$ to have (i,j)–coefficient given by $\left\langle e_i, e_j \right\rangle$.

For a Hermitian matrix $H \in \mathscr{H}^\times$ with associated Hermitian form $\langle \cdot,\cdot \rangle$, we define the *signature tuple* of H or $\langle \cdot,\cdot \rangle$ to be the order pair (p,q), where p is the number of positive eigenvalues of H and q is the number of negative eigenvalues of H. We note that it is a classical result that H can be diagonalized and the coefficients of the associated diagonal matrix are non-zero real number.

With H as above, we define the *H–unitary group* to be the subgroup of $\mathrm{GL}(n+1;\mathbf{C})$ consisting of matrices A such that for all $x,y \in \mathbf{C}^{n+1}$, $\langle Ax, Ay\rangle_H = \langle x,y\rangle_H$. We denote this group by $\mathrm{U}(H)$. The *special H–unitary group* $\mathrm{SU}(H)$ consists of those elements $A \in \mathrm{U}(H)$ such that $\det(A) = 1$.

3. Field of definition and equivalence

We say that $H \in \mathscr{H}^\times$ is defined over a subfield $k \subset \mathbf{C}$ if there exists a basis $\mathscr{B}$ such that in this basis H has coefficients in k. The group $\mathrm{SU}(H)$ in turn can be viewed as a k–defined real algebraic group.

We say that two Hermitian matrices are *isometric* if there associated Hermitian pairs on $\mathbf{C}^n$ are isometric. More generally, we say that two k–defined Hermitian matrices H_1, H_2 are *equivalent* if there exists $\alpha \in k^\times \cap \mathbf{R}$ such that αH_1 and H_2 are isometric.

EXAMPLE 4.1.

Let

$$H_{4,2,-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$H_{4,2,-p} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

for $p \in \mathbf{N}$. For any imaginary quadratic number field E, $H_{4,2,-1}, H_{4,2,-p}$ are E–defined. The reader can check that $H_{4,2,-1}$ and $H_{4,2,-p}$ are equivalent over E if and only if $p \in N_{E/\mathbf{Q}}(E^\times)$ (see Section 3 for the definition of $N_{E/\mathbf{Q}}$).

In Chapter 6 we return to the question of deciding when a pair of forms H_1, H_2 are equivalence over E. The importance of equivalence is seen in the following result (see [**36**]).

THEOREM 4.1. *Let H_1, H_2 be a pair of k–defined Hermitian matrices with associated unitary groups $\mathrm{SU}(H_1)$ and $\mathrm{SU}(H_2)$. Then H_1 and H_2 are equivalence over k if and only if $\mathrm{SU}(H_1) \cong \mathrm{SU}(H_2)$ are isomorphic as real $(k \cap \mathbf{R})$–defined algebraic groups.*

4. Diagonalizing forms; anisotropic and isotropic factors

We refer the reader to [**12**], [**23**], or [**29**] for the proofs of the results from this section. Additional, the reader can refer to [**47**].

For a Hermitian matrix $H \in \mathrm{M}(n+1;k)$, we say that H is *isotropic* if there exists $v \in \mathbf{C}^{n+1} \setminus \{0\}$ such that $\langle v,v\rangle_H = 0$ and say H is *anisotropic* otherwise. Associated to H is a direct sum decomposition $V_{an} \oplus V_{iso} = \mathbf{C}^{n+1}$ such that $H_{|V_{an}}$ is anisotropic and for every $v \in V_{iso}$, $\langle v,v\rangle_H = 0$.

The following is a standard application of the Gram-Schmidt process.

THEOREM 4.2. *Every Hermitian matrix H is equivalent to a diagonal form $H' = \mathrm{diag}(\alpha_1,\dots,\alpha_{n+1})$ with $\alpha_j \in \mathscr{O}_k$.*

In fact, more can be said about the isotropic factor H_{iso}.

THEOREM 4.3. *Up to k–isomorphism of associated unitary groups, H is equivalent to $H' = H_1 \oplus H_2$ where $H_1 = \mathrm{diag}(\alpha_1,\dots,\alpha_r)$ is anisotropic and*

$$H_2 = \bigoplus_{j=1}^{\frac{1}{2}\dim_{\mathbf{C}} V_{iso}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Given H, a nondegenerate k–defined Hermitian form and an isotropic k–vector v, we can construct a non-trivial unipotent element in $\mathrm{SU}(H;k)$. From Theorem 4.3, there exists a k–vector v_0 such that $\langle v,v_0\rangle_H = 1$, $\langle v,v\rangle_H = \langle v_0,v_0\rangle_H = 0$. Let $V_{0,\infty}$ denote the H–orthogonal complement of $\mathbf{C}[v,v_0]$. By Theorem 4.3, H restricted to the space $V_{0,\infty}$ is E–defined. Select a vector $w \in V_{0,\infty}$ such that $\langle w,w\rangle_H \in k$. Finally, we complete the set $\{w,v,v_0\}$ into an H–orthogonal basis which is defined over k. Thus for $t_w(\cdot) = \exp(\langle \cdot,v\rangle_H\, w - \langle \cdot,w\rangle_H\, v)$, we obtain the required unipotent element.

THEOREM 4.4. *If H is isotropic, then* $\mathrm{SU}(H;k)$ *contains a nontrivial unipotent element.*

5. Involutions associated to Hermitian matrices

By an *involution* on $\mathrm{M}(n+1;\mathbf{C})$, we mean an order two map $\star\colon \mathrm{M}(n+1;\mathbf{C}) \longrightarrow \mathrm{M}(n+1;\mathbf{C})$ such that

$$(A+B)^\star = A^\star + B^\star \text{ and } (AB)^\star = B^\star A^\star.$$

For a Hermitian matrix $H \in \mathscr{H}^\times$, we define an *associated involution* $\star_H$ by $\star_H = \mu_H \circ *$ where μ_H denotes conjugation by H. The following theorem is fundamental in the classification of arithmetic lattices in $\mathrm{SU}(n,1)$.

THEOREM 4.5 (Weil; [**45**]). *Every involution $\star$ on* $\mathrm{M}(n+1;\mathbf{C})$ *is equivalent to $\star_H$ for some Hermitian matrix.*

We say that $\star$ is of *first kind* if $\star$ is the identity on $\mathbf{C}$ and of *second kind* if $\star$ is complex conjugation on $\mathbf{C}$. Finally, the reader can verify that

$$\mathrm{SU}(H) = \{A \in \mathrm{GL}(n+1;\mathbf{C}) \ : \ AA^{\star_H} = I\}. \tag{1}$$

CHAPTER 5

Arithmetic lattices of the first type

In this section, we define arithmetic lattices of first kind in $\mathrm{SU}(n,1)$.

1. Subgroups arising from forms

Let E/F be a totally imaginary quadratic extension of a totally real number field F of degree s over $\mathbf{Q}$. Fix a complex embedding τ_1 of E and a compatible real embedding σ_1 of F. Using these embeddings, we can identify $E \subset \mathbf{C}$ and $F = E \cap \mathbf{R}$. Given an E–defined Hermitian matrix H of signature $(n,1)$, we have the associated group $\mathrm{SU}(H)$. For each embedding $\tau_j \neq \tau_1$ and compatible real embedding $\sigma_j \neq \sigma_1$, we obtain a new Hermitian matrix by applying τ_j to the coefficients of H. We denote the resulting Hermitian matrix by ${}^{\tau_j}H$ and new associated group $\mathrm{SU}({}^{\tau_j}H)$ by ${}^{\tau_j}\,\mathrm{SU}(H)$. In total, we obtain an injection

$$\mathrm{SU}(H) \longrightarrow \prod_{j=1}^{s} {}^{\tau_j}\,\mathrm{SU}(H)$$

via the diagonal embedding. The latter group is nothing more than $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(H))$, which is $\mathbf{Q}$–defined. By construction, $\mathrm{SU}(H;\mathscr{O}_E)$ and $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(H))(\mathbf{Z})$ are commensurable. Finally, observe that we have a natural projection $\pi\colon \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(H)) \longrightarrow \mathrm{SU}(H)$ with kernel

$$\ker \pi = \prod_{j=2}^{s} {}^{\tau_j}\,\mathrm{SU}(H).$$

Under the mapping π, $\pi(\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(H))(\mathbf{Z}))$ and $\mathrm{SU}(H;\mathscr{O}_E)$ are commensurable. To obtain lattices in $\mathrm{SU}(n,1)$, we can use the diagram

$$\begin{array}{ccccccccc} & & & & & & 1 & & \\ & & & & & & \downarrow & & \\ 1 & \longrightarrow & \prod_{j=2}^{s} {}^{\tau_j}\,\mathrm{SU}(H) & \longrightarrow & \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(H)) & \longrightarrow & \mathrm{SU}(H) & \longrightarrow & 1 \\ & & & & & & \downarrow & & \\ & & & & & & \mathrm{SU}(n,1) & & \\ & & & & & & \downarrow & & \\ & & & & & & 1 & & \end{array}$$

We will return to this later (see §2) when we discuss k–forms of a Lie group.

2. Admissible pairs and arithmetic lattices of the first type

The following lemma appears in [**24**, Lemma 8.1.3] and will be used repeatedly.

LEMMA 5.1. *Let X,Y, and Z be locally compact topological groups, $\Lambda < Z$ be a lattice, and*

$$1 \longrightarrow Y \xrightarrow{\ \iota\ } Z \xrightarrow{\ \pi\ } X \longrightarrow 1\ ,$$

a split exact sequence. Then $\pi(\Lambda) = \Gamma$ is a lattice in X if and only if Y is compact. In addition, if Y is compact, then Λ is cocompact if and only if Γ is cocompact.

We are now able to prove the following:

THEOREM 5.2. $\mathrm{SU}(H;\mathscr{O}_E)$ *is a lattice in* $\mathrm{SU}(H)$ *if and only if* ${}^{\tau_j}\,\mathrm{SU}(H)$ *is compact for all $j = 2,\dots,s$.*

PROOF. From our discussion above, we have the short exact sequence

$$1 \longrightarrow \prod_{j=2}^{s} {}^{\tau_j}\,\mathrm{SU}(H) \xrightarrow{\ \iota\ } \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(H)) \xrightarrow{\ \pi\ } \mathrm{SU}(H) \longrightarrow 1$$

and this has been constructed so that $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(H))$ is a $\mathbf{Q}$–real algebraic group and $\pi(\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(H))(\mathbf{Z}))$ and $\mathrm{SU}(H;\mathscr{O}_E)$ are commensurable. By Theorem 2.5, the group $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(H))(\mathbf{Z})$ is a lattice in $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(H))$. The theorem now follows from an application of Lemma 5.1. □

The reader can see that this construction is a generalization of the construction of the lattices $\mathrm{SU}(n,1;\mathscr{O}_E)$, where E is an imaginary quadratic number field. In this case, since E has only one complex embedding, up to complex conjugation, the compactness condition was vacuously satisfied. With this said, we call a pair $(H,E/F)$ *admissible* if ${}^{\tau_j}\,\mathrm{SU}(H)$ is compact for each $j=2,\dots,s$.

EXAMPLE 5.1.

Let $E=\mathbf{Q}(\sqrt{5},i)$, $F=\mathbf{Q}(\sqrt{5})$ and $H=\mathrm{diag}(1,1,\dots,1,-\sqrt{5})$. We assert that $\mathrm{SU}(H;\mathscr{O}_E)$ is a lattice in $\mathrm{SU}(n,1)$. To see this, first note that H has signature $(n,1)$. Under the other embedding of E, we obtain the Hermitian matrix ${}^{\tau}H=\mathrm{diag}(1,1,\dots,1,\sqrt{5})$ which has signature $(n+1,0)$. Thus, ${}^{\tau}\,\mathrm{SU}(H)$ is compact and so $\mathrm{SU}(H;\mathscr{O}_E)$ is a lattice in $\mathrm{SU}(H)\cong\mathrm{SU}(n,1)$.

COROLLARY 5.3. *If $(H,E/F)$ is admissible, then* $\mathrm{SU}(H;\mathscr{O}_E)$ *is noncompact if and only if E is an imaginary quadratic number field.*

PROOF. Since any unipotent in $\mathrm{SU}(H;\mathscr{O}_E)$ produces a unipotent in the group $\ker\pi$, it is enough to prove that compact algebraic groups cannot possess any nontrivial unipotent elements.

For a compact algebraic group $\mathbf{K}<\mathrm{GL}(n;\mathbf{C})$ with associated Lie algebra $\mathfrak{k}$ and a nontrivial unipotent element $x\in\mathbf{K}$, we can conjugate $\mathbf{K}$ in $\mathrm{GL}(n;\mathbf{C})$ such that $g^{-1}xg$ is upper triangular with ones along the diagonal. For $g^{-1}xg$, there exists $X\in\mathfrak{k}$ such that $\exp(X)=x$. In fact, $\mathbf{R}X$ maps embeds into $\mathbf{K}$ which violates the compactness of $\mathbf{K}$. The proof is completed by applying Theorem 2.6. □

Any lattice $\Lambda<\mathrm{SU}(n,1)$ which is commensurable in the wide sense with a lattice of the form $\mathrm{SU}(H;\mathscr{O}_E)$ for an admissible pair $(H,E/F)$ is called *an arithmetic lattice of first type*.

3. Commensurability classes of lattices of first type

We conclude this section with the following result.

THEOREM 5.4. *There exist infinitely many distinct commensurability classes of lattices of the first type in* $\mathrm{SU}(n,1)$.

PROOF. Let E/F be as above and set $H=(\alpha_1,\dots,\alpha_{n+1})$ for $\alpha_1,\dots,\alpha_{n+1}\in F$. By Theorem 2.1, we can select $\alpha_1,\dots,\alpha_{n+1}$ such that $\alpha_1<0$, $\alpha_j>0$, and for all $\sigma_\ell\neq\mathrm{id}_F$, $\sigma_\ell(\alpha_j)>0$ for $j=1,\dots,n+1$. By Theorem 5.2, $\mathrm{SU}(H;\mathscr{O}_E)$ is a lattice in $\mathrm{SU}(n,1)$. We assert that if two lattices $\mathrm{SU}(H_1;\mathscr{O}_{E_1})$ and $\mathrm{SU}(H_2;\mathscr{O}_{E_2})$ are commensurable in the wide sense, then $E_1\cong E_2$. This follows from the commensurability invariance of the invariant trace field (see §2)

$$k(\mathrm{SU}(H_j;\mathscr{O}_{E_j}))\overset{\mathrm{def}}{=}\left\{\mathrm{Tr}(\gamma^{n+1})\;:\;\gamma\in\mathrm{SU}(H_j;\mathscr{O}_{E_j})\right\}$$

in combination with $k(\mathrm{SU}(H_j;\mathscr{O}_{E_j}))=E_j$. □

CHAPTER 6

Classifying Hermitian structures over fields

In this chapter, we give a list of invariant which determine the similarity class of a Hermitian form on a vector space V. More generally, this holds for Hermitian structures on free modules of division algebras (see Chapter 11). The main reference for this section is [**36**, Ch. 10].

1. Signature and dimension

Let V be a finite dimensional E/F vector space equipped with a Hermitian pairing H. We denote the associated $\mathbf{C}$–vector space by $V_{\mathbf{C}}$ and the Hermitian pairing extended to $V_{\mathbf{C}}$ by H. Above, we associated to H a signature pair (p,q). A related invariant on the form H is the number $\sigma_H = |p-q|$. For two different pairings H_1 and H_2, we have the following (see [**12**]).

THEOREM 6.1. *As real algebraic groups* $\mathrm{SU}(H_1)$ *and* $\mathrm{SU}(H_2)$ *are real isomorphic if and only if* $\sigma_{H_1} = \sigma_{H_2}$.

For an E–defined form H on $V_{\mathbf{C}}$, we have the pair $(\dim_E V, \sigma_H)$.

2. Signatures at other archimedean places

For a different embedding τ_j of E into $\mathbf{C}$, we obtain a new signature for any Hermitian form H on V. If V_∞ denote the set of inequivalent complex embeddings of E, for each $\nu \in V_\infty$, we obtain a signature $\sigma_H(\nu)$ called the ν–signature.

3. Determinant

For a Hermitian form H on $V_{\mathbf{C}}$, by selecting an E–basis for V, we can associate to H a matrix $T_{H,\mathscr{B}}$. The determinant of this matrix is independent not independent of the selection of the basis $\mathscr{B}$ but is independent viewed as an element of $F^\times/N_{E/F}(E^\times)$. This produces an invariant $\det H \in F^\times/N_{E/F}(E^\times)$ which we call the *determinant of* H and denote by $\det H$.

4. Classifying equivalence classes of Hermitian forms

The following result can be found in [**36**].

THEOREM 6.2 (Classification of forms). *Two forms H_1 and H_2 on V and defined over E/F are equivalent if and only if $\sigma_{H_1}(\nu) = \sigma_{H_2}(\nu)$ for all $\nu \in V_\infty$ and* $\det(H_1) = \det(H_2)$.

As a consequence of this result, if we fix the signatures of at each $\nu \in V_\infty$, we see that there are precisely two classes of Hermitian forms.

For any form H over E/F on a $\mathbf{C}$–vector space V, we form the invariant tuple $(\dim V, \sigma_\nu(H), \det H)$ which takes values in $\mathbf{N}\times\mathbf{N}^{V_\infty(F)}\times F^\times/N_{E/F}(E^\times)$. If we denote this tuple by $\mathrm{Inv}(H)$, the classification theorem states that H_1 and H_2 are equivalent over E if and only if $\mathrm{Inv}(H_1) = \mathrm{Inv}(H_2)$.

5. A few examples

5.1. Signature $(2,1)$ forms over $\mathbf{Q}(\sqrt{-d})$. For an imaginary quadratic number field $E = \mathbf{Q}(\sqrt{-d})$, $V_\infty(E)$ consists of precisely one complex embedding (up to complex conjugation). So long as $d \neq 1$, $-1 \notin N_{E/\mathbf{Q}}(E^\times)$. According Theorem 6.2, over E we have the possible invariants $(3,3,1)$, $(3,3,-1)$, $(3,0,1)$, and $(3,0,-1)$. Of interest for us are the latter two, since these correspond to Hermitian forms of signature $(1,2)$ or $(2,1)$. For $(3,0,1)$, we take the Hermitian form

$$H_{3,0,1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $(3,0,-1)$, we take the Hermitian form

$$H_{3,0,-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

By Theorem 6.2 every form H on $\mathbf{C}^3$ defined over E is equivalent to one of these two forms. However, note that $H_{3,0,1} = -H_{3,0,-1}$. Thus, $\mathrm{SU}(H_{3,0,1})$ and $\mathrm{SU}(H_{3,0,-1})$ are isomorphic.

Over $E = \mathbf{Q}(\sqrt{-1})$, we take $H_{3,0,-1}$ again and the form

$$H_{3,0,-3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Since $-3 \notin N_{\mathbf{Q}(i)/\mathbf{Q}}(\mathbf{Q}(i)^{\times})$, these two forms represent all the possible classes over $\mathbf{Q}(i)$. On the other hand, $H_{3,0,-3}$ is equivalent to

$$H_{3,0,-27} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Since $H_{3,0,-27} = 3H_{3,0,-1}$, the groups $\mathrm{SU}(H_{3,0,-3})$ and $\mathrm{SU}(H_{3,0,-1})$ are isomorphic.

THEOREM 6.3. *For any imaginary quadratic number field $E/\mathbf{Q}$ and every Hermitian form H of signature $(2,1)$ defined over E, $\mathrm{SU}(H;\mathscr{O}_E)$ and $\mathrm{SU}(2,1;\mathscr{O}_E)$ are isomorphic as real $\mathbf{Q}$–algebraic groups.*

COROLLARY 6.4. *For E and H as above, every arithmetic lattice in $\mathrm{SU}(H)$ is commensurable in the wide sense with $\mathrm{SU}(2,1;\mathscr{O}_E)$.*

COROLLARY 6.5. *Every Hermitian form H of signature $(2,1)$ over an imaginary quadratic number field is isotropic.*

5.2. Signature $(n,1)$ forms over $\mathbf{Q}(\sqrt{-d})$: even n. For $E = \mathbf{Q}(\sqrt{-d})$, there are two classes of Hermitian forms over E of signature $(n,1)$ (or equivalently, $(1,n)$). As before, we have a pair of Hermitian forms

$$H_{n+1,0,-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 \end{pmatrix}$$

and

$$H_{n+1,0,(-1)^n} = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

For $d \neq 1$ and n even, the forms $H_{n+1,0,-1}$ and $H_{n+1,0,1}$ represent the two possible classes. For $E = \mathbf{Q}(\sqrt{-1})$, the remaining exceptional case can be handled as before.

THEOREM 6.6. *Let E be an imaginary quadratic number field, H a Hermitian form over E of signature $(n,1)$ with n even. Then $\mathrm{SU}(H)$ and $\mathrm{SU}(n,1)$ are isomorphic as real $\mathbf{Q}$–algebraic groups.*

COROLLARY 6.7. *For E and H as above, every arithmetic lattice in* $\mathrm{SU}(H)$ *is commensurable in the wide sense with* $\mathrm{SU}(n,1;\mathscr{O}_E)$.

COROLLARY 6.8. *Every Hermitian form H of signature $(n,1)$ over an imaginary quadratic number field with n even is isotropic.*

5.3. Signature $(n,1)$ forms over $\mathbf{Q}(\sqrt{-d})$: odd n. When n is odd, the above forms $H_{n+1,0,-1}$ and $H_{n+1,0,(-1)^n}$ represent the same class. In this case, it follows that $\det(H)$, viewed as an element of $\mathbf{Q}$, is negative. For fixed $d \neq 1$, select a prime integer p such that $p \nmid d$. Then $p \notin N_{\mathbf{Q}(\sqrt{-d})/\mathbf{Q}}(\mathbf{Q}(\sqrt{-d})^\times)$. It follows then that any form H with $\det(H) = -p$ will represent the trivial class, since neither p and -1 are norms. Define

$$H_{n+1,0,-p} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -p \end{pmatrix}.$$

Then $H_{n+1,0,-1}$ and $H_{n+1,0,-p}$ represent the two possible classes of Hermitian forms over E. When $d = 1$, -1 is a norm and for any selection of p, $H_{n+1,0,-p}$ represents the nontrivial class in $\mathbf{Q}^\times/N_{\mathbf{Q}(i)/\mathbf{Q}}(\mathbf{Q}(i)^\times)$. In this case, $H_{n+1,0,-1}$ represents the trivial class.

THEOREM 6.9. *Let E be an imaginary quadratic number field and $H_{n+1,0,-1}$ and $H_{n+1,0,-p}$ be as above. For any E–defined Hermitian form H of signature $(n,1)$,* $\mathrm{SU}(H)$ *is isomorphic as a real $\mathbf{Q}$–algebraic group to exactly one of the groups* $\mathrm{SU}(n,1)$ *or* $\mathrm{SU}(H_{n+1,0,-p})$.

COROLLARY 6.10. *For E and H as above, every arithmetic lattice in* $\mathrm{SU}(H)$ *is commensurable in the wide sense to either* $\mathrm{SU}(n,1;\mathscr{O}_E)$ *or* $\mathrm{SU}(H_{n+1,0,-p};\mathscr{O}_E)$.

6. Parity and describing commensurability classes

In this section we observe a general reduction of the equivalence classes of Hermitian forms, a special case of is Theorem 6.6. Note that Theorem 6.11 implies Theorem 6.9 whose proof was omitted.

Let E/F be a totally imaginary quadratic extension of a totally real number field with a fixed compatible embeddings $\sigma_1 : F \longrightarrow \mathbf{R}$ and $\tau_1 : E \longrightarrow \mathbf{R}$. As always, we now view $\sigma_1 = \mathrm{id}_F$ and $\tau_1 = \mathrm{id}_E$. Let H be a Hermitian form such that $\sigma_{\sigma_1}(H) = n-1$ and for all $\sigma_j \neq \sigma_1$, $\sigma_{\sigma_j}(H) = n+1$. According to Theorem 6.2, there are two equivalence classes of Hermitian forms on E/F with this prescribed signature.

THEOREM 6.11 (Parity theorem).

(a) *If n is even, there is precisely one wide commensurability class for arithmetic lattices of first type defined over E/F.*
(b) *If n is odd, there are precisely two wide commensurability classes of arithmetic lattices of first type defined over E/F.*

PROOF. For (a), it suffices to show that any Hermitian form H' over E/F with $(E/F,H')$ admissible, then $\mathrm{SU}(H')$ and $\mathrm{SU}(H)$ are isomorphic as real F–algebraic groups. Equivalently, we must show that H and H' are equivalent. For this, $\mathrm{Inv}(H)$ and $\mathrm{Inv}(H')$ can differ only on the determinant. By Theorem 6.2, if $\mathrm{Inv}(H) = \mathrm{Inv}(H')$, H' and H are equivalent. Thus, we may assume that $\det(H) \neq \det(H')$ viewed as elements of $F^\times/N_{E/F}(E^\times)$. Without loss of generality, assume that $\det(H)$ represents the nontrivial class in $F^\times/N_{E/F}(E^\times)$. It follows that H and $\det(H)H$ are equivalent. On the other hand, $\det(\det(H)H) = \det(H)^{n+2}$. Since n is even, $\det(H)^{n+2}$ represents the trivial class in $F^\times/N_{E/F}(E^\times)$. Thus by Theorem 6.2, $\det(H)H$ and H' are equivalent and so H and H' are equivalent.

For (b), let H and H' be such that $(E/F,H)$ and $(E/F,H')$ are admissible and $\det(H)$ and $\det(H')$ represent the trivial and nontrivial class, respectively. If H and H' are equivalent, there exists $\alpha \in F$ such that $\alpha H = H'$. However, $\det(\alpha H) = \alpha^{n+1}\det(H)$. Since n is odd, $\alpha^{n+1}\det(H) = \det(H)$ in $F^\times/N_{E/F}(E^\times)$. By assumption, $\alpha^{n+1}\det(H) = \det(H')$, which is a contradiction of our selection of H and H'. □

7. All Hermitian forms over $\mathbf{Q}(\sqrt{-d})$ are isotropic

In this section, we prove the following theorem.

THEOREM 6.12. *Let E be an imaginary quadratic number field, H an E–defined Hermitian form of signature $(n,1)$ over E. Then H is isotropic over E.*

The reader can compare this result with the fact that not every $\mathbf{Q}$–defined bilinear form of signature $(n,1)$ is isotropic over $\mathbf{Q}$. We will return to this in a moment.

For n even, we proved this result above and so it remains to treat the case when n is odd. This is achieved instantly for all but $n = 3$ from the following theorem of Kneser (see also [**36**]).

THEOREM 6.13 (Kneser; add reference). *Every $\mathbf{Q}$–defined bilinear form of signature $(n,1)$ with $n > 3$ is isotropic over $\mathbf{Q}$.*

As a result of Theorem 6.13, the form $H_{n+1,0,-p}$ and $H_{n+1,0,-1}$ are both isotropic viewed as bilinear forms on $\mathbf{Q}^{n+1}$, so long as $n > 3$.

The remaining case of $n = 3$ we only need to consider the form $H_{4,0,-p}$. There are several ways to deduce that this form is isotropic over E; for instance, we could simply find an isotropic vector. As it introduces an important tool, we introduce the associated quadratic form Q_H for a Hermitian form H and deduce that $H_{4,0,-p}$ is isotropic from Theorem 6.13.

For any Hermitian form H on V and any $v \in V$, $H(v,v) \in \mathbf{R}$. Viewing V as a real vector space of real dimension $2\dim_{\mathbf{C}} V$, we thus obtain a quadratic form

$$Q_H : V \longrightarrow \mathbf{R}$$

given by

$$Q_H(v) = H(v,v).$$

The following result in combination with Theorem 6.13 implies $H_{4,0,-p}$ is isotropic over E (see [**36**]

THEOREM 6.14. *Let H be a Hermitian form over E/F and Q_H be its associated quadratic form over F. Then H is isotropic over E if and only if Q_H is isotropic over F.*

8. Local theory for Hermitian forms

In this section, we discuss some local-to-global results for Hermitian structures on free modules over division algebras. Additionally, we state Witt's theorem.

CHAPTER 7

Cyclic division algebras

In this chapter, we introduce the required background needed to generalize the above construction.

Before introducing a multitude of new algebraic objects required in the construction, we motivate this by reviewing our construction of lattices of first type in $\mathrm{SU}(n,1)$.

We begin with a E–algebra $\mathrm{M}(n+1;E)$ together with an involution $\star_H$ determined by a Hermitian matrix of signature $(n,1)$. Using the involution $\star_H$, we obtain a real F–algebraic group $\mathrm{SU}(H)$ which is isomorphic to $\mathrm{SU}(n,1)$. In the case $E/\mathbf{Q}$ is an imaginary quadratic extension of $\mathbf{Q}$, we obtain a discrete subgroup of $\mathrm{SU}(n,1)$ by taking the discrete subring $\mathscr{O}_E$ of $\mathbf{C}$. When E is not an imaginary quadratic extension of $\mathbf{Q}$, to obtain a discrete subgroup of $\mathrm{SU}(n,1)$ from the indiscrete subring $\mathscr{O}_E \subset \mathbf{C}$, we further insist that the groups ${}^{\tau_j}\,\mathrm{SU}(H)$ be compact at the other complex embeddings of E.

1. Central simple algebras

Throughout, E will denote a number field. The reader should keep in mind that E will be a totally imaginary quadratic extension of a totally real number field F for our applications.

By an *E–algebra A* we mean an E–vector space equipped with an associative multiplicative structure. We say that an E–algebra is *simple* if there are no non-trivial two-sided ideals in A. We say that A is a *central E–algebra* if the center of A, denoted by $Z(A)$, is E. Finally, if every element of $A \setminus \{0\}$ is invertible in A, we call A a *division algebra*.

EXAMPLE 7.1.

- E is a central, simple E–algebra.
- $\mathrm{M}(n;E)$ is a central, simple E–algebra.
- Let A be a 4–dimensional E–vector space with the basis $1,x,y,z$ such that

$$x^2 = a, \quad y^2 = b, \quad xy = z, \quad yx = -xy.$$

The algebra A is a simple, central E–algebra and is called a *quaternion algebra*. Associated to A is the so-called *Hilbert symbol* $\left(\frac{a,b}{E}\right)$ which encodes all the date necessary to retrieve the algebra.

- For any central E–algebra A, $\mathrm{M}(r;A)$ is a central E–algebra.

2. The Skolem-Noether and Structure theorems

We require the following two fundamental results in the sequel. We refer the reader to [**22**], [**24**] or [**30**] for the proofs.

THEOREM 7.1 (Skolem-Noether theorem). *Let A be a simple k–algebra. Then* $\mathrm{Aut}_k(A) = \mathrm{Inn}(A)$.

A useful corollary of Theorem 7.1 is:

COROLLARY 7.2. *Let A be a simple k–algebra with simple subalgebra B. Then every automorphism of B extends uniquely to an automorphism of A.*

The second result required throughout the remainder of this note is the Wedderburn Structure Theorem.

THEOREM 7.3 (Wedderburn structure theorem). *Let A be a simple, central finite dimensional E–algebra. Then there exists a central E–division algebra D and an integer r such that $A \cong \mathrm{M}(r;D)$, as E–algebras.*

3. Cyclic algebras

Our primary interest is in a special type of algebra known as a *cyclic algebra*. Given a cyclic extension L/E of degree d with Galois group $\mathrm{Gal}(L/E) = \langle\theta\rangle$ and $\alpha \in E^{\times}$, we define a central simple E–algebra $(L/E,\theta,\alpha)$ by:

$$(L/E,\theta,\alpha) = \left\{ \sum_{j=0}^{d-1} \beta_j X^j \ : \ \beta_j \in L \right\}$$

subject to the relations

$$X^d = \alpha, \quad X\beta = \theta(\beta)X, \quad \beta \in L.$$

PROPOSITION 7.4. *$(L/E,\theta,\alpha)$ is a central simple E–algebra of dimension d^2 over E as a vector space.*

PROOF. That $(L/E,\theta,\alpha)$ is an E–algebra of dimension d^2 follows immediately from the definition of $(L/E,\theta,\alpha)$. To see that the center of this algebra is precisely E, note that the center must be contained in E since $\mathrm{Fix}(\theta) = E$. One can check using the relations above that E is central which demonstrates the reverse containment. To prove that E is simple, note that any two-sided ideal $I \subset A$ lifts to an ideal I_L in $A \otimes_E L$. However, this algebra is isomorphic, as an L–algebra, to $\mathrm{M}(d;L)$ which is simple. $\square$

EXAMPLE 7.2.

Let $F = \mathbf{Q}$, $E = \mathbf{Q}(i)$ and $L = E(\sqrt{d})$. Then $\theta(\sqrt{d}) = -\sqrt{d}$. For $\alpha \in E$,

$$(L/E, \theta, \alpha) = \left(\frac{\alpha, d}{E}\right).$$

The following two examples will be important in the sequel (see 3).

EXAMPLE 7.3.

Let $F = \mathbf{Q}$, $E = \mathbf{Q}(\sqrt{7})$, and $L = \mathbf{Q}(\zeta_7)$. Then $\theta(\zeta_7) = \zeta_7^2$. $A = (L/E, \theta, \sqrt{-7})$ is a cyclic E–algebra.

EXAMPLE 7.4.

Let $F = \mathbf{Q}(\sqrt{21})$, $E = \mathbf{Q}(\sqrt{21}, \zeta_3)$, $L = \mathbf{Q}(\zeta_{21})$. Then $\theta(\zeta_{21}) = \zeta_{21}^4$ and $A = (L/E, \theta, \zeta_3)$ is a cyclic E–algebra.

The *norm* of an element of L over E is define by

$$N_{L/E}(\beta) = \prod_{j=0}^{d-1} \theta^j(\beta).$$

The following theorem is due to Wedderburn (see [**30**, p. 278–279] for a proof).

THEOREM 7.5 (Wedderburn). (a) *If* $\alpha^j \notin N_{L/E}(L^\times)$ *for* $j = 1, \ldots, d-1$*, then* $(L/E, \theta, \alpha)$ *is a division algebra. In addition, if d is prime, this is necessary as well.*
(b) $\alpha \in N_{L/E}(L^\times)$ *if and only if* $(L/E, \theta, \alpha) \cong \mathrm{M}(d; E)$.

4. Splitting fields and degree

The next proposition allows us to view the cyclic algebra $A = (L/E, \theta, \alpha)$ inside $\mathrm{M}(d; L)$.

PROPOSITION 7.6. $A \otimes_E L \cong \mathrm{M}(d; L)$ *as central simple L–algebras.*

PROOF. To begin, we surely must map the field E to the scalar matrices. Specifically, for $\alpha \in E$,

$$\alpha \longmapsto \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha \end{pmatrix}.$$

To ensure that $L \subset A$ is not central, we can not be so free with how we define the homomorphism on L. For $\beta \in L$,

$$\beta \longmapsto \begin{pmatrix} \beta & 0 & 0 & \dots & 0 \\ 0 & \theta(\beta) & 0 & \dots & 0 \\ 0 & 0 & \theta^2(\beta) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta^{d-1}(\beta) \end{pmatrix}$$

Finally, for a general element

$$\beta_0 + \beta_1 X + \dots \beta_{d-1} X^{d-1},$$

we have

$$\begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{d-1} \\ \alpha\theta(\beta_{d-1}) & \theta(\beta_0) & \theta(\beta_1) & \dots & \theta(\beta_{d-2}) \\ \alpha\theta^2(\beta_{d-2}) & \alpha\theta^2(\beta_{d-1}) & \theta^2(\beta_0) & \dots & \theta^2(\beta_{d-3}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha\theta^{d-1}(\beta_1) & \alpha\theta^{d-1}(\beta_2) & \alpha\theta^{d-1}(\beta_3) & \dots & \theta^{d-1}(\beta_0) \end{pmatrix}$$

That this is an injective homomorphism is left for the reader. □

We say that an extension K of E *splits* A if $A \otimes_E K \cong \mathrm{M}(m;K)$.

PROPOSITION 7.7. *Every finite dimensional E–algebra splits over a finite extension of E.*

We refer the reader again to [**30**, p. 239] for a proof. A stronger result holds and can be useful in practise.

PROPOSITION 7.8. *Let $L \subset A$ be a maximal subfield of a simple E–algebra A. Then A splits over L.*

We note in passing that every finite dimensional algebra over E splits over **C**. To see this, we apply Proposition 7.7. Specifically, if a finite dimensional **C**–algebra A did not split over **C**, then there would be a nontrivial finite extension $L/\mathbf{C}$.

Using Proposition 7.7, we define the *degree* of A to be the minimum degree over E of a splitting field for A and denote this by $\deg_E(A)$.

PROPOSITION 7.9. *If $A = (L/E, \theta, \alpha)$ is a division algebra, then* $\deg_E(A) = [L:E]$. *More generally,* $\deg_E(A) \mid [L:E]$.

The *index* $\mathrm{ind}(A)$ is the unique positive integer such that

$$\mathrm{ind}_E(A)^2 \deg_E(A)^2 = \dim_E A.$$

By Theorem 7.3, $A = \mathrm{M}(m;D)$, for some central E–division algebra D. Alternatively, the index and degree of A are given by $\mathrm{ind}_E(A) = m$ and $\deg_E(A) = \deg_E(D)$.

PROPOSITION 7.10. *Let A be a simple E–algebra, L/E a finite extension, and* $\deg A = [L:E]$. *Then L is splits A if and only if A contains a maximal subfield isomorphic to L.*

5. Orders in algebras

In building lattices in SU$(n,1)$ using Hermitian matrices, our lattices were constructed, up to commensurability, as stabilizers of lattices in Hermitian vector spaces. Indeed, for an admissible pair $(E/F,H)$ where H has signature $(n,1)$, the group SU$(n,1;\mathscr{O}_E)$ is precisely the stabilizers of the lattices $\mathscr{O}_E^{n,1}$ in $\mathbf{C}^{n+1}$ under the action of the H–unitary group SU(H). We now introduce orders in algebras which will play the role of the lattice $\mathscr{O}_E^{n+1}$ in the vector space $\mathbf{C}^{n+1}$.

5.1. The basics. For an E–algebra A, by an *$\mathscr{O}_E$–order* in A, we mean a finitely generated subring $\mathscr{O}$ of A such that

(a) $\mathscr{O}$ is a finitely generated $\mathscr{O}_E$–module, and
(b) $A = \mathscr{O} \otimes_{\mathscr{O}_E} E$ as an $\mathscr{O}_E$–module.

EXAMPLE 7.5.

- $\mathscr{O}_E$ is a $\mathscr{O}_E$–order in E. For any finite extension L/E, $\mathscr{O}_L$ is a $\mathscr{O}_E$–order in L.
- $\mathrm{M}(d;\mathscr{O}_E)$ is an $\mathscr{O}_E$–order in $\mathrm{M}(d;E)$.
- For $A = \left(\frac{-1,-1}{\mathbf{Q}}\right)$, $\mathbf{Z}[1,i,j,k]$ is a $\mathbf{Z}$–order.
- For $A = (L/E,\theta,\alpha)$, $\mathscr{O} = \bigoplus_{j=0}^{d-1} \mathscr{O}_L X^j$ is a $\mathscr{O}_E$–order.

5.2. Existence of orders; maximal orders. In order to build lattices in SU$(n,1)$, we require the following existence theorem for orders in a central, simple E–algebra.

THEOREM 7.11 (Existence of orders; [**35**]). *For any E–algebra A, there exists a $\mathscr{O}_E$–order $\mathscr{O}$ of A.*

In describing commensurability classes, the existence of maximal order is quite useful. By a maximal order, the mean a $\mathscr{O}_E$–order $\mathscr{O}$ of an E–algebra A which is not properly contained in any $\mathscr{O}_E$–order of A.

THEOREM 7.12 (Existence of maximal orders; [**35**]). *Let E be a number field (or local field), A an E–algebra, and $\mathscr{O}$ a $\mathscr{O}_E$–order of* A. *Then $\mathscr{O}$ is contained in a maximal $\mathscr{O}_E$–order.*

CHAPTER 8

Involutions and Hermitian elements

1. Involutions of first and second kind

1.1. The basics. In order to relate cyclic central E–algebras A to the special unitary group SU$(n,1)$, we require an additional structure on A called an *involution of second kind*. An *involution* $*$ on A is a map $*\colon A \longrightarrow A$ such that

$$(x+y)^* = x^* + y^* \text{ and } (xy)^* = y^* x^*.$$

We say that $*$ is of *first kind* if $*_{|E} = \mathrm{id}_E$ and say that $*$ is of *second kind* otherwise.

Our interest is when E/F is a totally imaginary quadratic extension of a totally real number field F. In this case, if A is an cyclic E–algebra, we seek an involution of second kind $*$ which extends the Galois involution on E/F.

EXAMPLE 8.1.

For a cyclic extension L/E with E/F as above and $\alpha \in N_{L/E}(L^\times)$, the cyclic algebra $(L/E,\theta,\alpha)$ is isomorphic to $\mathrm{M}(d;E)$, where $d = [L:E]$. The algebra $\mathrm{M}(d;E)$ admits an involution of second kind which extends the Galois involution on E/F. Specifically, complex transposition is such an involution.

Given a cyclic E–algebra A equipped with an involution of second kind $*$ which extends the Galois involution on E, in the splitting $A \otimes_E L$, we obtain a new involution $*$ on $\mathrm{M}(d;L)$. By Theorem 4.5, $* = \star_H$, for some Hermitian matrix $H \in \mathscr{H}^\times$. We call $*$ *standard* if in the splitting $A \otimes_E \mathbf{C}$, $*$ produces complex transposition. The following result is due to Albert (see also [**22**] or [**36**]).

PROPOSITION 8.1 (Albert; [**1**]). *If A is a simple central E–algebra with an involution $*$ and L/E is any extension, then $A \otimes_E L$ also admits an involution which extends the involution $*$.*

By a *unitary algebra* over E/F, we mean a cyclic E–algebra A equipped with a standard involution of second kind. In the sequel, we require something more than. For each embedding τ_j of E, we insist that $*$ in $A \otimes_{\tau_j(E)} \mathbf{C}$

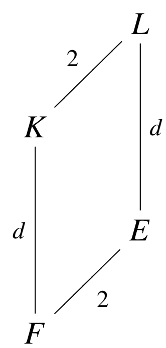
be complex transposition. In the remainder of this note we call such an involution *standard* and the pair $(A,*)$ a *unitary E–algebra*.

THEOREM 8.2 ([**36**]). *Any E–algebra A equipped with an involution of second kind admits a standard involution.*

1.2. Albert's criterion on existence of involutions. In order to construct unitary algebras, we require a means of deciding when a given cyclic E–algebra possesses an involution of second kind. We give the first of two theorems achieving this (see [**1**, p. 162] or [**22**, p. 36–39]).

Let $A = (L/E, \theta, \alpha)$ be a cyclic algebra and E/F be as before, a totally imaginary quadratic extension of a totally real number field. By Proposition 8.1, the Galois involution on E/F can be extended to an involution on $L = L \otimes_E E$ and we denote the totally real fixed field of this extension by K. This yields the lattice of fields

(2)



THEOREM 8.3 (Albert; [**1**]). $(L/E, \theta, \alpha)$ *admits an involution of second kind extending the involution* $* \in \mathrm{Gal}(E/F)$ *if and only if there exists* $\beta \in K^\times$ *such that* $N_{K/F}(\beta) = N_{E/F}(\alpha)$.

PROOF. Since in application one seeks an explicit involution, we prove the converse direction whose proof is constructive (see also [**15**]). Given $\beta \in K$ such that $N_{K/F}(\beta) = N_{E/F}(\alpha)$, we construction an involution which extends $*$ on E/F. Using β, we define $\star\colon (L/E, \theta, \alpha) \longrightarrow (L/E, \theta, \alpha)$ by

$$(X^j)^\star = \beta\theta(\beta)\dots\theta^{j-1}(\beta)(X^j)^{-1}, \quad \beta^\star = \beta^*,\ \beta \in L.$$

With this, the extension to all of $(L/E, \theta, \alpha)$ is given by

$$\left(\sum_{j=0}^{d-1} \beta_j X^j\right)^\star = \sum_{j=0}^{d-1} \beta_j^* (X^j)^\star.$$

To see that $\star$ is an involution, we must show $(Y^\star)^\star = Y$ for all $Y \in A$. For this, we need to know $(X^j)^{-1}$. Since $X^d = \alpha$, $X^{-1} = \alpha^{-1}X^{d-1}$ and more generally $(X^j)^{-1} = \alpha^{-1}X^{d-j}$. It suffices to shows that $(X^\star)^\star = X$. Indeed, we have

$$\begin{aligned}(X^\star)^\star &= (\beta\alpha^{-1}X^{d-1})^\star \\ &= \alpha^{-1}\beta\theta(\beta)\ldots\theta^{d-1}(\beta)\alpha^{-1}X \\ &= N_{K/F}(\beta)N_{E/F}(\alpha^{-1})X \\ &= \frac{N_{K/F}(\beta)}{N_{E/F}(\alpha)}X = X.\end{aligned}$$

□

2. Hermitian elements in unitary algebras and involutions

Let $(A,*)$ will denote a unitary algebra over E/F. We say that $h \in A$ is *Hermitian* (or $*$–Hermitian) if $x^* = x$ and *skew-Hermitian* if $x^* = -x$. We denote the F–vector subspace of Hermitian and skew-Hermitian elements by $\mathscr{H}$ and $\mathscr{S}\mathscr{H}$, respectively.

PROPOSITION 8.4. *As an F–vector space, $A = \mathscr{H} \oplus \mathscr{S}\mathscr{H}$.*

2.1. Hermitian involutions. Given a Hermitian element $h \in \mathscr{H}^\times$, we associate to h an involution $\star_h$ on A by $x^{\star_h} = \mu_h \circ *$. By Theorem 4.5 applied to $A \otimes_E L$, we obtain an involution $\star_h$ which is equivalent to $\star_H$ for some Hermitian matrix H. Since $*$ is standard, h viewed as an element of $\mathrm{M}(d;L)$ is a Hermitian matrix and is precisely the Hermitian matrix which yields the involution $\star_h$. We define the *signature* of h to be the signature of H.

PROPOSITION 8.5. *The signature of h is independent of the splitting.*

PROOF. Let L_1, L_2 be two minimal splitting fields of A and $h \in A$ a Hermitian element. We assume in both splittings of A that $*$ produces complex transposition. Notice first that if we compute the signature of h in $\mathrm{M}(d;L_j)$ or $\mathrm{M}(d;L_j) \otimes_{L_j} \mathbf{C}$, we obtain the same answer. Let (p_1,q_1) and (p_2,q_2) be the signatures of h computed in $\mathrm{M}(d;L_1)\otimes\mathbf{C}$ and $\mathrm{M}(d;L_2)\otimes\mathbf{C}$. These algebras are both isomorphic to $\mathrm{M}(d;\mathbf{C})$. By Theorem 7.1 this isomorphism can be viewed as an inner automorphism of $\mathrm{M}(d;\mathbf{C})$. However, the signature of a Hermitian matrix is certainly preserved under conjugation. □

2.2. Unitary groups. Using (1) as a guide, for a Hermitian element $h \in \mathscr{H}^\times$, we define the *h–special unitary group* by

$$\mathrm{SU}(h) \stackrel{\text{def}}{=} \left\{ x \in (A \otimes_E \mathbf{C})^\times \ : \ xx^{\star_h} = 1 \right\}.$$

Since $(A \otimes_E \mathbf{C})^\times \cong \mathrm{GL}(d;\mathbf{C})$, if h has signature $(n,1)$, $\mathrm{SU}(h) \cong \mathrm{SU}(n,1)$. More generally, $\mathrm{SU}(h) \cong \mathrm{SU}(p,q)$, where $p+q=d$ and h has signature (p,q).

CHAPTER 9

Arithmetic lattices in SU$(n,1)$

In this section, we give the most general construction of arithmetic lattices in SU$(n,1)$.

1. Second type lattices over Q

For lattices of first type in SU$(n,1)$, when $E/\mathbf{Q}$ was an imaginary quadratic extension, the groups SU$(n,1;\mathscr{O}_E)$ were lattices in SU$(n,1)$. However, when E was a larger extension of $\mathbf{Q}$, the indiscreteness of $\mathscr{O}_E \subset \mathbf{C}$ complicated the matter considerably. This persists in our constructions of lattices in this section. For this reason, we begin with the case $E/\mathbf{Q}$ is an imaginary quadratic extension and generalize the construction to arbitrary totally imaginary quadratic extensions of totally real number fields for which this construction is a special case.

Let A be a unitary division algebra of degree $n+1$ over an imaginary quadratic extension E of $\mathbf{Q}$. For a Hermitian element $h \in \mathscr{H}^\times$ of signature $(n,1)$, the group SU(h) is isomorphic to SU$(n,1)$ via the splitting isomorphism $A \otimes_E \mathbf{C} \cong \mathrm{M}(n+1;\mathbf{C})$. Given an $\mathscr{O}_E$–order $\mathscr{O}$ in A, we form the group

$$\mathrm{SU}(h;\mathscr{O}) \stackrel{\text{def}}{=} \left\{x \in \mathscr{O}^\times \ : \ xx^{\star_h} = 1\right\}.$$

THEOREM 9.1. SU$(h;\mathscr{O})$ *is a cocompact lattice in* SU(h).

PROOF. That SU$(h;\mathscr{O})$ is a lattice follows from Theorem 2.5, since SU$(h;\mathscr{O})$ can be viewed as the $\mathbf{Z}$–points of the real $\mathbf{Q}$–algebraic group SU(h). We demonstrate cocompactness by arguing the contrapositive. If SU$(h;\mathscr{O})$ is not cocompact, by Theorem 2.6, there exists a nontrivial unipotent element $x \in \mathrm{SU}(h;\mathscr{O})$. This, in turn, produces the nontrivial nilpotent element $x-1 \in A$, which opposes our assumption that A is a division algebra. □

2. Second type lattices over arbitrary number fields

Let E/F be a totally imaginary quadratic extension of a totally real number field F. We denote the distinct complex embeddings of E be $\tau_1,\dots\tau_s$ and the compatible real embeddings of F by $\sigma_1,\dots,\sigma_s$. Via τ_1 and σ_1, we identify $E \subset \mathbf{C}$ and $F \subset \mathbf{R}$ such that $E \cap \mathbf{R} = F$. Given a cyclic extension L/E, for each embedding τ_j of E, we obtain a family of embeddings $\lambda_{1,j}, \dots, \lambda_{r_j,j}$. We select a fixed embedding λ_j for each τ_j and view $L \subset \mathbf{C}$ via λ_1.

Given a cyclic E–algebra $A = (L/E, \theta, \alpha)$ via the embedding $L \subset \mathbf{C}$ afforded by λ_1 and the splitting $A \otimes_E L \cong \mathrm{M}(d;L)$, for each embedding $\lambda_j \neq \lambda_1$, we obtain a new algebra

$$ {}^{\lambda_j}A = (\lambda_j(L)/\tau_j(E), \lambda_j \circ \theta \circ \lambda_j^{-1}, \tau_j(\alpha)). $$

Up to an E–algebra isomorphism, this algebra is independent of the selection of λ_j among the family $\lambda_{1,j}, \dots, \lambda_{r_j,j}$. We denote this algebra by ${}^{\tau_j}A$.

In addition, if $(A, *)$ is a unitary algebra with Hermitian element $h \in \mathscr{H}^\times$, for each $\tau_j \neq \tau_1$, we obtain a new Lie group ${}^{\tau_j}\,\mathrm{SU}(h) = \mathrm{SU}({}^{\tau_j}h)$. As before, applying $\mathrm{Res}_{F/\mathbf{Q}}$ we obtain the short exact sequence

$$ 1 \longrightarrow \prod_{j=2}^{s} {}^{\tau_j}\,\mathrm{SU}(h) \xrightarrow{\ \iota\ } \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(h)) \xrightarrow{\ \pi\ } \mathrm{SU}(h) \longrightarrow 1 \ . $$

We start with $(A, *)$, a unitary division algebra of degree $n+1$ over E with a Hermitian element h of signature $(n,1)$. Given an $\mathscr{O}_E$–order $\mathscr{O}$ in A, $\mathrm{SU}(h;\mathscr{O})$ maps to a subgroup of $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(h))$ which is commensurable with $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(h))(\mathbf{Z})$ and in the projection π, $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(h))(\mathbf{Z})$ maps to a subgroup which is commensurable with $\mathrm{SU}(h;\mathscr{O})$.

THEOREM 9.2. $\mathrm{SU}(h;\mathscr{O})$ *is a (cocompact) lattice in* $\mathrm{SU}(n,1)$ *if and only if* ${}^{\tau_j}\,\mathrm{SU}(h)$ *is compact for each* $j = 2, \dots, s$.

PROOF. That $\mathrm{SU}(h;\mathscr{O})$ is a lattice is identical to the proof of Theorem 5.2. For cocompactness, we can argue as in the proof of Theorem 9.1 or Corollary 5.3. □

We call the lattices in Theorem 9.2 *arithmetic lattices of second type*.

3. Mixed type lattices

Let $r, d \in \mathbf{N}$, $rd = n+1$, E/F a totally imaginary quadratic extension of a totally real number field, and $(A, *)$, a unitary algebra over E/F of degree d. The simple E–algebra $\mathrm{M}(r;A)$ admits an involution of second kind given by $*$–transposition. If L is a splitting field for A, then

$$ \mathrm{M}(r;A) \otimes_E L = \mathrm{M}(r; A \otimes_E L) = \mathrm{M}(r; \mathrm{M}(d;L)) = \mathrm{M}(rd;L). $$

We assume that the involution $*$ on $\mathrm{M}(r;A)$ is standard in this splitting.

For a Hermitian element $h \in \mathrm{M}(r;A)$ with associated twisted involution $\star$, set

$$\mathrm{SU}(h;A) = \{x \in \mathrm{M}(r;A) \ : \ xx^\star = I_r\}.$$

If in the splitting $\mathrm{M}(r;A) \otimes_E L$, h has signature $(n,1)$, $\mathrm{SU}(h;A \otimes_E \mathbf{C}) \cong \mathrm{SU}(n,1)$. Given a triple $(A,*,h)$ above for the pair (r,d), we say that $(A,*,h)$ is *admissible* if for all $\tau_j \neq \mathrm{id}_E$ the group ${}^{\tau_j}\,\mathrm{SU}(h;A \otimes_E \mathbf{C})$ is compact.

THEOREM 9.3. *Let $(A,*,h)$ be admissible over E/F with associated pair (r,d). Then for any $\mathcal{O}_E$–order $\mathcal{O}$,* SU$(h;\mathcal{O})$ *is a lattice in* SU$(n,1)$.

PROOF. This follows from Theorem 2.5 and Lemma 5.1. □

We call the lattices in Theorem 9.3 *arithmetic lattices of mixed type*. Note that both arithmetic lattices of first and second type are of mixed type.

EXAMPLE 9.1.

Let K be as in (2), $\alpha_1,\dots,\alpha_r \in K$, and $h = \mathrm{diag}(\alpha_1,\dots,\alpha_r)$. In the splitting given by Proposition 7.6,

$$h = \mathrm{diag}(\alpha_1, \theta(\alpha_1), \theta^2(\alpha_1), \dots, \theta^{d-1}(\alpha_1), \alpha_2, \dots, \theta^{d-1}(\alpha_r)).$$

By Theorem 2.1, there exists $\alpha_1,\dots,\alpha_r \in K$ such that

(1) $\alpha_1 < 0$ and $\alpha_j > 0$, for $j > 1$, and
(2) $\theta^\ell(\alpha_j) > 0$ for all $\ell = 1,\dots,d-1$ and $j = 1,\dots,r$.

Thus, $(A,*,h)$ is admissible and so by Theorem 9.3, SU$(h;\mathcal{O})$ is a lattice in SU$(n,1)$ for any $\mathcal{O}_E$–order $\mathcal{O}$ in A.

THEOREM 9.4. *If $r,d >,1$ and $(A,*,h)$ is an admissible triple over E/F for the associated pair (r,d), then* SU$(h;\mathcal{O})$ *is cocompact for any $\mathcal{O}_E$–order $\mathcal{O}$.*

PROOF. To begin, we note that Theorem 4.2, Theorem 4.3, and Theorem 4.4 hold in this setting. On the free A–module A^r, there exists a basis $e_1,\dots,e_r$ such that $h = h_{an} \oplus h_{iso}$ and this decomposition coincides with the A–module decomposition

$$A^r = A[e_1,\dots,e_{r-m}] \oplus A[e_{r-m+1},\dots,e_r] = A_{an} \oplus A_{iso}.$$

Since h is non-degenerate, $\dim_A A_{iso}$ is even. In particular, m is even. In the splitting afforded by L, h_{an} has signature say (p,q). It is a simple matter now to see that h has signature $(p+dm/2, q+dm/2)$. Note here that $p+q = d(r-m)$ and so

$$p+q+dm = d(r-m)+dm = dr$$

as expected. However, $d > 1$ and $h = (n,1)$, which is impossible since $dm/2 > 1$ and $q \geq 0$, unless $h = h_{an}$. By Theorem 2.6, if $\mathrm{SU}(h;\mathscr{O})$ is noncocompact, $\mathrm{SU}(h;\mathscr{O})$ contains a nontrivial unipotent element. However, by Theorem 4.4, the existence of such an element is equivalent to $h \neq h_{an}$. Therefore, by Theorem 2.6, $\mathrm{SU}(h;\mathscr{O})$ is cocompact. □

4. Admissible triples and the classification theorem

In total, we have the following theorem:

THEOREM 9.5 (Classification of arithmetic lattices). *Let $(A,*,h)$ be an admissible triple over E/F with associated pair (r,d) and $\mathscr{O}$ an $\mathscr{O}_E$–order in A. Then $\mathrm{SU}(h;\mathscr{O})$ is an lattice in $\mathrm{SU}(n,1)$ via the injection induced by the isomorphism of $\mathrm{SU}(h)$ with $\mathrm{SU}(n,1)$ given by the splitting isomorphism $\mathrm{M}(r;A) \otimes_E \mathbf{C} \longrightarrow \mathrm{M}(n+1;\mathbf{C})$. Moreover, if $r = 1$ and E is not an imaginary quadratic extension of $\mathbf{Q}$ or $r > 1$, then $\mathrm{SU}(h;\mathscr{O})$ is cocompact.*

REMARK. By Theorem 7.1, if we change the splitting (and hence the isomorphism between $\mathrm{SU}(h)$ and $\mathrm{SU}(n,1)$, we obtain a new lattice which is commensurable in the wide sense with the original one. Thus, up to wide commensurability, this is independent of the selection of a splitting isomorphism.

One corollary which the author made use of in [**25**] is:

COROLLARY 9.6 (Classification of noncocompact arithmetic lattices). *Let $\Lambda < \mathrm{SU}(n,1)$ be a noncocompact arithmetic lattice. Then there exists an imaginary quadratic extension $E/\mathbf{Q}$ and an E–defined signature $(n,1)$ Hermitian matrix H such that Λ is commensurable in the wide sense with $\mathrm{SU}(H;\mathscr{O}_E)$.*

In combination with Theorem 6.11, we have an very explicit description of the noncocompact arithmetic lattices in $\mathrm{SU}(n,1)$.

5. Tits symbols and a summary of the lattices

For an admissible triple $(A,*,h)$ with pair (r,d), the elements of reduced norm one, $(A \otimes_E \mathbf{C})^1$, is an *outer real F–form* of $\mathrm{SL}(n;\mathbf{C})$. In the notation of Tits (see [**6**]), these groups are denoted by ${}^2\mathrm{A}^d_{n,r-1}$. We associate a *Tits symbol* to each of the above groups and their lattices, namely $\mathrm{U}^d_{n,r-1}$.

(r,d)	**Symbol**	SL**–group**	**division algebra** A
$(3,1)$	$\mathrm{U}^1_{2,2}$	$\mathrm{SL}(3;A)$	number field
$(1,3)$	$\mathrm{U}^3_{2,0}$	$\mathrm{SL}(1;A)$	cyclic, degree 3
$(4,1)$	$\mathrm{U}^1_{3,3}$	$\mathrm{SL}(4;A)$	number field
$(1,4)$	$\mathrm{U}^4_{3,0}$	$\mathrm{SL}(1;A)$	cyclic, degree 4
$(2,2)$	$\mathrm{U}^2_{3,1}$	$\mathrm{SL}(2;A)$	quaternion algebra
$(5,1)$	$\mathrm{U}^1_{4,4}$	$\mathrm{SL}(5;A)$	number field
$(1,5)$	$\mathrm{U}^5_{4,0}$	$\mathrm{SL}(1;A)$	cyclic, degree 5
$(6,1)$	$\mathrm{U}^1_{5,5}$	$\mathrm{SL}(6;A)$	number field
$(1,6)$	$\mathrm{U}^6_{5,0}$	$\mathrm{SL}(1;A)$	cyclic, degree 6
$(3,2)$	$\mathrm{U}^2_{5,2}$	$\mathrm{SL}(3;A)$	quaternion algebra
$(2,3)$	$\mathrm{U}^3_{5,1}$	$\mathrm{SL}(2;A)$	cyclic, degree 3
$(7,1)$	$\mathrm{U}^1_{6,6}$	$\mathrm{SL}(7;A)$	number field
$(1,7)$	$\mathrm{U}^7_{6,0}$	$\mathrm{SL}(1;A)$	cyclic, degree 7
$(8,1)$	$\mathrm{U}^1_{7,7}$	$\mathrm{SL}(8;A)$	number field
$(1,8)$	$\mathrm{U}^8_{7,0}$	$\mathrm{SL}(1;A)$	cyclic, degree 8
$(4,2)$	$\mathrm{U}^2_{7,3}$	$\mathrm{SL}(4;A)$	quaternion algebra
$(2,4)$	$\mathrm{U}^4_{7,1}$	$\mathrm{SL}(2;A)$	cyclic, degree 4

CHAPTER 10

Examples for small pairs

In this chapter we give several different examples of arithmetic lattices in SU$(n,1)$.

1. Lattices of type $\mathrm{U}^1_{2,2}$

Our first class of examples appeared in our discussion of arithmetic lattices of first type. The lattices that arise from $\mathrm{U}^1_{2,2}$ are described as follows. For E/F and Theorem 6.11, there is at most one class of Hermitian matrix for each of the $2^{[F:\mathbf{Q}]}$ possible signatures. Without loss of generality, select $H = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$ for $\alpha_j \in \mathscr{O}_F$. Since Theorem 9.6 treated the case when $F = \mathbf{Q}$, we assume that $F \neq \mathbf{Q}$. To construct cocompact lattices, we apply Theorem 2.1. Specifically, select $\alpha_1, \alpha_2, \alpha_3$ such that

(1) $\alpha_1, \alpha_2 > 0$ and $\alpha_3 < 0$, and
(2) for all $\sigma_\ell \neq \mathrm{id}_F$, $\sigma_\ell(\alpha_j) < 0$ or $\sigma_\ell(\alpha_j) > 0$ for $j = 1,2,3$.

EXAMPLE 10.1.

Let $F = \mathbf{Q}(\sqrt{d})$, $d \in \mathbf{N}$ and square free, $E = \mathbf{Q}(\sqrt{d}, i)$. Set $H = \mathrm{diag}(1,1,-\sqrt{d})$. Then $\mathrm{SU}(H;\mathscr{O}_E)$ is a cocompact lattice in SU$(2,1)$.

2. Lattices of type $\mathrm{U}^1_{n,n}$

For E/F and a Hermitian form $H \in \mathrm{M}(n+1;E)$, up to E–isomorphism of the associated unitary groups, by Theorem 4.2, we can assume that $H = \mathrm{diag}(\alpha_1, \ldots, \alpha_{n+1})$, with $\alpha_j \in \mathscr{O}_F$. By Theorem 2.1, select $\alpha_1, \ldots, \alpha_{n+1}$ such that

(1) $\alpha_{n+1} < 0$, $\alpha_j > 0$ for $j \neq n+1$, and
(2) for all $\sigma_\ell \neq \mathrm{id}_F$, $\sigma_\ell(\alpha_j) > 0$ or $\sigma_\ell(\alpha_j) < 0$ for all j.

Then $\mathrm{SU}(H;\mathscr{O}_E)$ is a lattice in SU$(n,1)$.

THEOREM 10.1. *Every lattice of type* $\mathrm{U}^1_{n,n}$ *is commensurable in the wide sense to* $\mathrm{SU}(H;\mathscr{O}_E)$ *for some pair* $(H, E/F)$ *as above.*

EXAMPLE 10.2.

Let $F = \mathbf{Q}(\sqrt{d})$, $d \in \mathbf{N}$ and square free, $E = \mathbf{Q}(\sqrt{d}, i)$. Set $H = \mathrm{diag}(1,1,\ldots,1,-\sqrt{d})$. Then $\mathrm{SU}(H;\mathscr{O}_E)$ is a lattice in SU$(n,1)$.

3. Lattices of type $\mathrm{U}^3_{2,0}$

From several perspectives, the lattices of type $\mathrm{U}^1_{n,n}$ are well understood. The simplest lattices not of this type are of type $\mathrm{U}^3_{2,0}$.

For E/F, L/E a cyclic Galois extension of degree 3, and K as in (2), let A be a cyclic division algebra $A = (L/E, \theta, \alpha)$ which admits an involution of second kind $*$. For a Hermitian element $h \in A^\times$, we insist that h have signature $(2,1)$ and at each $\tau_\ell \neq \mathrm{id}_E$, we insist that ${}^{\sigma_\ell}h$ have signature $(3,0)$ or $(0,3)$. Then for any $\mathscr{O}_E$–order $\mathscr{O}$ of A, $\mathrm{SU}(h;\mathscr{O})$ is a lattice in $\mathrm{SU}(2,1)$.

THEOREM 10.2. *Every lattice of type* $\mathrm{U}^3_{2,0}$ *is commensurable in the wide sense to* $\mathrm{SU}(h;\mathscr{O})$ *for some* E/F *and* $(A,*,h,\mathscr{O})$ *as above.*

Our first concrete example produces a lattice whose associated arithmetic orbifold is commensurable with Mumford's fake $\mathbf{CP}^2$ (see [**28**]). We refer the reader to [**18**] for more on this.

EXAMPLE 10.3.

Let $E = \mathbf{Q}(\sqrt{-7})$, $F = \mathbf{Q}$, $L = \mathbf{Q}(\zeta_7)$, and $K = \mathbf{Q}(\cos(2\pi/7))$. Set

$$\lambda = \frac{-1+\sqrt{-7}}{2}, \quad \alpha = \lambda/\overline{\lambda}.$$

In addition, set θ to be the Galois automorphism

$$\theta \colon L \longrightarrow L, \quad \theta(\zeta_7) = \zeta_7^2.$$

The reader can check (using Theorem 7.5) that $A = (L/E, \theta, \alpha)$ is a division algebra.
According to [**18**], (see below)

$$\mathrm{Inv}_\lambda(A) = \frac{1}{3}, \quad \mathrm{Inv}_{\overline{\lambda}}(A) = -\frac{1}{3}$$

and any prime which does not divide 2, $\mathrm{Inv}_{\mathfrak{p}}(A) = 0$. Thus, by Theorem 12.8, A admits an involution of second type. Concretely, set

$$X^* = \overline{\alpha}X^2, \quad \beta^* = \overline{\beta}, \quad \beta \in L$$

$$h = (\lambda - \overline{\lambda}) - \overline{\lambda}X + \overline{\lambda}X^2$$

$$\mathscr{O} = \mathscr{O}_L \oplus \mathscr{O}_L\overline{\lambda}X \oplus \mathscr{O}_L\overline{\lambda}X^2.$$

Then $\mathrm{SU}(h;\mathscr{O})$ is an arithmetic lattice in $\mathrm{SU}(2,1)$.

By Yau's uniformization theorem, the manifold M_{Mum} Mumford constructed is a complex hyperbolic 2–manifold of minimal Euler characteristic. By Wang's theorem ([**41**]), M_{Mum} is a minimal volume complex hyperbolic 2–manifold.

THEOREM 10.3 (Kato; [**18**]). *Let* $M = \mathbf{H}^2_{\mathbf{C}}/\mathrm{SU}(h;\mathscr{O})$ *from (10.3). Then* M_{Mum} *and* M *are commensurable.*

More generally, Klingler [**20**] proved that every so-called fake $\mathbf{CP}^2$ is arithmetic of type $\mathrm{U}^3_{2,0}$. We refer the reader to [**17**], [**33**], [**38, 39**], and [**48**] for more on fake $\mathbf{CP}^2$ manifolds and p–adic uniformization.

Our next example comes from Amitsur [**3**] where finite subgroups of division algebras over fields of characteristic 0 are classified.

EXAMPLE 10.4.

Let $F = \mathbf{Q}(\sqrt{21})$, $E = \mathbf{Q}(\sqrt{21},\zeta_3)$, $L = \mathbf{Q}(\zeta_{21})$ and $K = \mathbf{Q}(\cos(2\pi/21))$. Then $A = (L/E,\theta,\zeta_3)$, where $\theta\colon L \longrightarrow L$, $\quad \theta(\zeta_{21}) = \zeta_{21}^4$ is a division algebra. In addition, using either Theorem 8.3 or Theorem 12.8, one can check that A admits an involution of second kind. Finally, set $h = \zeta_{21} + \zeta_{21}^{20} - \zeta_{21}^4 - \zeta_{21}^{17} \in K$. Having not given the involution $*$ on A, we can still verify that h is Hermitian since $*_{|L}$ is nothing more than the non-trivial Galois automorphism of $\mathrm{Gal}(L/K)$. To see that h has signature $(1,2)$, in the splitting $A \otimes_L E$ given by Proposition 7.6

$$h \longmapsto \begin{pmatrix} h & 0 & 0 \\ 0 & \theta(h) & 0 \\ 0 & 0 & \theta^2(h) \end{pmatrix}.$$

At the other embedding of F, h has signature $(3,0)$. Thus, by Theorem 9.5, $\mathrm{SU}(h;\mathscr{O})$ is a lattice in $\mathrm{SU}(2,1)$ for any $\mathscr{O}_E$–order $\mathscr{O}$ of A.

The algebra A in (10.4) is the group algebra $\mathbf{Q}G_{21,4}$, where

$$G_{m,r} = \left\{x,y \ : \ x^m = 1,\ y^n = x^t,\ yxy^{-1} = x^r\right\}$$

for any pair m,r of positive, relatively prime integers and

$$s = (r-1,m), \quad t = m/s$$

and n is the multiplicative order of r in $(\mathbf{Z}/m\mathbf{Z})^\times$.

More generally, we have the algebras

$$A_{m,r} = \left(\mathbf{Q}(\zeta_m)/\mathbf{Q}(\zeta_s), \zeta_m \longmapsto \zeta_m^r, \sum_{j=0}^{n} \zeta_m^{rj}\right)$$

which generalize (10.4).

Lattices of type $\mathrm{U}^3_{2,0}$ and $\mathrm{U}^1_{2,2}$ differ greatly (see [**26**]).

THEOREM 10.4. (a) *Every lattice of type* $\mathrm{U}^1_{2,2}$ *contains a totally geodesic real hyperbolic* 2*–orbifold group.*

(b) *No lattice of type* $\mathrm{U}^3_{2,0}$ *contains a totally geodesic real hyperbolic 2–orbifold group.*

PROOF. For (a), let $\Lambda = \mathrm{SU}(H;\mathscr{O}_E)$ for a pair $(H,E/F)$. As before, we may assume that $H = \mathrm{diag}(\alpha_1,\alpha_2,\alpha_3)$ with $\alpha_1 < 0$. Set $\Gamma = \mathrm{Stab}(\mathbf{C}[e_1,e_2])$. By construction, Γ is a lattice in $\mathrm{SU}(1,1)$. Alternatively, let Γ be the subgroup of Λ fixed under $*$. Then Γ is the lattice $\mathrm{SO}(H;\mathscr{O}_F)$, which by construction is in $\mathrm{SO}(2,1)$. For (b), given a totally geodesic real hyperbolic 2–orbifold group Γ in a lattice Λ of type $\mathrm{U}^3_{2,0}$, we obtain a quaternion subalgebra $D \subset A$, where A is the cyclic degree 3 division algebra used to Λ. However, $\dim D = 4$ while $\dim A = 9$, which is impossible. □

4. Lattices of type $\mathrm{U}^2_{3,1}$

Our next class of examples are the first ones of genuine mixed type. The lattices of type $\mathrm{U}^2_{3,1}$ are constructed as follows. Let A be a quaternion algebra over E/F equipped with an involution of second kind $*$. Let $\alpha_1,\alpha_2 \in A^\times$ such that $\alpha_j^* = \alpha_j$ and set $h = \mathrm{diag}(\alpha_1,\alpha_2)$. We select α_1,α_2 such that the signature of α_1 and α_2 in the splitting $A\otimes_E \mathbf{C}$ are $(1,1)$ and $(2,0)$, respectively. In addition, for each other embedding $\tau_\ell \neq \mathrm{id}_E$, we insist that α_1 and α_2 both have signature $(2,0)$. It follows that $\mathrm{SU}(h;A\otimes_E\mathbf{C})$ is isomorphic to $\mathrm{SU}(3,1)$ and by Theorem 9.5 for any $\mathscr{O}_E$–order $\mathscr{O}$ of A, $\mathrm{SU}(h;\mathscr{O})$ is a lattice in $\mathrm{SU}(3,1)$.

EXAMPLE 10.5.

Let D be a quaternion algebra with Hilbert symbol $\left(\frac{a,b}{\mathbf{Q}(i)}\right)$. Select $\alpha_1 \in D^\times$ such that $\mathrm{SU}(\alpha_1;D)$ is a Fuchsian group and $\alpha_2 = 1$. Setting $h = \mathrm{diag}(\alpha_1,\alpha_2)$, $\mathrm{SU}(h;\mathscr{O})$ is a lattice of type $\mathrm{U}^2_{3,1}$ for any $\mathscr{O}_E$–order $\mathscr{O}$ of D.

The lattices of type $\mathrm{U}^2_{3,1}$ differ from those of type $\mathrm{U}^4_{3,0}$ and $\mathrm{U}^1_{3,3}$.

THEOREM 10.5. (a) *Every lattice of type* $\mathrm{U}^1_{3,3}$ *contains a totally geodesic real hyperbolic s–orbifold group for* $s = 2,3$.

(b) *Every lattice of type* $\mathrm{U}^2_{3,1}$ *contains a totally geodesic real hyperbolic 2–orbifold group but not a totally geodesic real hyperbolic 3–orbifold group.*

(c) *There exist lattices of type* $\mathrm{U}^4_{3,0}$ *which do not contain a totally geodesic real hyperbolic s–orbifold group for* $s = 2,3$.

PROOF. For a lattice Λ of type $\mathrm{U}^1_{3,3}$, there exists a pair E/F and a Hermitian form $H \in \mathrm{M}(4;E)$ such that Λ is commensurable in the wide sense with $\mathrm{SU}(H;\mathscr{O}_E)$. As noted above, we can assume that H is diagonal and

$H \in \mathrm{M}(4;F)$. Consequently, we can form the group $\mathrm{SU}(H;\mathscr{O}_F)$. This group is a totally geodesic real hyperbolic 3–orbifold group, which in turn contains a totally geodesic real hyperbolic 2–orbifold group.

For a lattice Λ of type $\mathrm{U}^1_{3,3}$, we simply take the subgroup Γ of $\mathrm{SU}(H;\mathscr{O})$ consisting elements of the form $\mathrm{diag}(\beta,1)$. Visibly, Γ is a lattice in $\mathrm{SU}(1,1)$ which in turn produces a lattice in $\mathrm{SO}(2,1)$, since $\mathrm{SO}(2,1)$ and $\mathrm{SU}(1,1)$ are isogenous.[1]

For a lattice Λ of type $\mathrm{U}^1_{3,0}$, select A as to not contain a $*$–invariant quaternion subalgebra. The presence of such a totally geodesic subgroup in either case would produce such a quaternion subalgebra $B \subset A$ as in the proof of Theorem 4.2. By our selection of A, this is impossible. □

5. Lattices of type $\mathrm{U}^2_{5,2}$

The lattice of type $\mathrm{U}^2_{5,2}$ are similar to those of type $\mathrm{U}^2_{3,1}$. Let A be a quaternion algebra over E/F equipped with an involution of second type $*$. Select $\alpha_1,\alpha_2,\alpha_3$ such that $\alpha_j^* = \alpha_j$, α_1 has signature $(1,1)$, and α_j has signature $(2,0)$ for $j = 2,3$. In addition, we insist that at each embedding $\tau_\ell \neq \mathrm{id}_E$, each α_j have signature either $(2,0)$. Set $h = \mathrm{diag}(\alpha_1,\alpha_2,\alpha_3)$. By Theorem 9.5, for any $\mathscr{O}_E$–order $\mathscr{O}$ of A, $\mathrm{SU}(h;\mathscr{O})$ is a lattice in $\mathrm{SU}(5,1)$.

EXAMPLE 10.6.

Let D, α_1, and $\alpha_2 = 1$ be as in 10.5. Setting $h = \mathrm{diag}(\alpha_1,1,1)$, $\mathrm{SU}(h;\mathscr{O})$ is an arithmetic lattice of type $\mathrm{U}^2_{5,2}$ for any $\mathscr{O}_E$–order $\mathscr{O}$ of D.

6. Lattices of type $\mathrm{U}^3_{5,1}$

Let A be a cyclic division algebra over E/F of degree 3 equipped with an involution of second kind. Select $\alpha_1,\alpha_2 \in A^\times$ such that $\alpha_j^* = \alpha_j$, α_1 has signature $(2,1)$, and α_2 has signature $(3,0)$. Further, we insist that for each $\tau_\ell \neq \mathrm{id}_E$, both α_j have signature $(3,0)$. Set $h = \mathrm{diag}(\alpha_1,\alpha_2)$. By Theorem 9.5, for any $\mathscr{O}_E$–order $\mathscr{O}$ of A, $\mathrm{SU}(h;\mathscr{O})$ is a lattice in $\mathrm{SU}(5,1)$.

EXAMPLE 10.7.

Let $A = (L/E,\theta,\alpha)$ be the algebra from (10.3) related to Mumford's fake $\mathbf{CP}^2$. Let $\alpha_1 = h$ and $\alpha_2 = -1$. Setting $h = \mathrm{diag}(\alpha_1,-1)$ one can check that h has signature $(1,5)$. Taking the order $\mathscr{O}$ from (10.3), $\mathrm{SU}(h;\mathscr{O})$ is a lattice in $\mathrm{SU}(5,1)$.

As before, the lattices of type $\mathrm{U}^2_{5,2}$ and $\mathrm{U}^3_{5,1}$ differ.

[1] This groups contains **C**–Fuchsian but not **R**–Fuchsian geodesic subgroups.

THEOREM 10.6. (a) *Every lattice of type* $\mathrm{U}^2_{5,2}$ *contains a totally geodesic real hyperbolic* 2*–orbifold group. In addition, there exist lattices of type* $\mathrm{U}^2_{5,2}$ *which do not contain a totally geodesic complex hyperbolic* 2*–orbifold group.*

(b) *Every lattice of type* $\mathrm{U}^3_{5,1}$ *contains a totally geodesic complex hyperbolic* 2*–orbifold group (of type* $\mathrm{U}^3_{2,0}$*). In addition, there exist lattices of type* $\mathrm{U}^3_{5,1}$ *which do not contain a totally geodesic real hyperbolic* 2*–orbifold group.*

PROOF. The proof is similar to the ones given above and left as an exercise for the reader. □

CHAPTER 11

Classifying Hermitian structures over division algebras

In this chapter, we give the classification of Hermitian structures on a cyclic algebra $\mathrm{M}(r;A)$ over E/F which is in the same spirit as the classification given in Chapter 6.

1. Characteristic polynomials, reduced norms, and reduced traces

Before we can state the classification of Hermitian structure on free modules of division algebras, we first introduce to each element of an E–algebra a reduced norm, trace, and characteristic polynomial.

2. Invariants

3. The classification

4. A few examples

CHAPTER 12

Producing infinitely many commensurability class

The goal of this section is to produce infinitely many commensurability classes of arithmetic lattices of the second type.

THEOREM 12.1. *There exists infinitely many commensurability classes of lattices of the second type (type* $\mathrm{U}_{n,0}^{n+1}$*) in* $\mathrm{SU}(n,1)$ *for all n.*

1. The Brauer group

For a number field k, we say that a pair of simple, central k–algebras A_1 and A_2 are *Brauer equivalent* if there exists integers $r_1, r_2 \in \mathbf{N}$ such that $\mathrm{M}(r_1;A_1) \cong \mathrm{M}(r_2;A_2)$ as E–algebras. The Brauer equivalence classes of E–algebras is denoted by $\mathscr{B}_k$.

PROPOSITION 12.2 ([**30**]). *$\mathscr{B}_k$ is an abelian group with binary operation $\mathscr{B}_k \times \mathscr{B}_k \longrightarrow \mathscr{B}_k$ defined by $(A_1,A_2) \longmapsto A_1 \otimes_k A_2$.*

The inverse of a simple central E–algebra A is the *opposite algebra* A^{op}. Specifically, we have an isomorphism $A \otimes_k A^{\mathrm{op}} \longrightarrow \mathrm{End}_k(A)$ where A is viewed as a k–vector space.

REMARK. Every Brauer class contains a unique (up to E–algebra isomorphism) division algebra representative. To see this, for a central simple k–algebra A, by applying Theorem 7.3, we find $r \in \mathbf{N}$ and a k–division algebra D such that $A \cong \mathrm{M}(r;D)$. Consequently, there exists an alternative definition of the Brauer group in terms of only division algebras (see [**22**]).

We briefly review the structure of $\mathscr{B}_{k_\mathfrak{p}}$ for local fields, $\mathbf{R}$, and $\mathbf{C}$. For $\mathbf{C}$, since $\mathbf{C}$ is algebraically closed, $\mathscr{B}_\mathbf{C} = 1$. For $\mathbf{R}$, it is a classical theorem that the only central simple $\mathbf{R}$–algebras are $\mathbf{R}$ and $\mathbb{H}$, up to Brauer equivalence, and so the Brauer group $\mathscr{B}_\mathbf{R} = C_2$, the cyclic group of order two.

It remains to treat the nonarchimedean local fields and this is achieved with the following.

THEOREM 12.3. *For a local field $k_\mathfrak{p}$, $\mathscr{B}_{k_\mathfrak{p}} \cong \mathbf{Q}/\mathbf{Z}$.*

To prove Theorem 12.3, for $k_\mathfrak{p}$ and each $d \in \mathbf{N}$, there is a unique unramified cyclic extension $L_{\mathfrak{p},d}$ of $k_\mathfrak{p}$ of degree d whose Galois group $\mathrm{Gal}(L_{\mathfrak{p},d}/k_\mathfrak{p})$ is generated by the Frobenius automorphism (see [**30**] or [**46**]). Recall that the

Frobenius automorphism is unique lift of the Galois automorphism of the finite field $\mathscr{O}_{L,\mathfrak{p}}/\mathscr{O}_{k,\mathfrak{p}}$ given by $\sigma(x) = x^2$. To construct the asserted isomorphism $\mathscr{B}_{k_\mathfrak{p}} \longrightarrow \mathbf{Q}/\mathbf{Z}$, send $\frac{m}{n} + \mathbf{Z}$ to the algebra $(L_{\mathfrak{p},n}/k_\mathfrak{p}, \sigma, \pi^m)$ where σ is the Frobenius automorphism and π is a uniformizer of $k_\mathfrak{p}$. We refer the reader to [**30**, p. 338] for a proof of the injectivity of this map.

2. Local invariants

Given a central simple k–algebra A of degree d, for each valuation $\nu_\mathfrak{p}$, we obtain a central simple $k_\mathfrak{p}$–algebra by taking $A_\mathfrak{p} = A \otimes_k k_\mathfrak{p}$. By Theorem 12.3, this yields an element $\frac{m}{d} + \mathbf{Z} \in \mathbf{Q}/\mathbf{Z}$ which we call the *local invariant* at $\mathfrak{p}$ or $\nu_\mathfrak{p}$. and we denote this number by $\mathrm{Inv}_\mathfrak{p}(A)$. As mentioned above, at the archimedean places, there are two possibilities for $A \otimes_k k_\mathfrak{p}$. If $k_\mathfrak{p} = \mathbf{C}$, then $A \cong \mathrm{M}(d;\mathbf{C})$ and $\mathrm{Inv}_\mathbf{C}(A) = 0$. Otherwise, we set $\mathrm{Inv}_\mathbf{R}(A) = 1/2$ if $A \otimes_k \mathbf{R}$ is Brauer equivalent to $\mathbb{H}$, the quaternions, and $\mathrm{Inv}_\mathbf{R}(A) = 0$ otherwise. In total, the local invariants of A yield the *Hasse invariant* $\mathrm{Inv}(A) = \{\mathrm{Inv}_\mathfrak{p}(A)\}_\mathfrak{p}$.

3. A local-to-global theorem

For a given Brauer class $[A] \in \mathscr{B}_k$, we associated to A the Hasse invariant. Let

$$I(k) = \bigoplus_{\nu_\mathfrak{p} \in V_f} \mathbf{Q}/\mathbf{Z} \oplus \bigoplus_{\nu \in V_\mathbf{R}} \mathbf{Z}/2\mathbf{Z} \oplus \bigoplus_{\nu \in V_\mathbf{C}} 1.$$

The Hasse invariant yields a homomorphism $\mathscr{B}_k \longrightarrow I(k)$; in fact, we have a sequence

$$1 \longrightarrow \mathscr{B}_k \longrightarrow I(k) \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow 1 \tag{3}$$

where $I(k) \longrightarrow \mathbf{Q}/\mathbf{Z}$ is given by

$$(\dots, \iota_\nu, \dots) = \sum_{\nu \in V(k)} \iota_\nu.$$

THEOREM 12.4 (Local-to-global theorem). *The sequence (3) is exact.*

One of the main tools used in the proof of Theorem 12.4 is the Albert-Hasse-Brauer-Noether theorem.

THEOREM 12.5 (Albert-Hasse-Brauer-Noether theorem). *Let k be a number field and A a central simple k–algebra. If for all $\nu \in V(k)$, $A \otimes_k k_\nu$ is Brauer equivalent to k_ν, then A is Brauer equivalent to k.*

If we knew every simple central algebra was Brauer equivalent to a cyclic algebra, Theorem 12.5 can be proven by Theorem 7.5 in combination with the Hasse norm theorem (see [**4**]).

THEOREM 12.6 (Hasse norm theorem). *Let L/k be a cyclic extension and k a number field. For α in k, $\alpha \in N_{L/k}(L)$ if and only if for each $v \in V(k)$, $\alpha \in N_{L_v/k_v}(L_v)$.*

PROOF OF THEOREM 12.5 FOR CYCLIC ALGEBRAS. For cyclic k–algebras $A = (L/k, \theta, \alpha)$, by Theorem 7.5, A is Brauer equivalent to k if and only if $\alpha \in N_{L/k}(L^{\times})$. By Theorem 12.6, this is equivalent to α being a local norm for each $v \in V(k)$. Finally, applying Theorem 7.5 for each v, this is equivalent to A_v being Brauer equivalent to k_v for all $v \in V(k)$. □

The full proof of Theorem 12.5 uses a powerful result of Grunewald and Wang (see [**30**, p. 359] and [**4**], [**42**]).

THEOREM 12.7 (Grunewald-Wang; [**43**]). *Let E be a number field, $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be a finite set of prime ideals (possibly including infinite ones) and $m_1, \dots, m_r$ a collection of integers. Then there exists a cyclic extension L of degree n, where n is the least common multiple of the m_j, such that $[L_{\mathfrak{p}_j} : E_{\mathfrak{p}_j}] = m_j$ and $[L_{\mathfrak{q}} : E_{\mathfrak{q}}] = 1$ for all other primes.*

4. Local criterion for involutions of the second kind

The final piece required to prove Theorem 12.1 is a characterization on the existence of an involution of second kind based solely on the local invariants of the algebra A. Let E/F be a totally imaginary quadratic extension of a totally real number field F with non-trivial Galois involution $*$. For each prime ideal $\mathfrak{p} \in \mathscr{O}_E$, the involution $*$ applied to $\mathfrak{p}$ produces another prime ideal $\mathfrak{p}^*$.

For a central simple E–algebra A, we have local invariants at the ideals $\mathfrak{p}$ and $\mathfrak{p}^*$. The following theorem of Landherr (see [**36**, p. 355]) gives a necessary and sufficient condition for A to admit an involution which extends $*$ based solely on the arithmetic of the invariants $\mathrm{Inv}_{\mathfrak{p}}(A)$ and $\mathrm{Inv}_{\mathfrak{p}^*}(A)$.

THEOREM 12.8 (Landherr). *A admits an involution which extends $*$ on E/F if and only if*

$$\mathrm{Inv}_{\mathfrak{p}}(A) = 0, \quad \textit{for all } \mathfrak{p} = \mathfrak{p}^*$$

and

$$\mathrm{Inv}_{\mathfrak{p}}(A) + \mathrm{Inv}_{\mathfrak{p}^*}(A) = 0, \quad \textit{for all } \mathfrak{p} \neq \mathfrak{p}^*.$$

PROOF. We sketch a proof of the necessity of these restriction on the local invariants. For a prime ideal $\mathfrak{q}$ of F, there are two possibilities for the prime ideals over $\mathfrak{q}$ in E. If $\mathfrak{q}$ is unramified, then there exist a pair of distinct ideals over $\mathfrak{q}$ denoted by $\mathfrak{p}$ and $\mathfrak{p}'$, $\mathfrak{p}^* = \mathfrak{p}'$. In this case, the local algebra $A_{\mathfrak{p}}$, viewed as a $F_{\mathfrak{q}}$–algebra, is a direct product $B_{\mathfrak{p}} \times B_{\mathfrak{p}^*}$. The algebras $B_{\mathfrak{p}}$ and $B_{\mathfrak{p}^*}$ can be written as $A_{\mathfrak{p}}$ and $A_{\mathfrak{p}^*}$ and from this, it can be checked that $\mathrm{Inv}_{\mathfrak{p}}(A) = -\mathrm{Inv}_{\mathfrak{p}^*}(A)$. In the second case, since $\mathfrak{q}$ is ramified, $\mathfrak{p} = \mathfrak{p}^*$. The local algebra $A_{\mathfrak{p}}$ is a simple $E_{\mathfrak{p}}$–algebra which admits an involution of second kind. Every such algebra is Brauer equivalent to $E_{\mathfrak{p}}$ (see [**36**, p. 353]) and $\mathrm{Inv}_{\mathfrak{p}}(A) = 0$ as needed. □

For $E = F(\sqrt{\kappa})$ for $\kappa \in F$, we have:

PROPOSITION 12.9. *κ is a square in $F_{\mathfrak{q}}$ if and only if $\mathfrak{q}$ is unramified.*

Thus in determining the local invariants of an algebra A, the non-zero local invariants occur only at the local places $\mathfrak{q}$ for which κ is a square in $F_{\mathfrak{q}}$. Of course, only finitely many prime ideals $\mathfrak{q}$ are ramified.

5. Proof of Theorem 12.1

We now prove Theorem 12.1.

PROOF OF THEOREM 12.1. For each imaginary quadratic extension $E/\mathbf{Q}$, by Theorem 12.4 and Theorem 12.8, there exists a finite, non-empty collection of cyclic E–division algebras of degree $n+1$ which admit an involution of second kind extending the Galois involution on $E/\mathbf{Q}$. For each unitary pair $(A,*)$, let L denote a maximal cyclic extension of E contained in A and K, its totally real subfield (see (2)). It follows that each $h \in K^\times$ is a Hermitian element in A. Select h such that $\eta(h)$ is positive for every $\eta \in \mathrm{Gal}(L/E)$ except for one η_0. By construction, $(A,*,h)$ is an admissible triple for $E/\mathbf{Q}$.

It remains to prove:

PROPOSITION 12.10. *For K as above, there exists $h \in K$ such that $\eta(h)$ is positive for all but one element of* $\mathrm{Gal}(L/E)$.

PROOF. Let K^+ denote the subset of all elements $x \in K$ such that for all $\theta \in \mathrm{Gal}(L/E)$, $\theta(x) > 0$. It follows that $K^\times/K^+$ is a 2–group of order $2^{[K:\mathbf{Q}]}$. Specifically,

$$K^\times/K^+ = \bigoplus_{j=1}^{[K:\mathbf{Q}]} C_2$$

where each factor C_2 is associated to a distinct real embedding $\chi_j \colon K \longrightarrow \mathbf{R}$. Moreover, $\beta \in K^\times$ has a non-trivial component in the factor group C_2 associated to χ_j if and only if $\chi_j(\beta) < 0$. We assert that for any j, there exists $\beta_j \in K^\times$ such that β_j has a non-trivial component in the factor group C_2 associated to χ_j only. Otherwise, $K^\times/K^+$ can be generated by $[K:\mathbf{Q}]-1$ elements, which is impossible since this group has rank $[K:\mathbf{Q}]$. Setting $\beta_j = \eta$ completes the proof. □

To see that this indeed produces infinitely many distinct commensurability classes in SU$(n,1)$, note that the field E is a commensurability invariant for the lattice SU$(h;\mathscr{O})$ (see Proposition 15.2), for any $\mathscr{O}_E$–order in A. In fact, A is a commensurability invariant (see Proposition 15.3) for any of these lattices and so we obtain infinitely many commensurability classes by ranging over the imaginary quadratic extensions E of $\mathbf{Q}$. □

THEOREM 12.11. *For a fixed totally real number field F, there exist infinitely many commensurability classes of lattice of second type over F in* SU$(n,1)$ *for all $n > 0$.*

Finally, the most general result:

THEOREM 12.12. *Let (r,d) be a pair, F, a totally real number field. Then there exist infinitely many non-commensurable lattices* SU$(h;\mathscr{O})$ *arising from admissible triples $(A,*,h)$ over F with associated pair (r,d).*

PROOF. As in the proof of Theorem 12.1, it suffices to show that there exist infinitely many distinct algebras A that occur in admissible tuples $(A,*,h)$ for the pair (r,d) over E/F. It follows that over E/F, there exist infinitely many distinct cyclic division algebras A of degree d which admit an involution of second kind. For K as above, by Theorem 2.1, we can select $\alpha_1,\dots,\alpha_d \in K$ such that $\alpha_1 < 0$, $\alpha_j > 0$ for $j > 1$ and for all $\sigma_\ell \neq \mathrm{id}_K$, $\sigma_\ell(\alpha_j) > 0$ for $j = 1,\dots,d$. Set $h = \mathrm{diag}(\alpha_1,\dots,\alpha_d)$. By Theorem 9.5, SU$(h;\mathscr{O})$ is an arithmetic lattice of type $\mathrm{U}^d_{n,r-1}$ for any $\mathscr{O}_E$–order $\mathscr{O}$ of A. □

CHAPTER 13

Unitary algebras over local fields and reductions

CHAPTER 14

Arithmetic lattices in Lie groups: a general definition

1. Arithmetic lattices in Lie groups

Let G be a real Lie group with discrete subgroup Λ. We say that Λ is a *arithmetic lattice* if there exists a $\mathbf{Q}$–algebraic group $\mathbf{G}$, compact Lie groups K_1 and K_2, and an exact sequence

$$1 \longrightarrow K_1 \longrightarrow \mathbf{G}(\mathbf{R}) \stackrel{\psi}{\longrightarrow} G \longrightarrow K_2 \longrightarrow 1$$

such that $\psi(\mathbf{G}(\mathbf{Z}))$ is commensurable with Λ. It follows from a simple generalization of Lemma 5.1 and Theorem 2.5 that Λ is indeed a lattice in G.

REMARK. When G is semisimple, we can assume that $K_2 = 1$.

Recall that in the most general construction of lattices in $\mathrm{SU}(n,1)$, we had an admissible triple $(A,*,h)$ with pair (r,d) and an exact sequence

$$1 \longrightarrow \prod_{j=2}^{s} \tau_j \mathrm{SU}(h) \stackrel{\iota}{\longrightarrow} \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(h)) \stackrel{\pi}{\longrightarrow} \mathrm{SU}(h) \longrightarrow 1$$

such that $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(h))$ is the real points of a $\mathbf{Q}$–algebraic group and for an $\mathscr{O}_E$–order $\mathscr{O}$ in A, $\pi(\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SU}(h))(\mathbf{Z}))$ is commensurable with $\mathrm{SU}(h;\mathscr{O})$. The admissibility assumption ensures that $\ker\pi$ is compact.

THEOREM 14.1. *The lattices constructed in Theorem 9.5 are arithmetic.*

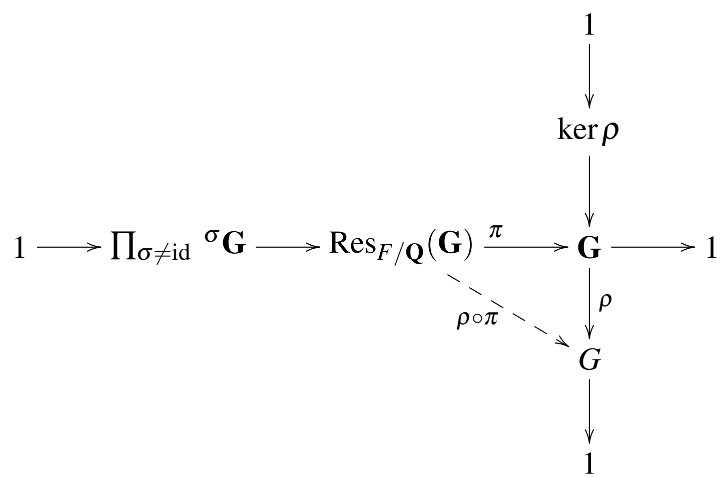
2. k–forms of a Lie group and lattices

Given a Lie group G and a field k, we say that $\mathbf{G}$ is a k–form of G if $\mathbf{G}$ is a k–algebraic group and there exists a Lie epimorphism $\psi\colon \mathbf{G} \longrightarrow G$ with compact kernel. Note in the case that G is a real Lie group like $\mathrm{SU}(n,1)$, we allow for $\mathbf{G}$ to be a k–defined real algebraic group.

EXAMPLE 14.1.

$\mathrm{SU}(n,1)$ is a $\mathbf{Q}$–defined real algebraic group and thus a $\mathbf{Q}$–form of $\mathrm{SU}(n,1)$. More generally, for a k–defined Hermitian form H, $\mathrm{SU}(H)$ is a k–defined real algebraic group. Provided H has signature $(n,1)$, $\mathrm{SU}(H)$ is a k–form of $\mathrm{SU}(n,1)$.

Let G be a real Lie group and $\mathbf{G}$ a real k–form of G. By applying restriction of scalars to $\mathbf{G}$, we obtain a real $\mathbf{Q}$–algebraic group $\mathrm{Res}_{F/\mathbf{Q}}(\mathbf{G})$. This produces the diagram:



By Lemma 5.1, when $\ker \pi$ is compact, $\mathbf{G}(\mathscr{O}_k)$ is a lattice in $\mathbf{G}$. Another application of Lemma 5.1proves that $\rho(\mathbf{G}(\mathscr{O}_k))$ is an arithmetic lattice in $\mathbf{G}$. We call such a k–form of G an *admissible k–form*. Crucial to the classification theorem is that every arithmetic lattice in G arises from an admissible k–form which follows from the definition. Indeed, in our most general construction, we obtained lattices precisely from the admissible F–forms of $\mathrm{SU}(n,1)$, where E/F is the given pair of fields in our construction. Consequently, the classification of k–forms of $\mathrm{SU}(n,1)$ together with a means of decide when a given k–form is admissible yields the classification of arithmetic lattices, up to wide commensurability.
Below we discuss the classification of F–forms of $\mathrm{SU}(n,1)$ for a pair of fields E/F. For a complete proof we refer the reader to [**31**, p. 78–92].

3. L/k–forms

Let $\mathbf{G}$ be an k–algebraic group and L/k, a finite Galois extension. By an *L/k–form* of $\mathbf{G}$, we mean a k–algebraic group $\mathbf{H}$ together with an L–algebraic isomorphism $\rho : \mathbf{G} \longrightarrow \mathbf{H}$.

For a L/k–form $\mathbf{H}$ of $\mathbf{G}$ with L–morphism ρ, we obtain a new L–morphism for each $\eta \in \mathrm{Gal}(L/k)$ by $\rho^\eta = \eta \circ \rho$. Additionally, we have an associated k–automorphism $\alpha_\eta = \rho^{-1} \circ \rho^\eta$. Altogether this yields a set map

$$\mathrm{Gal}(L/k) \longrightarrow \mathrm{Aut}_k(\mathbf{G})$$

defined by $\eta \longmapsto \alpha_\eta$. In fact, this is a noncommutative 1–cocycle and yields for $F(L/k;BG) = \{L/k\text{–forms}\}$

$$\varphi\colon F(L/k;BG) \longrightarrow H^1(\mathrm{Gal}(L/k);\mathrm{Aut}_k(\mathbf{G}))$$

where the L/k–forms are taken up to k–isomorphism.

THEOREM 14.2 ([**31**]). *φ is a bijection.*

By consider the collection of all finite Galois extensions L/k we obtain an inverse limit system and a map

$$F(k;\mathbf{G}) = \{k\text{–forms of } \mathbf{G}\} \longrightarrow H^1(\Gamma;\mathrm{Aut}_k(\mathbf{G})).$$

Note that

$$\begin{aligned} H^1(\Gamma;\mathrm{Aut}_k(\mathbf{G})) &= \varinjlim H^1(\mathrm{Gal}(L/k);\mathrm{Aut}_k(\mathbf{G})) \\ &= H^1(\varprojlim \mathrm{Gal}(L/k);\mathrm{Aut}_k(\mathbf{G})). \end{aligned}$$

THEOREM 14.3 ([**31**]). *The k–forms of $\mathbf{G}$, up to k–isomorphism, are in bijection with $H^1(\Gamma;\mathrm{Aut}_k(\mathbf{G}))$.*

For completeness, we explain why the set map in Theorem 14.2 is surjective. Before doing so, we briefly review non-abelian cohomology and Galois cohomology.

4. Non-abelian cohomology and surjectivity of φ

By a *profinite groups*, we mean a projective limit of finite groups. Let G be a group which is either profinite or discrete and A a discrete group with a continuous G–action by automorphisms; we call A a *G–group*. A continuous map $f\colon G \longrightarrow A$ is called a *1–cocycle* provided

$$f(st) = f(s)s \cdot f(t),$$

and we denote the set of 1–cocycles by $Z^1(G;A)$. We say that a pair of 1–cocycles $f,g \in Z^1(G;A)$ are *equivalent* if there exists $a \in A$ such that $f(s) = a^{-1}g(s)s \cdot a$. The *first cohomology of G* with this prescribed action and coefficients in A is the set of equivalence classes of 1–cocycles and is denoted by $H^1(G;A)$.

More generally, let G and A be as above and F a set with a continuous G–action and a compatible A–action. Namely

$$g \cdot (a \cdot f) = g \cdot a(g \cdot f), \quad g \in G,\ a \in A,\ f \in F.$$

Given a 1–cocycle $\lambda \in Z^1(G;A)$, we define a new G–action on F, denoted $*$, by $g * f = \lambda(g) \cdot f$.

When $G = \mathrm{Gal}(L/k)$, $A = \mathrm{Aut}_k(\mathbf{G})$, and $F = \mathbf{G}$, a k–algebraic group, $\mathbf{G}$ is a G–set. By twisting the G–action on $\mathbf{G}$ by a 1–cocycle as described above we obtain a new G–set ${}_{\lambda}\mathbf{G}$. Indeed, ${}_{\lambda}\mathbf{G}$ is a new k–algebraic group. We briefly explain this. Let $k[\mathbf{G}] = k[T]/\mathfrak{a}_k$ denote the k–algebra of regular functions on $\mathbf{G}$. As a k–affine variety, $\mathbf{G}$ is completely determined by this algebra. Additionally, set $L[\mathbf{G}] = L[T]/\mathfrak{a}_L$ to be the L–algebra of regular functions on $\mathbf{G}$. The action of $\mathrm{Gal}(L/k)$ on $\mathbf{G}$ given by λ need not be trivial on $k[\mathbf{G}] \subset L[\mathbf{G}]$. The new k–algebra $\lambda(k[\mathbf{G}])$ is denoted by $k[{}_{\lambda}\mathbf{G}]$. Over L, we still have $L[{}_{\lambda}\mathbf{G}] = L[\mathbf{G}]$ and according to Borel [**8**], $\mathbf{G}$ is completely determined as a k–algebraic linear group by the Hopf algebra structure (see [**8**] or [**22**], [**44**]) on $k[\mathbf{G}]$. Thus, ${}_{\lambda}\mathbf{G}$ is the k–algebraic group whose k–algebra of regular functions is $k[{}_{\lambda}\mathbf{G}]$ with its Hopf structure.

For $\mathbf{G}$ be a k–algebraic group and $\mathbf{H} \in F(L/k)$ we associated to $\mathbf{H}$ a 1–cocycle $\beta \in Z^1(\mathrm{Gal}(L/k); \mathrm{Aut}_k(\mathbf{G}))$. To see that this map is surjective, we will use the twisting construction above to construct an L/k–form for a given 1–cocycle. With $G = \mathrm{Gal}(L/k)$, $A = \mathrm{Aut}_k(\mathbf{G})$, and $F = \mathbf{G}$, given a 1–cocycle $\lambda \in Z^1(\mathrm{Gal}(L/k); \mathrm{Aut}_k(\mathbf{G}))$ we obtain a new k–algebraic group ${}_{\lambda}\mathbf{G}$. By construction this is an L/k–form of $\mathbf{G}$ which maps to λ. We refer the reader to [**31**] for the proof that φ is well defined and injective.

5. The classification theorem of F–forms of $\mathrm{SU}(n,1)$

THEOREM 14.4 (Classification of F–forms of $\mathrm{SU}(n,1)$). *Let E/F be a pair of fields with F totally real and E, a totally imaginary quadratic extension of F. Up to F–algebraic isomorphism the F–forms of* $\mathrm{SU}(n,1)$ *are given by triples $(A,*,h)$, where A is a cyclic central E–algebra with an involution of second kind extending the nontrivial Galois involution on E/F, and h is a $*$–Hermitian element of signature* $(n,1)$.

Given the material above, its actually not too difficult to prove this (see [**31**, p.87–88]).

SKETCH OF PROOF. Set $A = \mathrm{M}(n+1;\mathbf{C}) \oplus \mathrm{M}(n+1;\mathbf{C})$ and equip A with the involution $*$ defined by $(X,Y)^* = (Y^T, X^T)$. We can embed $\mathrm{GL}(n;\mathbf{C})$ into A via $X \longmapsto (X, (X^T)^{-1})$. Under this embedding, $\mathrm{SL}(n;\mathbf{C})$ is identified with $X \in A$ such that $XX^* = 1$, which we denote by $\mathbf{SU}$. By the Theorem 7.1, $\mathrm{Aut}(A) = \langle \mathbf{SU}, \tau \rangle$, where $\tau(X,Y) = (Y,X)$. To see this, note that if an automorphism preserves each factor, then by Theorem 7.1 it must be an inner automorphism. Thus, an outer automorphism must be τ, up to an inner automorphism.

Set $\mathbf{G} = \mathbf{SU}$ and let $\lambda \in H^1(\Gamma; \mathrm{Aut}_{\overline{F}}(A))$ be a 1–cocycle which is not contained in $H^1(\Gamma;\mathbf{G})$ and let B be the twisted algebra given by ${}_{\lambda}A$ and let $C = \mathrm{Fix}_{\Gamma}(B)$. It can be shown (see [**31**]) that C admits an involution which

restricts to $Z(C)$ nontrivially. By Theorem 7.3 $C = \mathrm{M}(r;D)$ for some division algebra D of degree d over $Z(C) = E$.
Assume ${}_{\lambda}\mathbf{G}$ is a F–form of $\mathrm{SU}(n,1)$. By Theorem 4.5, the involution on C is induced by some Hermitian element h. Therefore, ${}_{\lambda}\mathbf{G} = \mathrm{SU}(h;D)$ and it follows that $rd = n+1$ since $\mathrm{SU}(h;D) \cong \mathrm{SU}(n,1)$. Thus, $(D,*,h)$ is a unitary algebra with associated pair (r,d), as needed. $\square$

CHAPTER 15

Appendix

1. Basic X–hyperbolic geometry

In this section, we give a basic introduction to complex hyperbolic geometry. We refer the read to [**13**], [**9**], or [**14**].

1.1. X–hyperbolic n–space. For a general reference on this material, see [**9**, II.10]. In all that follows, we let $X = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$ and $\ell = \dim_{\mathbf{R}} X$.

Equip X^{n+1} with a Hermitian form H of signature $(n,1)$. We define *X–hyperbolic n–space* to be the (left) X–projectivization of the H–negative vectors with the Bergmann metric associated to H. We denote X–hyperbolic n–space together with this metric by $\mathbf{H}^n_X$ and say that $\mathbf{H}^n_X$ is *modelled on H* or call H a *model form*.

The *boundary* of $\mathbf{H}^n_X$ in PX^{n+1} is the X–projectivization of the H–null vectors. We denote this set by $\partial \mathbf{H}^n_X$, which is topologically just $S^{\ell n}$ (see [**9**, p. 265]) and call the elements of the boundary *light-like vectors*.

1.2. The isometry group and lattices. The isometry group of $\mathbf{H}^n_X$ is denoted by $\mathrm{Isom}(\mathbf{H}^n_X)$. In each setting, $\mathrm{Isom}(\mathbf{H}^n_X)$ is locally isomorphic to $\mathrm{U}(H)$. Specifically,

$$\mathrm{Isom}(\mathbf{H}^n_X) = \begin{cases} \langle \mathrm{PU}(H)_0, \iota \rangle, & X = \mathbf{R}, \mathbf{C} \\ \mathrm{PU}(H), & X = \mathbf{H}, \end{cases}$$

where ι is an involution induced by inversion in the real case and complex conjugation in the complex case. The usual trichotomy for isometries holds in $\mathrm{Isom}(\mathbf{H}^n_X)$ (see [**34**, p. 180–185], [**13**, p. 203], [**19**]). Specifically, every (nontrivial) isometry is either *elliptic*, *parabolic*, or *loxodromic*.

That finite volume manifolds exist, both compact and noncompact, was established by Borel [**7**].

The spaces constructed in this way yield every locally symmetric space of rank–1 except for those modelled on the exceptional *Cayley hyperbolic plane* $\mathbf{H}^2_{\mathbf{O}}$. We shall only make use of the fact that $\mathrm{Isom}(\mathbf{H}^2_{\mathbf{O}})$ has a faithful linear representation and refer the reader to [**2**] for more on the Cayley hyperbolic plane.

1.3. The trichotomy for isometries. Since every isometry of $\mathbf{H}^n_X$ extends to a continuous mapping of the boundary, by the Browder fixed point theorem, every isometry has a fixed point in $\mathbf{H}^n_X \cup \partial \mathbf{H}^n_X$. We say that γ is *elliptic* if γ has a fixed point in $\mathbf{H}^n_X$. Otherwise, if γ has exactly one fixed point, we say that γ is *parabolic*, while if γ has two fixed points, we say that γ is *loxodromic*. That every isometry is precisely one of the above three is well known (see [**34**], [**13**], and [**19**]).

1.4. Parabolic isometries. Let $v \in X^{n+1}$ be a nonzero H–null vector. As such, we can identify $[v]$, the X–projective class of v, as point of $\partial \mathbf{H}^n_X$. By selecting another nonzero H–null vector v_0 such that $[v] \neq [v_0]$, we can construct parabolic isometries fixing $[v]$ as follows. Let V_∞ be the H–unitary complement of the X–span of v and v_0 and let H_∞ denote the model form H restricted to V_∞. For $x \in V_\infty$, define

$$dx\colon X^{n+1} \longrightarrow X^{n+1}$$

by

$$dx(\cdot) = H(\cdot, v)x - H(\cdot, x)v.$$

Since H is a Hermitian form, the reader can easily verify that $dx \in Mat(X; n+1)$. That is, dx is an endomorphism of X^{n+1}. For any endomorphism A, we define

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

provided this exists. It is a straightforward venture to show that $\exp(dx) \in \mathrm{SU}_0(H)$ and is a parabolic isometry fixing $[v]$. We call any isometry of the form $\exp(dx)$ a *horizontal translation*. For notational brevity, we denote $\exp(dx)$ simply by x.

For $\mu \in \mathrm{Im} X$, define

$$d\mu\colon X^{n+1} \longrightarrow X^{n+1}$$

by

$$d\mu(\cdot)\mu H(\cdot, v)v.$$

As before, $d\mu \in \mathrm{M}(X; n+1)$. As above, $\exp(d\mu) \in \mathrm{SU}_0(H)$ and is a parabolic isometry fixing $[v]$. We call any isometry of the form $\exp(d\mu)$ a *vertical translation*. Again, for notational brevity, we denote $\exp(d\mu)$ simply by μ. Any isometry of the form $x\mu$ (taken as a product of X–matrices or composition of isometries) is called a *pure parabolic isometry*.

To obtain every parabolic isometry from the pure parabolic isometries defined above, we introduce two additional twist factors. Let $S \in \mathrm{U}(H_\infty)$ and define $S \in \mathrm{U}(H)$ by $S(v) = v$ and $S(v_0) = v_0$. When xS is a parabolic isometry, we call xS an *ellipto-parabolic isometry of the first kind*. When $X = \mathbf{R}$

or **C**, it follows (by considering the Iwasawa decomposition) that every parabolic isometry is of this form.

For $X = \mathbb{H}$, we have an additional twist factor which can be introduced. Let $\tau \in X^1$, the set of unit modulus elements. Define

$$\tau \colon X^{n+1} \longrightarrow X^{n+1}$$

by τI. When x is a parabolic isometry, we call $x\tau$ an *ellipto-parabolic isometry of the second kind*. When $X \neq \mathbb{H}$, such isometries as pure.

Finally, we call any isometry of the form $x\tau S$ an *ellipto-parabolic isometry* or *screw parabolic isometry*.

Altogether, we have

$$(x,\mu,\tau,S)(\cdot) = \tau S + \tau H(\cdot,v)x + \tau\left(\mu H(\cdot,v) - H(S(\cdot),x)v - \frac{1}{2}H(x,x)H(\cdot,v)\right)v.$$

1.5. Loxodromic isometries. In this subsection, we give a basis free description of a generic loxodromic isometry.

Let v, v_0 be a pair of H–null vectors. Define

$$dt \colon X^{n+1} \longrightarrow X^{n+1}$$

by

$$dt(\cdot) = t[H(\cdot,v)v_0 + H(\cdot,v_0)v].$$

One can verify that $\exp(dt) \in \mathrm{U}(H)$ and is loxodromic. We call such an isometry a *pure loxodromic isometry*. As in the case of parabolic isometries, we can introduce a twist factor. Let $S \in \mathrm{U}(H_\infty)$ with H_∞ as before. However, we will not consider such isometries here. For a pure loxodromic isometry x, we have:

$$x(\cdot) = I + \frac{H(\cdot,v_0)}{H(v,v_0)}\left(e^{tH(v_0,v)} - 1\right)v + \frac{H(\cdot,v)}{H(v_0,v)}\left(e^{tH(v,v_0)} - 1\right)v_0.$$

Since we will not consider

1.6. Screw boundary elliptic isometries. In §3.2, we described generic parabolic isometries which have an associated triple (ξ,μ,S). As we will only treat the case when S is of finite order, we now make this assumption throughout the remainder of this article. In this case, define

$$L_S = \sum_{j=0}^{|S|-1} S^j.$$

If $\mu = 0$ and $\xi \in \ker L_S$, then the resulting isometry will be elliptic. We call such isometries *screw boundary elliptic isometries*. Most important for us is that the above description for parabolic isometries holds in this setting.

2. Invariant trace fields and algebras

In this section, we define the invariant trace field and algebra for a subgroup of $\mathrm{SU}(2,1)$. This can easily be generalized to subgroups of $\mathrm{SU}(n,1)$, as noted below.

2.1. Invariant trace field. Let Γ be a finitely generated subgroup of $\mathrm{SU}(2,1)$ and let $\Gamma^{(3)}$ be the subgroup generated by all cubes. That is

$$\Gamma^{(3)} = \left\langle \gamma^3 \;:\; \gamma \in \Gamma \right\rangle.$$

For any group $\Lambda < \mathrm{SU}(2,1)$, set

$$\mathrm{Tr}(\Lambda) = \mathbf{Q}(\mathrm{Tr}\,\gamma \;:\; \gamma \in \Lambda).$$

We define the *trace field* for any group Γ to be $\mathrm{Tr}(\Gamma^{(3)})$ ad denote this by $k\Gamma$.

2.2. Invariant algebra. We define the *invariant algebra* of Γ to be

$$A\Gamma = \left\{ \sum_{j=1}^{r} \alpha_j \gamma_j \;:\; \alpha_j \in k\Gamma,\ \gamma_j \in \Gamma^{(3)} \right\}.$$

More generally, set

$$A(\Gamma) = \left\{ \sum_{j=1}^{r} \alpha_j \gamma_j \;:\; \alpha_j \in \mathrm{Tr}(\Gamma),\ \gamma_j \in \Gamma \right\}.$$

Then $A\Gamma = A(\Gamma^{(3)})$.

LEMMA 15.1. *$A\Gamma$ is a central simple algebra of degree three.*

2.3. Commensurability invariance of $k\Gamma$ and $A\Gamma$.

PROPOSITION 15.2. *$k\Gamma$ is a commensurability invariant.*

PROOF. As in the case of lattices in $\mathrm{PSL}(2;\mathbf{R})$ and $\mathrm{PSL}(2;\mathbf{C})$ (see [**24**]), it suffices to check that for any finite index subgroup $\Gamma_1 < \Gamma$, $k\Gamma < \mathrm{Tr}(\Gamma_1)$. Without loss of generality, we may assume that Γ_1 is normal (simply pass to the normal core).

CLAIM. For $\gamma \in \Gamma$, $\gamma^3 \in A(\Gamma_1)$.

PROOF OF CLAIM. Since Γ_1 is normal, conjugation by γ induces an automorphism of Γ_1 and $A(\Gamma_1)$. Of course, $A(\Gamma_1)$ is a simple algebra, therefore by Skolem-Noether, $\mu_\gamma = \mu_\alpha$, for some $\alpha \in A(\Gamma_1)^\times$, where μ_τ denotes conjugation by τ. In $A(\Gamma_1) \otimes_{\mathrm{Tr}(\Gamma_1)} \mathbf{C} \cong \mathrm{M}(3;\mathbf{C})$, $\gamma^{-1}\alpha$ is central and thus a scalar multiple of the identity, say βI_3. Therefore

$$\beta^3 = \det(\gamma^{-1}\alpha) = \det(\gamma^{-1})\det(\alpha) = \det(\alpha).$$

Thus

$$\beta^3 = \det(\alpha) = a^3 - \mathrm{Tr}(a)a^2 + \lambda a \in A(\Gamma_1)$$

since $P_a(t) \in \mathrm{Tr}(\Gamma_1)[t]$ (see [**35**]). However $\gamma^3 = \beta^{-3}\alpha^3 \in A(\Gamma_1)$, as required. □

With this claim, $\Gamma^{(3)} \subset A(\Gamma_1)$. Since the trace of any element $\tau \in A(\Gamma_1)$ is contained in $\mathrm{Tr}(\Gamma_1)$, we see that $\mathrm{Tr}(\Gamma^{(2)}) = k\Gamma \subset \mathrm{Tr}(\Gamma_1)$. □

PROPOSITION 15.3. *$A\Gamma$ is a commensurability invariant.*

PROOF. If Γ_1 and Γ_2 are commensurable, $k\Gamma_1 = k\Gamma_2$ be Proposition 15.2. Since $\Gamma_1^{(3)}$ and $\Gamma_2^{(3)}$ are both lattices, so is $\Gamma_1^{(3)} \cap \Gamma_2^{(3)}$. Moreover, both $A\Gamma_1$ and $A\Gamma_2$ are generated by any pair of non-commuting loxodromic isometries. Selecting such a pair to be in the intersection $\Gamma_1^{(3)} \cap \Gamma_2^{(3)}$ completes the proof. □

REMARK. The solution to the Burnside problem for $n = 3$ implies that $\Gamma^{(3)}$ is a finite index subgroup of Γ. This is false for $n = 4$. One can modify this by adding commutators to this generating set; we simply take our finite index subgroup to be the kernel of the surjective homomorphism of Γ onto the largest possible finite abelian group of exponent n.

REMARK. We can define $k\Gamma$ and $A\Gamma$ using

$$\mathrm{Ad}\colon\ \mathrm{SU}(2,1) \longrightarrow \mathrm{Aut}(\mathfrak{su}(2,1))$$

as done by Vinberg [**40**] (see also [**11**] and [**10**]).

3. Shimura varieties and other arithmetic moduli spaces

In this section, we give a brief introduction to a class of Shimura varieties related to certain arithmetic lattices constructed in this note.

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