Landau-Siegel Zeros and Cusp Forms

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$\oint$ **0.** L**-functions.** For every $m \geq 1$, let $\mathcal{D}_m$ denote the class of *Dirichlet series* $L(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$, absolutely convergent in $Re(s) > 1$ with an Euler product $\prod_p P_p(p^{-s})^{-1}$ of degree m there, extending to whole $s-plane$ as a meromorphic function of bounded order, in fact with no poles anywhere except at $s = 1$, and satisfying (relative to another Dirichlet series $L^\vee(s)$ in $\mathcal{D}_m$) a functional equation of the form

$$L_\infty(s)\, L(s) = W N^{\frac{1}{2}-s} L_\infty^\vee(1-s) L^\vee(1-s),$$

where $W \epsilon\, \mathbb{C}^*$, $N \epsilon \mathbb{Z}_{>0}$, with the "archimedean factor" $L_\infty(s)$ being $(\pi^{\frac{-sm}{2}}) \prod_{j=1}^m \Gamma(\frac{s+bj}{2})$, for sure $(bj)\,\epsilon\, \mathbb{C}^m$.

Put $\mathcal{D} = \bigcup_{m \geq 1} \mathcal{D}_m$. One says that $L(s)$ is self-dual if $L(s) = L^\vee(s)$, in which case $W \epsilon\, \{\pm 1\}$.

Examples:

(0) $\zeta(s)$, Dirichlet and Hecke L-functions.

(1) g a holomorphic newform of weight $k \geq 1$ and level N,

$$L(s) = L(s + \frac{k-1}{2}, g),\ L_\infty(s) = \pi^{-s}\Gamma(\frac{s+(k-1)/2}{2})\Gamma(\frac{s+(k+1)/2}{2}).$$

(2) $\phi =$ Maass form of level N, an eigenfunction of Hecke operators.

$$L(s) = L(s, \phi),\ L_\infty(s) = \pi^{-s}\Gamma(\frac{s+\delta+w}{2})\Gamma(\frac{s+\delta-w}{2}),\ \delta \epsilon \{0,1\}.$$

(3) $L(s) = L(s, \pi_f)$, $\pi = \pi_\infty \otimes \pi_f$ unitary cusp form on GL_m/F, $[F:Q] < \infty$, $N = D_F^m Norm_{F/Q}(cond(\pi))$; generalizes $(0)-(2)$.

(3) (Rankin - Selberg) $L(s) = L(s, \pi_{1,f} \times \pi_{2,f})$, π_j cusp form on $GL(m_j)$.

If

$$L(s, \pi_{1,p}) = \prod_{i=1}^{m_1}(1-\alpha_i p^{-s})^{-1} \ \&\ L(s, \pi_{2,p}) = \prod_{j=1}^{m_2}(1-\beta_j p^{-s})^{-1},$$

then

$$L(s, \pi_{1,p} \times \pi_{2,p}) = \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} (1 - \alpha_i \beta_j p^{-s})^{-1}$$

(4) A typical member in the Selberg class. (Selberg requires Ramanujan, which we don't, while he requires Euler product only in $Re(s) >> 1$)

(5) (Conjectural) $L(s) = L(s, \rho)$ (Artin L-function)

where $\rho : \mathrm{Gal}(\mathbb{Q}/\mathbb{Q}) \to GL_m(\mathbb{C})$ is a continous homomorphism.

$$L(s, \rho) = \prod_p \det(1 - Fr_p t | V_\rho^{I_p})|_{t=p^{-s}}^{-1}, \quad Fr_p : Frobenius, I_p : Inertia$$

Everything known except for holomorphy at $s \neq 1$, which is a big open problem.

(6) (Highly conjectural) M is a motive $\mathbb{Q}$ with coefficients in $E \hookrightarrow \mathbb{C}$ of weight w, $L(s) = L(s + \frac{w}{2}, M)$. (Known cases: CM ab. vars.: Shimura-Taniyama, Elliptic curve $\mathbb{Q}$: Wiles, Breuil-Conrad-Diamond-Taylor).

Hope/Conjecture I: *Every $L(s) \epsilon \mathcal{D}_m$ is quasi-automorphic, ie, there exists an automorphic form π on $GL_m/\mathbb{Q}$ s.t. $L_p(s)_= L(s, \pi_p)$ for almost all p. Moreover, $L(s)$ is primitive iff π is cuspidal.*

– Compatible with Langlands philosophy ([La]) and with the conjecture of Cogdell-Piatetski-Shapiro. ([CoPS])

– This cannot be correctly formulated over number fields. Also, there exists an example of Patterson over function fields $\mathbb{F}_q$ satisfying analogous conditions, but with zeros on the lines $Re(s) = \frac{1}{4}$ and $Re(s) = \frac{3}{4}$.

Hope/Conjecture II: *For any $L(s) \epsilon \mathcal{D}$, if it has a pole of order r at $s = 1$, then $\zeta(s)^r | L(s)$, ie., $L(s) = \zeta(s)^r L_1(s)$, with $L_1(s) \epsilon \mathcal{D}$.*

This is compatible with the conjectures of Selberg, Tate and Langlands.

∮ 1. Landau-Siegel Zeros Let $L(s) \epsilon \mathcal{D}_m$ with $L_\infty(s) = \pi^{-ms} \prod_{j=1}^{m} \Gamma(\frac{s+bj}{2})$. Define its *thickened conductor* to be

$$\tilde{N} = N(1 + \sum_{j=1}^{m} |bj|).$$

Definition. *Let $c > 0$. Then we say that $L(s)$ has a Landau-Siegel zero relative to c if $L(\beta) = 0$ for some $\beta \epsilon (1 - \frac{c}{\log \tilde{N}}, 1)$.*

Definition. *Let $\mathcal{F}$ be a family, ie., a subclass of L-functions in $\mathcal{D}$ with $\tilde{N} \to \infty$ in $\mathcal{F}$. We say that $\mathcal{F}$ admits no Landau-Siegel Zero if there exists an effective constant $c > 0$ such that no $L(s)$ in $\mathcal{F}$ has a zero in $(1 - \frac{c}{\log \tilde{N}}, 1)$.*

Conjecture: *Let $\mathcal{F}$ be a family in $\mathcal{D}$. Then $\mathcal{F}$ admits no Landau-Siegel zero.*

One reason for interest in this is that the lack of such a zero implies a good lower bound for $L(s)$ at $s = 1$.

For example, consider the family $\mathcal{F} = \{L(s, \chi_D) | D\ negative,\ fund.\, discriminant\}$.

$$Dirichlet:\ L(1, \chi_D) = \frac{2\pi h_D}{w_D \sqrt{|d|}},\ where\, w_D = \#\, roots\, of\ 1\ in\, Q(\sqrt{D})$$

and $h_D = \textit{class number}$ of $Q(\sqrt{D})$. If $\mathcal{F}$ has no $L-S$ zero then one can show: $(\forall \varepsilon > 0)$ $L(1, \chi_D| \geq C|D|^{\varepsilon}$, for an effective $C > 0$. Consequently, $h_D \geq C_1 |D|^{\frac{1}{2} - \varepsilon}$. $(*)$

This will solve Gauss's class # problem for imaginary quadratic fields.

Siegel proved $(*)$ with an *ineffective constant* C_2. He was influenced by Landau's earlier work on the problem giving $|D| < C' h_D^8 \log^3(3h_D) \Rightarrow h_D > C_\varepsilon |D|^{\frac{1}{8} - \varepsilon}$, for any $\varepsilon > 0$. The natural limit of Landau's method seems to be $|D|^{\frac{1}{4}}$ (mod logarithmic terms). Gross-Zagier-Goldfeld: $h_D \geq C \log |D|$ for an effective $C > 0$.

Suppose we consider the family $\{L(s, \chi_D)\}$ attached to the set of positive fundamental discriminants D. Then the expression for $L(1, \chi_D)$ involves $h_D R_D$, where $R_D = \log |u_D|$ is the regulator, with u_D denoting a fundamental unit of $Q(\sqrt{D})$. So we get a lower bound for $h_D R_D$ and not for h_D itself. This is related to Gauss's conjecture that $\mathcal{F}$ infinitely many $D > 0$ such that $h_D = 1$.

Very generally, let $L(s)\, (= L(s + \frac{w}{2}, M)$, for a motive M/Q of even weight w. Call $s = 1$ (Deligne) *critical* for $L(s)$ if neither $L_\infty(s)$ nor $L_\infty^\vee(1 - s)$ has a pole at $s = 1$.

- $L(s, \chi_D)$ is critical at $s = 1$ iff $D < 0$.
- If g is a holomorphic new form of wt. $2k + 1$, then $s = 1$ is critical for $L(s) = L(s + \frac{k-1}{2}, g)$
- If g is a holomorphic new form of wt. 2, then $s = 1$ is critical for $sym^{2r}(g)$, for any $r \geq 1$.
- if ϕ is a Maass form of weight o associated to an even Galois representation ρ; λ will be $\frac{1}{4}$.

When M is critical with coefficients in Q, Deligne's conjecture predicts

$$L(1, M) \,\epsilon\, (period\, of\, M)\, \mathbb{Q}^*.$$

The transendental part of the period of M should be constant in the family $\{M \otimes \chi_D | D > 0\}$.

Bloch-Kato conjectures: describe the rational #'s; they invlove a generalized class #'s, order of $Sha(M)$.

Upshot: If $\{M \otimes \chi_D\}$ has no Landau-Siegel zeros, and if the B-K conjectures hold , then one should have a good lower bound for the order of $Sha(M_D)$, as $D \to \infty$.

Another reason for being interested in $L(1)$ is the following:

Let π be a cusp form on GL_n/Q. Then $L(s, \pi \times \overset{\vee}{\pi})$ has a simple pole at $s = 1$ and its residue there is essentially given by (φ, φ) for a new vector $\varphi \epsilon V_\pi$. On the other hand, $L(s, \pi \times \overset{\vee}{\pi}) = \zeta(s) L(s, \pi, Ad)$; so one gets control of $\|\varphi\|$ by bounding $L(1, \pi, Ad)$ from below. For $n = 2$ and π self-dual, $L\,(1, \pi, Ad)$ is simply $L\,(1, sym^2(\pi))$.

∮ 3. Known Cases

(0) (Siegel, –) Fix $R > 0$, and consider $\mathcal{F}_R = \{L(S, \chi_D) |\, D < 0, |D| \leq R\}$. Then there exists an effective $C > 0$ such that there is at most one $\chi_D \,\epsilon\, \mathcal{F}_R$ which has a zero in $(1 - \frac{c}{\log_R}, 1)$.

(1) (Stark) Let F be a finite Galois extension of $\mathbb{Q}$. Then the Dedekind zeta function $\zeta_F(s)$ has no Landau-Siegel zero unless F contains a quadratic extension of $\mathbb{Q}$. This holds for non-normal extensions as well if one assumes the Artin conjecture.

(2) (Goldfeld-Hoffstein-Lockhart-Lieman) Let $\mathcal{F} = \{L(s, sym^2(\pi_f))\} | \pi$ non-dihedral cusp form on $GL_2\mathbb{Q}\}$. Then $\mathcal{F}$ admits no Landau-Siegel zero.

(3) (Hoffstein-Ramakrishnan)

(i) Let $\mathcal{F} = \{L(s, \pi_f) | \pi$ cusp form on $GL_2/F\}$. Then $\mathcal{F}$ admits no $L - S$ zero.

(ii) Analog of (0) holds for $\{L(s, \pi) | \pi$ cusp form on GL_m, all m.$\}$

(iii) Assume the Rankin-Selberg L-functions are modular. Then the analog of (i) holds for GL_m, for any $m \geq 1$.

Remark: *For $GL(3)$, the paper [HR] gave a reduction, which was established by W. Banks to give a non-existence result for $L - S$ zeros (for $GL(3)$) ([Ba]).*

∮ 4. Idea of proof of the results of [HR]:

The starting point is the following general principle, well known to experts in the area:

LEMMA *Let $L(s)\,\epsilon\,\mathcal{D}_m$ is a positive Dirichlet series having a pole of order $r \geq 1$ at $s = 1$, with $L'(s)/L(s) < 0$ for $s\epsilon(1, \infty)$. Then there exists an effective constant $C > 0$, depending only on m and r, such that $L(s)$ has most r real zeros in $(1 - \frac{C}{\log \tilde{N}}, 1)$.*

So, given $L_1(s)\,\epsilon\,\mathcal{D}$, one can rule out L-S zeros for $L_1(s)$ if we can find a positive $L(s)$ with order of pole $r \geq 1$ such that $L(s) = L_1(s)^k L_2(s)$, **with** $k > r$ and $L_2(s)$ holo in $(t, 1)$ for a fixed t.

LEMMA *$L(s, \pi \times \pi^\vee)$ is a positive Dirichlet series, $\forall$ auto form π, with $L'(s)/L(s) < 0$ for real $s\,\epsilon\,(0, \infty)$.*

[GHLL] case: π self-dual, non-dihedral cusp form on $GL(2)$, $S^2 = sym^2$;

$$L(s) = \zeta(s)L(s, S^2(\pi))^3 L(s, S^2(S^2(\pi))),\ r = 2.$$

[HR] case: π self-dual, non-dihedral cusp form on $GL(2)$;

$$L(s) = \zeta(s)^2 L(s, \pi)^4 L(s, S^2/\pi))^3 L(s, S^3(\pi)) L(s, S^2(\pi) \times S^2(\pi)),\ r = 3.$$

Crucial fact: *$L(s, sym^3(\pi))$ is holomorphic in $(\frac{3}{4}, 1)$ which had been known at the time by Bump-Ginzburg-Hoffstein ([BGH]). Now one can also appeal to Kim-Shahidi who prove that $L(s, sym^3(\pi))$ is even entire ([KSh]).*

General principle: *(in terms of Galois representations of Artin type)*

Let $\rho\,\epsilon\,\mathrm{Hom}_{cont}(\mathrm{Gal}, GL_n(\mathbb{C})$ be irreducible. We will assume ρ is self-dual; the general case can be reduced to this.

Elementary, but key point: If $n > 1$, $\exists$ irreducible $\tau \neq 1$ of Gal *occuring in $\rho \otimes \rho$.*

Case i *Can take $\tau = \rho$. (Notation: $\underline{1}$ = trivial representation)*

Write

$$\rho^{\otimes 2} = \underline{1} \oplus k\rho \oplus \rho',\ \ \rho \not\subset \rho',\ \ k \geq 1.$$

Then

$$\eta := (1 + \rho)^{\otimes 2} = 2.\underline{1} \oplus (2 + k)\rho \oplus \rho',\ \ k \geq 1.$$

$$\Rightarrow\ \ L(s, \eta) = \zeta(s)^2 L(s, \rho)^{2+k}\ L(s, \rho').$$

Done if $L(s,\rho')$ is holomorphic to the left of $s=1$. (This corresponds to the [GHLL] -case).

Case ii $\tau \neq \rho$.

Write

$$\rho^{\otimes 2} = \begin{cases} 1 \oplus k\tau \oplus k\tau^\vee \oplus \tau', \, if \ \tau \neq \tau^\vee \\ 1 \oplus l\tau \oplus \tau', if \, \tau \neq \tau^\vee, \end{cases}$$

$$with\ k,\, l \geq 1,\, \tau,\, \tau^\vee \not\subset \tau'.$$

Then

$$\eta := (1 \oplus \rho \oplus \tau)^{\otimes 2} = 3.\underline{1} \oplus 2\rho \oplus \tau'' \oplus (\rho \otimes \tau^\vee) \oplus (\rho \otimes \tau),$$

for some τ''. Note that ρ occurs in $\rho \otimes \tau^\vee$.

$$Hence\ L(s,\eta) = \zeta(s)^3 L(s,\rho)^4 L(s,\rho').$$

Done if $L(s,\rho')$ is holo. near 1.

(This corresponds to Hoffstein-Ramakrishnan case.)

Suppose we assume the functoriality principle of Langlands ([La]). More specifically, suppose we assume that the Rankin-Selberg L-functions $L(s,\pi \times \pi')$ on $GL(n) \times GL(m)$ are modular, ie., given as the standard L-series $L(s,\pi \boxtimes \pi')$ for automorphic representations $\pi \boxtimes \pi'$ of $GL(nm)$. For $n=m=2$, this is carried out in [Ra]. Then we can transport this Galois argument over to the automorphic side and conclude that for any cusp form π on $GL(n)$, $n>1$, $L(s,\pi)$ admits no Landau-Siegel zero.

But such a result can be proved without any hypothesis for $L(s,\pi)$ ([HR]) and $L(s,S^2(\pi))$ ([GHLL]) for any cusp form π on $GL(2)$. We refer to [HR] for further details.

Selected References

[Ba] W. Banks, *Twisted symmetric square L-functions and the non-existence of Siegel zeros on $GL(3)$*, Duke Math Journal, **87** (1997), no. 2, 343–353.

[BGH] D. Bump, D. Ginzburg and J. Hoffstein, *The symmetric cube*, Invent. Math. **125** (1996), no. 3, 413–449.

[CoPS] J. Cogdell and I. Piatetski-Shapiro, *Converse Theorems for GL_n*, Publications Math. IMES **79** (1994), 157–214.

[GHLL] D. Goldfeld, J. Hoffstein and D. Liemann, *An effective zero free region*, appendix to: J. Hoffstein, P. Lockhart, *Coefficients of Maass forms and the Siegel zero*, Ann. Math. (2) **140** (1994), 161–181.

[HR] J. Hoffstein and D. Ramakrishnan, *Siegel zeros and cusp forms*, IMRN (1995), no. 6, 279–308.

[KSh] H. Kim and F. Shahidi, *Symmetric cube L-functions of GL_2 are entire,* Annals of Math. (to appear)

[La] R. P. Langlands, *Automorphic Representations, Shimura varieties and motives.* Ein Marchen, Proc. Symp, Pure Math. **33** (1979), part 2, 205–246.

[L] E. Landau, *Bemer Rungen Zum Heilbronnschen Saltz*, Acta Arithmetica **1** (1936), 1–18.

[Ra] D. Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and Multiplicity one for* $SL(2)$, to appear in Annals of Math.

[S] C. L. Siegel, *Über die Classenzahl quadratischer Zahlkörper*, Acta Arithmetica **1**, (1936), 83–86.

[St] H. M. Stark, *Some effective cases of the Brauer-Siegel theorem*, Invent. math. **23** (1974), 135–152.