# MODULARITY OF SOLVABLE ARTIN REPRESENTATIONS OF GO(4)-TYPE

DINAKAR RAMAKRISHNAN

*To my teacher, Hervé Jacquet*

## Introduction

Let $F$ be a number field, and $(\rho, V)$ a continuous, $n$-dimensional representation of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ on a finite-dimensional $\mathbb{C}$-vector space $V$. Denote by $L(s,\rho)$ the associated $L$-function, which is known to be meromorphic with a functional equation. Artin's conjecture predicts that $L(s,\rho)$ is holomorphic everywhere except possibly at $s = 1$, where its order of pole is the multiplicity of the trivial representation in $V$. The *modularity conjecture* of Langlands for such representations ([La3]), often called the *strong Artin conjecture*, asserts that there should be an associated (isobaric) automorphic form $\pi = \pi_\infty \otimes \pi_f$ on $\mathrm{GL}(n)/F$ such that $L(s,\rho) = L(s,\pi_f)$. Since $L(s,\pi_f)$ possesses the requisite properties ([JS]), the modularity conjecture implies the Artin conjecture.

For any field $k$, let $\mathrm{GO}(n,k)$ denote the subgroup of $\mathrm{G}L(n,k)$ consisting of *orthogonal similitudes*, i.e., matrices $g$ such that ${}^tgg = \lambda_g I$, with $\lambda_g \in k^*$.

We will say that a $k$-representation $(\rho, V)$ is *of GO(n)-type* iff $\dim(V) = n$ and it factors as

$$\rho = [\mathrm{G}al(\overline{F}/F) \xrightarrow{\sigma} \mathrm{GO}(n,k) \subset \mathrm{GL}(V)].$$

In this article we prove

**Theorem A** *Let $F$ be a number field and let $(\rho, V)$ be a continuous, 4-dimensional $\mathbb{C}$-representation of $Gal(\overline{F}/F)$ whose image is solvable and lies in $GO(4,\mathbb{C})$. Then $\rho$ is modular, i.e., $L(s,\rho) = L(s,\pi_f)$ for some isobaric automorphic representation $\pi = \pi_\infty \otimes \pi_f$ of $GL(4,\mathbb{A}_F)$. Moreover, $\pi$ is cuspidal iff $\rho$ is irreducible.*

Among the finite groups $G$ showing up as the images of such $\rho$ are those fitting into an exact sequence

$$1 \to C \to H \times H \to G \to \{\pm 1\} \to 1,$$

where $H$ is any finite solvable group in $\mathrm{GL}(2,\mathbb{C})$ with scalar subgroup $C$, the embeding of $C$ in $H \times H$ is given by $x \to (x, x^{-1})$, and the action of $\{\pm 1\}$ on $(H \times H)/C$ is induced by the permutation of the two factors of $H \times H$. This is due to the well known fact (see section 1) that $\mathrm{GO}(4,\mathbb{C})$ contains a subgroup of index 2 which is a quotient of $\mathrm{GL}(2,\mathbb{C})\times$ $\mathrm{GL}(2,\mathbb{C})$ by $\mathbb{C}^*$. Of particular interest are the examples where $H$ is a central extension of $S_4$ or $A_4$ (cf. section 8).

PARTIALLY SUPPORTED BY THE NSF GRANT DMS-9801328

One can ask if this helps furnish new examples of Artin's conjecture, and the answer is yes.

**Corollary B** *Let $F$ be a number field, and let $\rho, \rho'$ be continuous $\mathbb{C}$-representations of Gal$(\overline{F}/F)$ of solvable GO(4)-type. Then Artin's conjecture holds for $\rho \otimes \rho'$.*

We will show in section 8 that in fact there is, for each $F$, a doubly *infinite* family of such examples where the representations $\rho \otimes \rho'$ are *irreducible* and *primitive* (i.e., not induced) of dimension 16. Primitivity is important because Artin $L$-functions are inductive, and one wants to make sure that these examples do *not* come by induction from known (solvable) cases in low dimensions. We will then show (in secton 9) that, given any $\rho$ as in Theorem A with corresponding extension $K/F$, the *strong Dedekind conjecture* holds for certain non-normal extensions $N/F$ contained in $K/F$, for instance when $[N : F] = 3^a$, $a \geq 1$ (see Theorem 9.3); the assertion is that the ratio $\zeta_N(s)/\zeta_F(s)$ is the standard $L$-function of an isobaric automorphic form $\eta$ on $\mathrm{GL}(3^a - 1)/F$. It implies that for any cusp form $\pi$ on $\mathrm{GL}(n)/F$, the formally defined Euler product $L(s, \pi_N)$ (see the discussion before *Corollary 9.4*) admits a meromorphic continuation and functional equation, and more importantly, it is divisible by $L(s, \pi)$. It should be noted that we do not know if $\pi_N$ is automorphic. For a curious consequence of this result see Remark 9.6.

It has been known for a long time, thanks to the results of Artin and Hecke, that monomial representations of $\mathrm{Gal}(\overline{F}/F)$, i.e., those induced by one-dimensional representations of $\mathrm{Gal}(\overline{F}/K)$ with $K/F$ finite, satisfy Artin's conjecture. (In fact this holds for any multiple of a monomial representation.) But the strong Artin conjecture is still open for these except when $K/F$ is normal and solvable ([AC]) and when $[K : \mathbb{Q}] = 3$ ([JPSS1]. The work [AC] of Arthur and Clozel implies that the strong conjecture holds for $\rho$ with nilpotent image, in fact whenever $\rho$ is *accessible* ([C]), i.e., a positive integral linear combination of representations induced from linear characters of open, subnormal subgroups. It should however be noted that Dade has shown ([Da]) that all accessible representations of solvable groups are monomial.

The odd dimensional orthogonal representations with solvable image are simpler than the even dimensional ones. Indeed we have

**Proposition C** *Let $\rho$ be a continuous, irreducible, solvable $\mathbb{C}$-representation of Gal$(\overline{F}/F)$ of GO$(n)$-type with $n$ odd. Then $\rho$ is monomial and hence satisfies the Artin conjecture. If $\rho$ is in addition self-dual, it must be induced by a quadratic character.*

By the groundbreaking work of Langlands in the seventies ([La1]), and the ensuing theorem of Tunnell in 1980 ([Tu]), one knows that the strong Artin conjecture holds for all two-dimensional representations $\sigma$ with solvable image. The *raison d'etre* for this article is the desire to find irreducible, *solvable*, but non-monomial, even primitive, examples in higher dimensions satisfying the Artin cojecture. We were led to this problem a few years ago by a remark of J.-P.Serre. One could look at the symmetric powers of a two-dimensional *solvable* $\sigma$, but they are all in the linear span of monomial representations and 1-dimensional twists of $\sigma$. The facts that the symmetric square of $\sigma$ is modular (by Gelbart-Jacquet [GeJ]) and monomial (for $\sigma$ solvable) were used crucially in [La1] and [Tu].

In the *non-solvable* direction, which is orthogonal (no pun intended) to the one pursued in this paper, there has been some spectacular progress recently. For *odd* 2-dimensionals $\rho$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with projectivization $\overline{\rho}$ of $A_5$-type, a theorem of Buzzard, Dickinson, Shepherd-Barron and Taylor ([BDST]) establishes the modularity conjecture assuming the following: (i) $\overline{\rho}$ is unramified at 2 and 5, and (ii) $\overline{\rho}(\mathrm{Frob}_2)$ has order 3. In a sequel ([T]), this is shown with different ramification conditions, namely that under $\overline{\rho}$ (i') the inertia group at 3 has odd order, and (ii') the decomposition group at 5 is unramified at 2. See also [BS] for some explicit examples. Some positive examples were given earlier in [Bu], and then in [Fr]. Moreover, the very recent theorems of Kim and Shahidi ([KSh]), and Kim ([K]), establishing the automorphy of the symmetric cube, and the symmetric fourth power, on GL(2), establishes the strong Artin conjecture for $\mathrm{sym}^3(\sigma)$, and $\mathrm{sym}^4(\sigma)$, for all the $\sigma$ proved modular by [BDST].

If $\sigma, \sigma'$ are Galois representations which are modular, then one can deduce the Artin conjecture for $\sigma \otimes \sigma'$ by applying the Rankin-Selberg theory on $\mathrm{GL}(m) \times \mathrm{GL}(n)$ developed in the works of Jacquet, Piatetski-Shapiro, Shalika ([JPSS2], [JS]) and of Shahidi ([Sh1,2]); see also [MW]. This explains why the strong form of Artin's conjecture is really a bit stronger than the original conjecture that Artin made, at least given what one knows today.

If $\sigma, \sigma'$ are 2-dimensional representations which are modular, then the strong Artin conjecture for $\sigma \otimes \sigma'$ follows from the main theorem of [Ra1], hence the Artin conjecture holds, by the remark above, for 4-fold tensor products of such representations. Now let $K/F$ be a quadratic extension with non-trivial automorphism $\theta$, and let $\sigma^\theta$ denote the $\theta$-twisted representation defined by $x \to \sigma(\tilde{\theta} x \tilde{\theta}^{-1})$, where $\tilde{\theta}$ is any lift to $\mathrm{Gal}(\overline{F}/F)$. (The equivalence class of $\sigma^\theta$ is independent of the choice of $\tilde{\theta}$.) Given any irreducible, non-monomial 2-dimensional representation $\sigma$ of $\mathrm{Gal}(\overline{F}/K)$ which is not isomorphic to any one-dimensional twist of $\sigma^\theta$ (see section 3), there is an irreducible 4-dimensional representation $As(\sigma)$ of $\mathrm{Gal}(\overline{F}/F)$ whose restriction to $\mathrm{Gal}(\overline{F}/K)$ is isomorphic to $\sigma \otimes \sigma^\theta$. When $\sigma$ is of solvable type, one can combine the theorem of Langlands-Tunnell with that of Asai ([HLR]) to deduce the Artin conjecture for $As(\sigma)$.

One of the main steps in our proof of Theorem A is that even modularity can be established for any Asai representation $As(\sigma)$ (in the solvable case). To begin, there exists, by Langlands-Tunnell, a cusp form $\pi$ on $\mathrm{GL}(2)/K$ such that $L(s, \sigma) = L(s, \pi)$. It follows that $L(s, \sigma \otimes \sigma^\theta)$ equals the Rankin-Selberg $L$-function $L(s, \pi \times (\pi \circ \theta))$. By [Ra1], there exists an automorphic form $\pi \boxtimes (\pi \circ \theta)$ on $\mathrm{GL}(4)/K$ such that $L(s, \sigma \otimes \sigma^\theta) = L(s, \pi \boxtimes (\pi \circ \theta))$. Since $\pi \boxtimes (\pi \circ \theta)$ is $\theta$-invariant, one can now deduce the existence of a cusp form $\Pi$ on $\mathrm{GL}(4)/F$ whose base change to $K$ is $\pi \boxtimes (\pi \circ \theta)$, but even when $\pi \boxtimes (\pi \circ \theta)$ is cuspidal, $\Pi$ is unique only up to twisting by the quadratic character $\delta$, say, of the idele class group of $F$ corresponding to $K/F$ by class field theory. But it is not at all clear that $\Pi$, or $\Pi \otimes \delta$, should correspond precisely to $As(\sigma)$, with an identity of the corresponding $L$-functions. (It is easy to see that the local factors agree at half the places.) Put another way, one can construct an irreducible admissible representation $As(\pi)$ of $\mathrm{GL}(4, \mathbb{A}_F)$ which has the same local factors as does $As(\sigma)$. But the problem is that there is no simple reason why $As(\pi)$ should be automorphic, *even* when $\pi$ is dihedral. (Recall that $\pi$ is dihedral, or of CM type, iff it is associated to a representation $\sigma$ of the global Weil group $W_K$ induced by a character $\chi$ of a quadratic extension $M$ of $K$, which we denote by $\mathrm{Ind}_M^K(\chi)$.) Anyhow we manage to solve this problem and establish the following result, where $\mathrm{Res}_M^K$ denotes, for any extension $M/K$ of number fields, the restriction of representations of $W_K$ to $W_M$:

**Theorem D** *Let $K/F$ be a quadratic extension of number fields with non-trivial automorphism $\theta$, and let $\pi$ be a cuspidal automorphic representation of $GL(2,\mathbb{A}_K)$. Then*

(a) *$As(\pi)$ is automorphic;*
(b) *If $\pi$ is non-dihedral, then $As(\pi)$ is a cusp form iff $\pi\circ\theta$ is not isomorphic to $\pi\otimes\mu$, for any idele class character $\mu$ of $K$;*
(c) *If $\pi$ is dihedral, i.e., associated to a representation $\sigma = \mathrm{Ind}_M^K(\chi)$ of $W_K$ for a character $\chi$ of a quadratic extension $M$ of $K$, then $As(\pi)$ is cuspidal iff $M/F$ is non-Galois and the representation $\mathrm{Res}_M^K(\sigma^\theta)\otimes\chi$ of $W_M$ does not extend to a representation of $W_F$.*

It may be helpful for the reader to note the following concrete description of the *Asai L-function* in a special case. Suppose $F=\mathbb{Q}$, $K$ a real quadratic field of class number 1 with ring of integers $\mathfrak{O}_K$, and $\pi$ the representation defined by a holomorphic Hilbert modular newform $f$ of weight 2 with

$$L(s,\pi) \,=\, \sum_{\mathfrak{a}}{}' \, c(\mathfrak{a})N(\mathfrak{a})^{-s}.$$

Then we have

$$L(s,As(\pi)) \,=\, \zeta(2s-2)\sum_{m\geq 1} c(m\mathfrak{O}_K)m^{-s}.$$

The interest in this comes, on the analytic side, from the fact that one sums over a sparse subset of the non-zero ideals $\mathfrak{a}$ of $\mathfrak{O}_K$ to get this $L$-function, and on the geometric side, from the fact that $L(s,As(\pi))$ is a factor of the degree 2 $L$-function of the associated Hilbert modular surface (cf. [Ra2], sec. 4, for example). It will be natural if one is reminded of the symmetric square $L$-function of a cusp form on GL(2)/$\mathbb{Q}$, where effectively one sums over the squares of positive integers. Indeed, for any quadratic extension $K/F$ of number fields and for any cusp form $\pi$ on GL(2)/$K$, if $\pi$ is the base change of a form $\pi_0$ on GL(2)/$F$, $L(s,As(\pi))$ factors as $L(s,\mathrm{sym}^2(\pi_0)\otimes\delta)L(s,\omega_0\delta)$, where $\omega_0$ is the central character of $\pi_0$.

We will now make some remarks about the proofs of these results.

In section 3 we show how to reduce the proof of Theorem A to the following

**Theorem A′** *Fix a quadratic extension $K/F$ of number fields with associated quadratic character $\delta$ of Gal($\overline{F}/F$). Let $\rho$ be an irreducible 4-dimensional, solvable $\mathbb{C}$-representation of Gal($\overline{F}/F$) whose restriction $\rho_K$ to Gal($\overline{F}/K$) is a tensor product of two 2-dimensional representations. Then*

(a) *$\rho$ is modular, i.e., there is a cuspidal automorphic representation $\Pi$ of $GL(4,\mathbb{A}_F)$ such that $L(s,\rho)=L(s,\Pi_f)$;*
(b) *If $\rho_K$ is irreducible, one of the following happens:*
 (i) *$\rho\simeq\tau\otimes\tau'$ over $F$, with dim($\tau$) =dim($\tau'$) = 2;*
 (ii) *$\rho\simeq \mathrm{Ind}_L^F(\eta)$, with $L/F$ quadratic, $L\neq K$, and $\eta$ a 2-dimensional representation of Gal($\overline{F}/L$);*
 (iii) *$\rho\simeq As(\sigma)\otimes\beta$, with $\sigma$ a 2-dimensional representation of Gal($\overline{F}/K$) and $\beta$ a character of Gal($\overline{F}/F$).*

Then we show, still in section 3, how to deduce Theorem A$'$ modulo Theorem D; in fact we need Theorem D only in the (crucial) case when (iii) occurs. When (i) or (ii) occurs, the modularity already follows from Theorem M of [Ra1] and base change ([AC]), with the desired $\Pi$ being in case (i) (resp. (ii)) the Rankin-Selberg product $\pi(\tau) \otimes \pi(\tau')$ (resp. the automorphic induction $I_L^F(\pi(\eta))$); here $\tau \to \pi(\tau)$ the Langlands-Tunnell map on solvable 2-dimensional Galois representations.

The proof of Theorem D is accomplished in sections 4 through 7. The approach is similar to, but somewhat more subtle than, the proof of the existence of $\boxtimes$ : $\mathcal{A}(\mathrm{GL}(2)/F) \times \mathcal{A}(\mathrm{GL}(2)/F) \to \mathcal{A}(\mathrm{GL}(4)/F)$ in [Ra1]. (For any $n \geq 1$, $\mathcal{A}(\mathrm{GL}(n)/F)$ denotes the set of isomorphism classes of isobaric automorphic representations of $\mathrm{GL}(n, \mathbb{A}_F)$.) In section 4 we show why it suffices to have the requisite properties at almost all places. Then in section 5 the *distinguished* case, i.e., when $\pi \circ \theta$ is an abelian twist of $\pi$, is treated separately. In the *general* situation, i.e., when $\pi \boxtimes (\pi \circ \theta)$ is cuspidal, we crucially use the converse theorem for GL(4) due to Cogdell and Piatetski-Shapiro ([CoPS1], which requires knowledge of the *niceness* of the twisted $L$-functions $L(s, As(\pi) \times \pi')$ for automorphic forms $\pi'$ of $\mathrm{GL}(2)/F$ for a suitable class of $\pi'$. Many properties of certain closely related functions, to be denoted $L_1(s, As(\pi) \times \pi')$, were established by Piatetski-Shapiro and Rallis ([PS-R]), and by Ikeda ([Ik1,2]), via an integral representation, which we use. There is another possible approach to studying $L(s, As(\pi) \times \pi')$ via the Langlands-Shahidi method [Sh1], which yields another family of closely related $L$-functions, denoted $L_2(s, As(\pi) \times \pi')$, with boundedness properties established recently in [GeSh], but we will not use it and our argument here is hewed to make use of the integral representation.

For almost all finite places $v$, the local factors $L(s, As(\pi_v) \times \pi'_v)$ and $L_1(s, As(\pi_v) \times \pi'_v)$ agree. But a thorny problem arises however, due to our inability to identify the bad and archimedean factors. In fact, when $F_v = \mathbb{R}$, one does not even have a computation of the corresponding $L_1$-factor when $\pi, \pi'$ are unramified. Luckily, things simplify quite a bit under suitable, solvable base changes $K/F$ with $K$ *totally complex*, and after constructing the base-changed candidates $As(\pi)_K$ for an infinitude of such $K$, we descend to $F$ as in sections 3.6, 3.7 of [Ra1]. We also have to control the intersection of the ramification loci of $As(\pi)$, $\pi'$ and $K/F$.

One of the reasons why we have to work with $L(s, As(\pi) \times \pi')$ is that we know how its local $\varepsilon$-factors behave under twisting by a highly ramified character, and this is not apriori the case with $L_j(s, As(\pi) \times \pi')$ for $j = 1$ or 2. Indeed this will present a difficulty for the lifting of (generic) automorphic forms from $\mathrm{GO}(2n)$ to $\mathrm{GL}(2n)$, and for large $n$ one cannot, as of yet, make use of base change and descent as we do here. For the lifting from $\mathrm{GO}(2n+1)$ to $\mathrm{GL}(2n)$, see [CoKPSSh].

We would like to thank R.P. Langlands and the Institute for Advanced Study, Princeton, for their hospitality during the year 1999-2000, and the American Institute of Mathematics, Palo Alto, for support during the month of August in 1999. This project was partially funded by the NSF and AMIAS. We also want to acknowledge with thanks some helpful conversations with Michael Aschbacher and David Wales concerning the representations of finite groups. The main result here was presented at a conference on Automorphic Forms in Luminy, France (May 1999), and then at the IAS, Princeton (April 2000), and ICTP, Trieste, Italy (August 2000), where we received some helpful feedback. We would like to thank a number of people, including Jim Arthur, Don Blasius, Kevin Buzzard, Jim Cogdell, Bill Duke, Farshid Hajir, Henry Kim, Jeff Lagarias, Nick Katz, Dipendra Prasad, A.Raghuram, Peter Sarnak, Jean-Pierre Serre, Freydoon Shahidi and Joe Shalika who have shown interest and/or made comments.

## 1. **Preliminaries on orthogonal similitude groups**

Here we collect some basic facts, which we will need.

Let $k$ be a field of characteristic different from 2, with separable algebraic closure $\overline{k}$. If $V$ is a finite dimensional vector space with a non-degenerate, symmetric bilinear form $B$, the associated orthogonal similitude group is

(1.1)
$$\mathrm{G}O(V,B) := \{g \in \mathrm{G}L(V) \,|\, B(gv,gw) = \lambda(g)B(v,w), \text{ with } \lambda(g) \in k^*, \forall v,w \in V\}.$$

The character $\lambda : \mathrm{G}O(V,B) \to k^*$, $g \to \lambda(g)$, is the *similitude factor*. The kernel of $\lambda$ is the *orthogonal group* $\mathrm{O}(V,B)$, whose elements necessarily have determinant $\pm 1$, and the kernel of det is the *special orthogonal group* $\mathrm{SO}(V,B)$.

If $V = k^n$ with $B$ the standard bilinear form $B_0 : (v,w) \to {}^tvw$, then one writes $\mathrm{GO}(n,k)$, $\mathrm{O}(n,k)$ and $\mathrm{SO}(n,k)$ instead of $\mathrm{GO}(V,B)$, $\mathrm{O}(V,B)$ and $\mathrm{SO}(V,B)$. Denote by $Z_n(k)$ the *center* of $\mathrm{GO}(n,k)$ consisting of all the scalar matrices $cI_n$, $c \in k^*$. Clearly, $\lambda(cI_n) = c^2$, so that $k^{*2}$ is in the image of $\lambda$. The *odd dimensional case* is relatively simple. One has

**Lemma 1.2** *If $n$ is odd and $k = \overline{k}$, then we have the direct product decomposition*

$$GO(n,k) = SO(n,k) \times Z_n(k).$$

Indeed, as $k = \overline{k}$, $\lambda(Z_n(k))$ is all of $k^*$, and since $\mathrm{O}(n,k)$ is by definition the kernel of $\lambda$, $\mathrm{GO}(n,k)$ is generated by the normal subgroups $\mathrm{O}(n,k)$ and $Z_n(k)$. On the other hand, the intersection of these two groups is simply $\{\pm I_n\}$. Since $n$ is odd, the image of *det* : $\mathrm{O}(n,k) \to \{\pm 1\}$ is the same as that of $\{\pm I_n\}$. The assertion follows.

Note that $\mathrm{GO}(1,k) = Z_1(k) = k^*$. There is a useful description in the $n = 3$ case, which we will now recall. The *adjoint representation*

$$\mathrm{A}d : \mathrm{P}GL(2,k) \to \mathrm{G}L(3,k)$$

is irreducible and self-dual with determinant 1. This identifies the image of Ad with $\mathrm{SO}(3,k)$, thanks to the simplicity of the latter. By abuse of notation, we will also write Ad for its composition with the canonical map from $\mathrm{GL}(2,k)$ onto $\mathrm{PGL}(2,k)$. This gives rise to the short exact sequence

$$1 \to k^* \to \mathrm{G}L(2,k) \to \mathrm{SO}(3,k) \to 1, \tag{1.3}$$

where the maps in the middle are $c \to cI_2$ and $g \to \mathrm{A}d(g)$.

The *even dimensional case* $n = 2m$ is more interesting. Since for any $g$ in $\mathrm{GO}(2m,k)$, the square of its determinant is $\lambda(g)^{2m}$, we can define a homomorphism, called the *similitude norm*

$$\nu : GO(2m,k) \to \{\pm 1\}, \tag{1.4}$$

by sending $g$ to $\lambda(g)^{-m}\mathrm{d}et(g)$.

The kernel of $\nu$, denoted $\mathrm{SGO}(2m,k)$, is called the *special orthogonal similitude group*. (Some people write $\mathrm{GSO}(2m,k)$ instead.) The map $\nu$ does not split.

Since $\nu$ is just the determinant map on $\mathrm{O}(2m,k)$, the intersection of $\mathrm{SGO}(2m,k)$ with $\mathrm{O}(2m,k)$ is $\mathrm{SO}(2m,k)$. When $k=\mathbb{C}$, $\mathrm{SGO}(2m,k)$ (resp. $\mathrm{SO}(2m,k)$) is the connected component of $\mathrm{GO}(2m,k)$ (resp. $\mathrm{O}(2m,k)$). $\mathrm{SGO}(2,k)$ is a torus.

Note that $\nu(cI_{2m})$ is 1, and that $\mathrm{SGO}(2m,k)$ is generated by $\mathrm{SO}(2m,k)$ and $Z_{2m}(k)$; but their intersection is $\{\pm I_{2m}\}$.

We will conclude this section by recalling a low dimesional isomorphism for $k=\overline{k}$, which we will need, between $\mathrm{SGO}(4,k)$ and a quotient of $\mathrm{GL}(2,k)\times \mathrm{G}L(2,k)$.

Let $W$ be $k^2$ with the standard symplectic form given by the determinant. Then the induced bilinear form $B$ on the tensor product $W\otimes W$ is non-degenerate and symmetric. There is an isometry between $(W\otimes W,B)$ and $(k^4,B_0)$. Since $\mathrm{GL}(2,k)$ is the symplectic similitude group of $(W,\mathrm{d}et)$, we get an exact sequence

$$(1.5)\qquad 1\to k^*\to \mathrm{G}L(2,k)\times \mathrm{G}L(2,k)\to GO(4,k),$$

where the map on $k^*$ is just given by $c\to(cI_2,c^{-1}I_2)$. The map $\beta$, say, on the right can be described explicitly as follows. The quadratic space $(k^4,B_0)$ is also isometric to $(M_2(k),B_1)$, where $B_1$ is the symmetric bilinear map $(X,Y)\to \mathrm{tr}({}^tXY)$. Under this identification, $\beta(g,g')$ is, for all $g,g'$ in $\mathrm{GL}(2,k)$, the automorphism of $k^4$ given by $X\to {}^tgXg'$. Clearly the kernel of $\beta$ consists of pairs $(cI_2,c^{-1}I_2)$ with $c\in k^*$, proving the requisite exactness.

Note that $\lambda(\beta(g,g'))$ is $\det(g)\det(g')$, while the determinant of $\beta(g,g')$ is its square. Hence $\nu$ is trivial on the image of $\beta$. It is easy to see that $Z_4(k)$ lies in the image of $\beta$, and that $\beta(\mathrm{SL}(2,k)\times\mathrm{SL}(2,k))$ is a subgroup of $\mathrm{SO}(4,k)$ properly containing $\{\pm I_4\}$. The abelianization of $\mathrm{SO}(4,k)$ is $k^*/{k^*}^2$ ([D], p. 57), and since we have assumed that $k=\overline{k}$, $\mathrm{SO}(4,k)$ is perfect, i.e., it equals its own commutator subgroup. Then by the discussion on page 59 of *loc. cit*, $\mathrm{SO}(4,k)/\{\pm I_4\}$ is isomorphic to $\mathrm{PSL}(2,k)\times\mathrm{PSL}(2,k)$. It follows that $\beta$ maps $\mathrm{SL}(2,k)\times\mathrm{SL}(2,k)$ onto $\mathrm{SO}(4,k)$ with kernel $\{\pm(I_2,I_2)\}$. Putting everything together, we obtain

$$(1.6)\qquad \beta(\mathrm{G}L(2,k)\times\mathrm{G}L(2,k)) \;=\; \mathrm{S}GO(4,k).$$

## 2. **The reducible case**

Suppose we are given a representation $\rho$ as in the statement of Theorem A, which is reducible. Thanks to Maschke's theorem we may write $\rho\simeq\oplus_j\rho_j$, with each $\rho_j$ irreducible of dimension $n_j$, and $\sum_j n_j=4$. Suppose we have found, for each $j$, a cuspidal automorphic representation $\pi_j=\pi_{j,\infty}\otimes\pi_{j,f}$ of $\mathrm{GL}(n_j,\mathbb{A}_F)$ such that $L(s,\rho_j)=L(s,\pi_{j,f})$. Then we can consider the *isobaric sum* of Langlands ([La2], [JS])

$$(2.1)\qquad \pi \;=\; \boxplus_j\pi_j,$$

which is automorphic and satisfies

$$L(s,\pi)\;=\;\prod_j L(s,\pi_j).$$

Since the $L$-functions of Artin are also additive, we get $L(s,\rho)=L(s,\pi_f)$ as desired. So it remains to find the $\pi_j$.

Note that cuspidal automorphic representations of $\mathrm{GL}(1,\mathbb{A}_F)$ are just idele class characters of $F$. So when $n_j=1$, the existence of $\pi_j$ follows from class field theory.

Since the image of $\rho$ is by hypothesis solvable, the same will be true for each $\rho_j$. So if $n_j = 2$, we may apply the celebrated theorem of Langlands ([La1]) and Tunnell ([Tu]) to conclude the existence of $\pi_j$.

It remains to consider the case when $n_j$ is 3 for some $j$, say for $j = 1$. Then we must have a decomposition

$$\rho \simeq \rho_1 \oplus \rho_2,$$

with $\rho_1$ (resp. $\rho_2$) irreducible of dimension 3 (resp. 1). Since by hypothesis, the image of $\rho$ lands is $\mathrm{GO}(4, \mathbb{C})$, and since there can be no intertwining between $\rho_1$ and $\rho_j$, we must have

$$im(\rho_1) \subset \mathrm{G}O(3, \mathbb{C}).$$

Thanks to Lemma 1.2, $\mathrm{GO}(3, \mathbb{C})$ is $\mathrm{SO}(3, \mathbb{C}) \times \mathbb{C}^*$. So we may write

$$\rho_1 \simeq \rho' \otimes \chi, \tag{2.2}$$

where $\chi$ is a character $\mathfrak{G}_F \to \mathbb{C}^*$, and $\rho'$ is a 3-dimensional representation of $\mathfrak{G}_F$ with image in $\mathrm{SO}(3, \mathbb{C})$.

Moreover, the exact sequence (1.3), which can be viewed as an exact sequence of trivial modules under $\mathfrak{G}_F = \mathrm{Gal}(\overline{F}/F)$, furnishes the cohomology exact sequence

$$\mathrm{H}om(\mathfrak{G}_F, \mathrm{G}L(2, \mathbb{C})) \to \mathrm{H}om(\mathfrak{G}_F, \mathrm{S}O(3, \mathbb{C})) \to H^2(\mathfrak{G}_F, \mathbb{C}^*), \tag{2.3}$$

with $\rho'$ belonging to the middle group. On the other hand, a theorem of Tate (see [Se], for a proof) asserts that the group on the right hand side of (2.3) is trivial as $F$ is a number field. Thus we may lift $\rho'$ to an element of the left hand side group of (2.3). In other words, we can find a (non-unique) 2-dimensional representation $\tau_1$ of $\mathfrak{G}_F$ such that

$$\rho_1 \simeq \mathrm{A}d(\tau_1) \otimes \chi. \tag{2.4}$$

Since $\rho_1$ has solvable image, $\tau_1$ is also forced to have the same property. Applying Langlands-Tunnell once again, we get an isobaric automorphic representation $\eta_1$, which must in fact be cuspidal as $\rho_1$ and hence $\tau_1$ are irreducible, such that $L(s, \tau_1)$ equals $L(s, \eta_{1,f})$.

By [GeJ] one knows that, given any cuspidal automorphic representation $\eta$ of $\mathrm{GL}(2, \mathbb{A}_F)$, there exists a functorially associated (isobaric) automorphic representation $\mathrm{Ad}(\eta)$ such that

$$L(s, \mathrm{A}d(\eta)) = \prod_v L(s, \mathrm{A}d(\sigma_v(\eta))), \tag{2.5}$$

where the product is over all the places $v$ of $F$, and $\sigma_v(\eta)$ (resp. $\sigma(\eta_v)$) is the 3-dimensional representation of the Weil group (resp. Weil-Deligne group) $W_{F_v}$ (resp. $W'_{F_v}$) when $v$ is archimedean (resp. non-archimedean), associated to $\eta_v$ by the local Langlands correspondence for $\mathrm{GL}(n)$ ([HaT], [He]).

Then it follows that

$$L(s, \rho_1) = L(s, (\mathrm{A}d(\eta_1) \otimes \chi)_f).$$

So we are done by setting $\pi_1 = \mathrm{A}d(\eta_1) \otimes \chi$.

□

## 3. Modularity modulo Theorem D

In this section we will show how to prove Theorem A if we admit the truth of Theorem D. We will also need to make use of the main theorem of [Ra1]. Thanks to the discussion in the previous section, we may assume that $\rho$ is irreducible.

By hypothesis, the image of $\rho$ lies in $\mathrm{GO}(4,\mathbb{C})$. Recall from section 1 the definion of the subgroup $\mathrm{SGO}(4,\mathbb{C})$, which is the kernel of the *similitude norm*

$$\nu : \mathrm{G}O(4,\mathbb{C}) \to \{\pm 1\}.$$

Let $K$ be the extension of $F$ defined by the kernel of $\nu \circ \rho$. Then $[K : F] \leq 2$. Write $\rho_K$ for the restriction of $\rho$ to $\mathfrak{G}_K = \mathrm{Gal}(\overline{F}/K)$. Thanks to (1.5) and (1.6), one has the following short equence of trivial Galois modules:

$$(3.1) \qquad 1 \to \mathbb{C}^* \to \mathrm{G}L(2,\mathbb{C}) \times \mathrm{G}L(2,\mathbb{C}) \to \mathrm{S}GO(4,\mathbb{C}) \to 1,$$

where the maps in the middle are $c \to (cI_2, c^{-1}I_2)$ and $(g, g') \to (X \to {}^t gXg')$. The associated (continuous) cohomology exact sequence gives
(3.2)

$$\mathrm{H}om(\mathfrak{G}_K, \mathbb{C}^*) \to \mathrm{H}om(\mathfrak{G}_K, \mathrm{G}L(2,\mathbb{C}) \times \mathrm{G}L(2,\mathbb{C})) \to \mathrm{H}om(\mathfrak{G}_K, \mathrm{S}GO(4,\mathbb{C})) \to H^2(\mathfrak{G}_K, \mathbb{C}^*),$$

with $\rho_K$ belonging to the second group from the right. Recall Tate's theorem which says that the first group on the right is trivial. So we may find $\sigma, \sigma'$ in $\mathrm{H}om(\mathfrak{G}_K, \mathrm{G}L(2,\mathbb{C}))$ such that

$$(3.3) \qquad \rho \simeq \sigma \otimes \sigma'.$$

Note that this lifting is unique only up to flipping the two factors and changing $(\sigma, \sigma')$ by $(\sigma \otimes \mu, \sigma' \otimes \mu^{-1})$, for any character $\mu \in \mathrm{H}om(\mathfrak{G}_K, \mathbb{C}^*)$.

Since the image of $\rho$ was assumed to be solvable, we see easily that the images of $\sigma, \sigma'$ should also be solvable. And since $\rho$ is irreducible, the same should hold for $\sigma$ and $\sigma'$. So we may apply the theorem of Langlands and Tunnell to deduce the existence of cuspidal automorphic representations $\pi, \pi'$ of $\mathrm{GL}(2, \mathbb{A}_F)$, respectively associated to $\sigma, \sigma'$.

Now suppose the image of $\rho$ lands in $\mathrm{SGO}(4,\mathbb{C})$ itself, in which case $K = F$. Then by Theorem M (in section 3) of [Ra1], we know the existence of an isobaric automorphic representation $\pi \boxtimes \pi'$ of $\mathrm{GL}(4, \mathbb{A}_F)$ such that

$$(3.4) \qquad L(s, \pi \boxtimes \pi') = L(s, \pi \times \pi'),$$

where the $L$-function on the right is the Rankin-Selberg $L$-function associated to the pair $(\pi, \pi')$. In addition, we have at any place $v$, the local factors of $\pi \boxtimes \pi'$ identify functorially with those of the tensor product $\sigma_v(\pi) \otimes \sigma_v(\pi')$ of the local Langlands parameters $\sigma(\pi), \sigma(\pi')$ ([HaT], [He]), proved long ago for GL(2) by P. Kutzko. Since $\pi$ (resp. $\pi'$) is associated to $\sigma$ (resp. $\sigma'$), the local representations $\sigma_v(\pi)$ (resp. $\sigma_v(\pi')$) are isomorphic to the ones defined by the restriction at $v$ of $\sigma$ (resp. $\sigma'$). Thus the automorphic representation $\Pi$ of $\mathrm{GL}(4, \mathbb{A}_F)$, whose existence is predicted by Theorem A, is none other than $\pi \boxtimes \pi'$. The cuspidality criterion of [Ra1] (Theorem M) shows easily that, since $\rho$ is irreducible, $\Pi$ must be cuspidal. We are done in this case.

We may henceforth assume that $[K : F] = 2$, which is the subtler case. Denote by $\theta$ the non-trivial automorphism of $K$ over $F$. We can again find cuspidal automorphic representations $\pi, \pi'$ of $\mathrm{GL}(2, \mathbb{A}_K)$ such that

$$L(s, \rho_K) \,=\, L(s, \pi_f \boxtimes \pi'_f), \tag{3.5}$$

which proves that the restriction $\rho_K$ of $\rho$ to $\mathfrak{G}_K$ is modular.

We will now explain why this case is difficult. The identity (3.5) implies that $\pi \boxtimes \pi'$ is $\theta$-invariant, so by the base change theorem of Arthur and Clozel ([AC]), we can find an isobaric automorphic representation $\Pi$ of $\mathrm{GL}(4, \mathbb{A}_F)$ such that its base change $\Pi_K$ is isomorphic to $\pi \boxtimes \pi'$. One can also see easily that the local factors of $\Pi$ and $\rho$ agree at all the places of $F$ which split in $K$. But one is stuck at this point and cannot easily deduce the requisite identity at the *inert* places, except when $\rho_K$ is no longer irreducible.

It is now clear that Theorem A is a consequence of Theorem *A′*. So we will address the following:

*Proof of Theorem A′ modulo Theorem D*:

Suppose $\rho_K$ is *reducible*. Then it must contain an irreducible summand $\tau$ of dimension $\leq 2$. By Frobenius reciprocity, $\rho$ should intertwine with the induction $\mathrm{I}nd_K^F(\tau)$ of $\tau$ to $\mathfrak{G}_F$. As $\rho$ is irreducible of dimension 4 and $\mathrm{I}nd_K^F(\tau)$ is at most of dimension 4, we are forced to have

$$\rho \,\simeq\, \mathrm{I}nd_K^F(\tau),$$

with $\dim(\tau) = 2$. The solvability of the image of $\rho$ implies the same about that of $\tau$, and so we may apply Langlands-Tunnell to get a cuspidal automorphic representation $\eta$ of $\mathrm{GL}(2, \mathbb{A}_K)$ associated to $\tau$, and we are done by taking $\Pi$ to be the automorphically induced representation $I_K^F(\tau)$ (see [AC], and also [Ra1], sec. 2). Since $\rho$ is irreducible, $\tau$, and hence $\eta$, cannot be $\theta$-invariant, where $\theta$ is the non-trivial automorphism of $K/F$. Consequently, $I_K^F(\eta)$ must be cuspidal.

So we may, and we will, assume that $\rho_K$ is irreducible. Since it is the restriction of $\rho$, we have

$$(\sigma \otimes \sigma')^\theta \,\simeq\, \rho_K^\theta \,\simeq\, \rho_K \,\simeq \sigma \otimes \sigma^\theta,$$

where $\theta$ is the nontrivial automorphism of $K$ over $F$.

We will first prove part (b) of Theorem A′.

**Lemma 3.6** The irreducibility of $\rho_K = \sigma \otimes \sigma'$ implies that at least one of the representations $\sigma, \sigma'$ is non-dihedral, and $\sigma'$ cannot be a one-dimensional twist of $\sigma$. Furthermore, since $\rho_K$ is $\theta$-invariant, one of the following must happen:

(I) There exists a character $\nu$ of $\mathrm{Gal}(\overline{F}/K)$ such that

$$\sigma^\theta \,\simeq\, \sigma \otimes \nu \quad \text{and} \quad (\sigma')^\theta \,\simeq\, \sigma' \otimes \nu^{-1};$$

(II) There exists a character $\mu$ of $\mathrm{Gal}(\overline{F}/K)$ such that

$$\sigma^\theta \,\simeq\, \sigma' \otimes \mu \quad \text{and} \quad (\sigma')^\theta \simeq \sigma \otimes \mu^{-1}.$$

*Proof of Lemma 3.6.* Let $\omega$, resp. $\omega'$, denote the determinant of $\sigma$, resp. $\sigma'$. Clearly, if $\sigma' \simeq \sigma \otimes \lambda$ for some character $\lambda$, then $\rho_K$ is reducible, as $\sigma \otimes \sigma'$ will then be $(\mathrm{sym}^2(\sigma)\otimes\lambda)\oplus\omega\lambda$. We will now show that not both $\sigma,\sigma'$ can be dihedral. We have

$$\mathrm{End}(\rho_K)\,\simeq\,\rho_K\otimes\rho_K^\vee\,\simeq\,\rho_K\otimes(\sigma\otimes\omega^{-1})\otimes(\sigma'\otimes{\omega'}^{-1})\,\simeq\,\rho_K^{\otimes 2}\otimes(\omega\omega')^{-1}.$$

Since $\rho_K$ is irreducible, any linear character occurring in $\mathrm{End}(\rho_K)$ must have multiplicity one. We claim that if $\sigma$ and $\sigma'$ are both dihedral, then $\mathrm{End}(\rho_K)$ contains the trivial representation with multiplicity $>1$. Indeed suppose $\sigma=\mathrm{Ind}_M^K(\chi)$ and $\sigma'=\mathrm{Ind}_N^K(\chi')$, where $M,N$ are quadratic extensions of $K$, and $\chi,\chi'$ characters of $\mathfrak{G}_M$, $\mathfrak{G}_N$ respectively. Since $\rho_K^{\otimes 2}$ is $\mathrm{sym}^2(\rho_K)\oplus\Lambda^2(\rho_K)$, it suffices to prove that $\omega\omega'$ occurs twice in $\mathrm{sym}^2(\rho_K)$. Note that

$$\mathrm{sym}^2(\sigma)\,\simeq\,\mathrm{Ind}_M^K(\chi^2)\oplus\omega\epsilon\quad\text{and}\quad\mathrm{sym}^2(\sigma')\,\simeq\,\mathrm{Ind}_M^K({\chi'}^2)\oplus\omega'\epsilon',$$

where $\epsilon$ (resp. $\epsilon'$) is the quadratic character of $\mathfrak{G}_K$ corresponding to $M/K$ (resp. $N/K$). Since

$$\mathrm{sym}^2(\sigma\otimes\sigma')\,\simeq\,\mathrm{sym}^2(\sigma)\otimes\mathrm{sym}^2(\sigma')\oplus\Lambda^2(\sigma)\otimes\Lambda^2(\sigma'), \tag{3.7}$$

we get

$$\mathrm{sym}^2(\rho_K)\,\simeq\simeq\,\mathrm{Ind}_M^K(\chi^2)\otimes\mathrm{Ind}_M^K({\chi'}^2)\oplus\mathrm{Ind}_M^K(\chi^2)\otimes\omega'\epsilon'\oplus\mathrm{Ind}_M^K({\chi'}^2)\otimes\omega\epsilon\oplus(\omega\omega')^{\oplus 2},$$

as asserted. So at least one of $\sigma,\sigma'$ must be non-dihedral. Finally by $\theta$-invariance,

$$\sigma^\theta\otimes(\sigma')^\theta\,\simeq\,\sigma\otimes\sigma',$$

which evidently implies, by using irreducibility, that we must be in case (I) or (II). □

**Lemma 3.8** *Suppose we are in case (I) of Lemma 3.6. Then there exist characters $\chi,\chi'$ of $\mathfrak{G}_K$ such that $\sigma\otimes\chi$ and $\sigma'\otimes\chi'$ are both $\theta$-invariant.*

*Proof of Lemma 3.8.* We may assume, after interchanging $\sigma,\sigma'$ if necessary, that $\sigma'$ is non-dihedral. We claim that

$$(\omega\omega')^\theta\,=\,\omega\omega'. \tag{3.9}$$

To begin, note that the $\theta$-invariance of $\rho_K$ implies the same for its contragredient and its symmetric and exterior powers. The determinant of $\rho_K$ is easily seen to be $(\omega\omega')^2$, and this shows that

$$\beta:=\,(\omega\omega')^\theta/\omega\omega'$$

has order $\le 2$. Suppose the order is 2. Then the identity (3.7) shows that

$$\beta\,\subset\,\mathrm{Ad}(\sigma)\otimes\mathrm{Ad}(\sigma'),$$

where for any 2-dimensional $\tau$,

$$\mathrm{Ad}(\tau):=\,\mathrm{sym}^2(\tau)\otimes\omega_\tau^{-1},$$

the adjoint representation. The self-duality of $\mathrm{Ad}(\sigma)$, plus the irreducibility of $\mathrm{Ad}(\sigma')$, then implies that $\mathrm{Ad}(\sigma')$ is isomorphic to $\mathrm{Ad}(\sigma)\otimes\beta$, which is the same, as $\beta^2=1$, as $\mathrm{Ad}(\sigma\otimes\beta)$. But we have the following

**Theorem 3.10** *Let $\tau,\tau'$ be irreducible, 2-dimensional representations of $\mathfrak{G}_K$ with isomorphic adjoint representations. Then there exists a character $\chi$ of $\mathfrak{G}_K$ such that*

$$\tau'\,\simeq\tau\otimes\chi.$$

For a proof see [Ra2]; it is important to note that it holds whether or not $\tau$ is dihedral. This is the Galois version of the so called *multiplicity one for SL*(2). Its automorphic version was proved in [Ra1] (see Theorem 4.1.2).

Applying this with $\tau = \sigma \otimes \beta$ and $\tau' = \sigma'$, we see that $\sigma'$ is a one dimensional twist of $\sigma$, which contradicts the irreducibility of $\rho_K$. Hence $\beta$ must be 1 and the claim is proved.

Next we claim that

$$(3.11)\quad (\mathrm{sym}^2(\sigma)\otimes\omega')^\theta \simeq \mathrm{sym}^2(\sigma)\otimes\omega' \quad \text{and} \quad (\mathrm{sym}^2(\sigma')\otimes\omega)^\theta \simeq \mathrm{sym}^2(\sigma')\otimes\omega.$$

Indeed, since $\sigma^\theta$ is by hypothesis $\sigma\otimes\mu$, we have

$$(3.12)\quad (\mathrm{sym}^2(\sigma)\otimes\omega')^\theta \simeq \mathrm{sym}^2(\sigma\otimes\mu)\otimes(\omega')^\theta \simeq (\mathrm{sym}^2(\sigma)\otimes\omega')\otimes\mu^2\left(\frac{(\omega')^\theta}{\omega'}\right).$$

On the other hand, comparing the determinants of $\sigma^\theta$ and $\sigma\otimes\mu$, we get

$$(3.13)\qquad \mu^2 = \frac{\omega^\theta}{\omega}.$$

The first half of the asserted identity (3.11) now follows by combining (3.9), (3.12) and (3.13). The proof of the second half is the same.

We will now prove the $\theta$-invariance of $\sigma\otimes\chi$ for a suitable $\chi$. The case of $\sigma'$ is similar and will be left to the reader.

Note that $\mathrm{sym}^2(\sigma\otimes\omega')$ is of GO(3)-type. Explicitly,

$$\mathrm{sym}^2(\mathrm{sym}^2(\sigma)\otimes\omega') \simeq (\mathrm{sym}^4(\sigma)\oplus\omega^2)\otimes{\omega'}^2,$$

and so $(\omega\omega')^2$ occurs in the symmetric square of $\mathrm{sym}^2(\sigma)\otimes\omega'$. Since $\omega\omega'$ is $\theta$-invariant by (3.9), it extends to a character of $\mathfrak{G}_F$. Moreover, the $\theta$-invariance of $\mathrm{sym}^2(\sigma)\otimes\omega'$ (cf. (3.11)) gives the existence of a 3-dimensional representation $\eta$ of $\mathfrak{G}_F$ such that

$$\mathrm{Res}^F_K(\eta) \simeq \mathrm{sym}^2(\sigma)\otimes\omega'.$$

Since the restriction of the symmetric square of $\eta$ to $\mathfrak{G}_K$ contains the $\theta$-invariant character $(\omega\omega')^2$, there will be a character $\lambda$, say, of $\mathfrak{G}_F$, such that

$$\lambda \subset \mathrm{sym}^2(\eta) \quad \text{and} \quad \mathrm{Res}^F_K(\lambda) = (\omega\omega')^2.$$

In other words, $\eta$ is also of GO(3)-type. By Lemma 1.2 and the exact sequence (1.3), we then get the existence of a 2-dimensional representation $\tau$, and a character $\alpha$, of $\mathfrak{G}_F$ such that

$$\eta \simeq \mathrm{Ad}(\tau)\otimes\alpha.$$

This yields the isomorphism

$$\mathrm{Ad}(\mathrm{Res}^F_K(\tau))\otimes\mathrm{Res}^F_K(\alpha) \simeq \mathrm{sym}^2(\sigma)\otimes\omega'.$$

Since $\mathrm{sym}^2(\sigma)$ is $\mathrm{Ad}(\sigma)\otimes\omega$, we get

$$\mathrm{Ad}(\sigma) \simeq \mathrm{Ad}(\mathrm{Res}^F_K(\tau\otimes\nu)),$$

for a character $\nu$ of $\mathfrak{G}_F$ satisfying

$$\mathrm{Res}^F_K(\nu/\alpha) = (\omega\omega')^{-1}.$$

The existence of $\nu$ comes from the $\theta$-invariance of $\omega\omega'$. Applying Theorem 3.10 again, now with $\tau = \sigma$ and $\tau' = \mathrm{Res}^F_K(\tau\otimes\nu)$, we get the existence of a character $\chi$ of $\mathfrak{G}_F$ such that

$$\sigma\otimes\chi \simeq \mathrm{Res}^F_K(\tau\otimes\nu).$$

Done with the proof of Lemma 3.8.

□

**Lemma 3.14** *Suppose we are in case (I) of Lemma 3.6. Then the following hold:*

(i) *If $\sigma, \sigma'$ are both non-dihedral, then there exist irreducible, 2-dimensional representations $\tau, \tau'$ of $\mathfrak{G}_F$ such that*

$$(*) \qquad \rho \simeq \tau \otimes \tau';$$

(ii) *If either $\sigma$ or $\sigma'$ is dihedral, then either (*) holds or there exists a quadratic extension $L/F$ and a 2-dimensional representation $\eta$ of $\mathfrak{G}_L$ such that*

$$\rho \simeq \mathrm{Ind}_L^F(\eta).$$

*Proof of Lemma 3.14.* Again by the irreducibility of $\sigma \otimes \sigma'$, we may assume that $\sigma'$ is non-dihedral. By Lemma 3.8, there exist characters $\chi, \chi'$of $\mathfrak{G}_K$ such that $\sigma \otimes \chi$ and $\sigma' \otimes \chi'$ are $\theta$-invariant. We first claim that

$$(3.15) \qquad \nu^2 = 1 \quad \text{where} \quad \nu = (\chi\chi')^\theta/\chi\chi'.$$

Indeed, since we are in case (I), $\sigma^\theta$ is of the form $\sigma \otimes \mu$ for some character $\mu$, so that

$$\sigma \simeq (\sigma \otimes \chi)^\theta \otimes \chi^{-1} \simeq \sigma \otimes \mu\chi^\theta/\chi.$$

When $\sigma$ is non-dihedral, we must have $\mu = \chi/\chi^\theta$, and when $\sigma$ is dihedral, $\mu\chi^\theta/\chi$ must be trivial or quadratic. So

$$(\mu\chi^\theta/\chi)^2 = 1$$

in either case. Similarly, $(\mu^{-1}{\chi'}^\theta/\chi')^2$ is 1. The claimed identity (3.15) is then a consequence.

This argument in fact shows that if $\sigma$ is *non-dihedral*, then $\nu = 1$. So we may choose a character $\gamma$ of $\mathfrak{G}_F$ such that

$$\chi\chi' = \mathrm{Res}_K^F(\gamma).$$

On the other hand, since $\sigma \otimes \chi$ and $\sigma' \otimes \chi'$ are $\theta$-invariant, there are irreducible 2-dimensional repesentations $\tau, \tau'$ of $\mathfrak{G}_F$ such that

$$\sigma \otimes \chi \simeq \mathrm{Res}_K^F(\tau) \quad \text{and} \quad \sigma' \otimes \chi' \simeq \mathrm{Res}_K^F(\tau').$$

Putting all these together we get

$$\mathrm{Res}_K^F(\rho) \simeq \mathrm{Res}_K^F(\tau \otimes \tau' \otimes \gamma).$$

Also, $\tau$ cannot be a one-dimensional twist of $\tau'$ as it would make $\sigma \otimes \sigma'$ reducible. Then it follows that

$$\rho \simeq \tau \otimes (\tau' \otimes \gamma'),$$

where $\gamma'$ is either $\gamma$ or $\gamma\delta$. So we get (i).

So we may assume (for this Lemma) that $\sigma$ is *dihedral* and that $\nu$ is a non-trivial quadratic character of $\mathfrak{G}_K$. The $\theta$-invariance of $\sigma \otimes \sigma' \otimes \chi\chi'$ then implies that $\sigma \otimes \sigma'$ is isomorphic to $\sigma \otimes \sigma' \otimes \nu$. Consequently,

$$\nu \subset (\sigma^\vee \otimes \sigma) \otimes ({\sigma'}^\vee \otimes \sigma') \simeq \mathrm{Ad}(\sigma) \otimes \mathrm{Ad}(\sigma') \oplus \mathrm{Ad}(\sigma) \oplus \mathrm{Ad}(\sigma') \oplus 1.$$

Recall that for any irreducible 2-dimensional representation $\tau$, $\mathrm{Ad}(\tau)$ is reducible iff $\tau$ is dihedral. Since $\sigma'$ (resp. $\sigma$) is non-dhedral (resp. dihedral), we see that $\mathrm{Ad}(\sigma) \otimes \mathrm{Ad}(\sigma')$ and $\mathrm{Ad}(\sigma')$ have no one-dimensional summands. So we must have

$$\nu \subset \mathrm{Ad}(\sigma).$$

This implies that $\sigma \otimes \nu \simeq \sigma$, i.e., that $\sigma$ is induced by a character of $\mathfrak{G}_M$, if $M$ denotes the quadratic extension of $K$ cut out by $\nu$. On the other hand, since $\theta^2 = 1$ and $\nu^2 = 1$,

$$\nu^\theta \,=\, \left(\frac{(\chi\chi')^\theta}{\chi\chi'}\right) \,=\, \nu^{-1} \,=\, \nu.$$

So $\nu$ is the restriction to $K$ of a character $\nu_0$, say, of $\mathfrak{G}_F$. Denote by $M_0$ the quadratic extension of $F$ cut out by $\nu_0$. Then $M$ is a biquadratic extension of $F$ containing $K$ and $M_0$. Note that

$$\mathrm{Res}^F_K(\rho) \,\simeq\, \mathrm{Ind}^K_M(\chi) \otimes \sigma' \,\simeq\, \mathrm{Ind}^K_M(\chi \otimes \mathrm{Res}^K_M(\sigma')).$$

Thus

$$\mathrm{Ind}^F_M(\chi \otimes \mathrm{Res}^K_M(\sigma')) \,\simeq\, \rho \,\oplus\, \rho \otimes \delta.$$

The left hand side is isomorphic to $\mathrm{Ind}^F_{M_0}(\mathrm{Ind}^{M_0}_M(\chi \otimes \mathrm{Res}^K_M(\sigma')))$, which is invariant under twisting by $\nu_0$. Thus

$$\rho \,\oplus\, \rho \otimes \delta \,\simeq\, \rho \otimes \nu_0 \,\oplus\, \rho \otimes \delta\nu_0.$$

Since $\rho$ is irreducible, we must have

$$\rho \,\simeq\, \rho \otimes \beta, \quad \text{where} \quad \beta \in \{\nu_0, \nu_0\delta\}.$$

Denote by $L$ the quadratic extension of $F$ cut out by $\beta$. (It must be either of the quadratic extensions which are contained in $M$ and different from $K$.) Then there exists an irreducible 2-dimensional rpesentation $\eta$ of $\mathfrak{G}_L$ such that

$$\rho \,\simeq\, \mathrm{Ind}^F_L(\eta),$$

giving (ii). Lemma 3.14 is now proved.

□

*Proof of Theorem A′ modulo Theorem D* (contd.):

Since (i), (ii) of Lemma 3.14 coincide with (i), (ii) (respectively) of Theorem A′, we may asume from here on that we are in case (II) (of Lemma 3.6), with $\sigma'$ non-dihedral. But as $\sigma^\theta \simeq \sigma' \otimes \mu$ for a character $\mu$, $\sigma$ is dihedral iff $\sigma'$ is. Indeed, if $\sigma$ were dihedral, it wold admit a self-twist by a quadratic character $\nu \neq 1$ and this would consequently force $\sigma'$ to admit self-twist by $\nu^\theta$. So neither $\sigma$ nor $\sigma'$ is dihedral. We have

$$\sigma \,\simeq\, (\sigma^\theta)^\theta \,\simeq\, (\sigma')^\theta \otimes \mu^\theta \,\simeq\, \sigma \otimes (\mu^\theta/\mu).$$

Since $\sigma$ is not dihedral, it does not admit any non-trivial self-twist by a character, and so we must have $\mu = \mu^\theta$. So there exists a character $\nu$ of $\mathfrak{G}_F$ such that $\mu$ is the restriction $\nu_K$ of $\nu$ to $\mathfrak{G}_K$. Thus we get (from (3.4),

$$(\rho \otimes \nu)_K \,\simeq\, \sigma \otimes (\sigma' \otimes \mu) \,\simeq\, \sigma \otimes \sigma^\theta.$$

It suffices to show that some character twist of $\rho$ is modular. So we may, after replacing $\rho$ by its twist by $\nu^{-1}$, that

$$\rho_K \,\simeq\, \sigma \otimes \sigma^\theta. \tag{3.16}$$

If $\delta$ denotes the quadratic character of $\mathfrak{G}_F$ corresponding to $K/F$, then $\rho$ and $\rho \otimes \delta$ are the only representations for which (3.16) holds.

It is easy to see that the induction (to $\mathfrak{G}_F$) of the exterior square of $\sigma$, i.e., $\det(\sigma)$, is a summand of the exterior square of the induction of $\sigma$. Thanks to semisimplicity, we may then define the *Asai representation* of $\sigma$, denoted $As(\sigma)$, by the decomposition

$$\Lambda^2(\mathrm{I}nd_K^F(\sigma)) \simeq As(\sigma) \oplus \mathrm{I}nd_K^F(\det(\sigma)). \tag{3.17}$$

**Lemma 3.18** *$\rho$ is isomorphic to $As(\sigma)$ or $As(\sigma)\otimes\delta$.*

*Proof of Lemma.* Let $\beta$ denote the tensor square representation of $\mathrm{I}nd_K^F(\sigma)$, so that

$$\beta = \Lambda^2(\mathrm{I}nd_K^F(\sigma)) \oplus sym^2(\mathrm{I}nd_K^F(\sigma)).$$

We can also write

$$\beta \simeq \mathrm{I}nd_K^F(\sigma \otimes \mathrm{Res}_K^F(\mathrm{I}nd_K^F(\sigma))), \tag{3.19}$$

which implies that

$$\beta \simeq \beta\otimes\delta. \tag{3.20}$$

So $As(\sigma)\otimes\delta$ must also occur in $\beta$.

On the other hand, since the restriction to $\mathfrak{G}_K$ of $\mathrm{I}nd_K^F(\sigma)$ is $\sigma\oplus\sigma^\theta$, we get from (3.19),

$$\beta \simeq \mathrm{I}nd_K^F(sym^2(\sigma)\oplus\Lambda^2(\sigma)\oplus(\sigma\otimes\sigma^\theta)). \tag{3.21}$$

Since $\rho_K$ is (by (3.16)) isomorphic to $\sigma\otimes\sigma^\theta$, it must occur in the induction of the latter to $\mathfrak{G}_F$; ditto for the twist of $\rho$ by $\delta$. Hence, by the additivity of induction, the representation on the right of (3.21) is forced to be

$$\mathrm{I}nd_K^F(sym^2(\sigma))\oplus\mathrm{I}nd_K^F(\Lambda^2(\sigma))\oplus\rho\oplus(\rho\otimes\delta).$$

The lemma now follows in view of (3.17).

□

So we may, after possibly replacing $\rho$ by $\rho\otimes\delta$, assume that

$$\rho \simeq As(\sigma), \tag{3.22}$$

for an irreducible 2-dimensional, continuous $\mathbb{C}$-representation $\sigma$ of $\mathfrak{G}_K$ with solvable image.

For any cuspidal automorphic representation $\pi$ of $\mathrm{GL}(2,\mathbb{A}_K)$, one may associate the following *Asai L-function*:

$$L(s,\pi,r) = \prod_v L(s, As(\sigma_w(\pi))), \tag{3.23}$$

where $v$ runs over the set $\Sigma_F$ of all the places of $F$, and for each $v\in\Sigma_F$, $w$ denotes a place of $K$ above $v$ and $As(\sigma_w(\pi))$ denotes the Asai representation associated to $\sigma_w(\pi)$. Note that the definition of $As(\sigma_w(\pi))$ is independent of the choice of $w$ above $v$. When $v$ splits in $K$, $K_v = K_w\times K_{\theta w}$, and $As(\sigma_v(\eta))$ simply means the tensor product $\sigma_w(\eta)\otimes\sigma_{\theta w}(\eta)$.

The $L$-function on the left of (3.23) looks like a *Langlands L-function*, and we need to explain why we are justified in adopting such a notation. For this recall that the $L$-group of the restriction of scalars of $\mathrm{GL}(2)/K$ to $F$ is the semidirect product

$${}^L(R_{K/F}\mathrm{G}L(2)/K) = (\mathrm{G}L(2,\mathbb{C})\times\mathrm{G}L(2,\mathbb{C}))\times\mathrm{G}al(K/F), \tag{3.24}$$

where $\theta$ acts by interchanging the two factors. One defines a representation

$$r : {}^L(R_{K/F}\mathrm{G}L(2)/K) \to \mathrm{G}L(\mathbb{C}^2\otimes\mathbb{C}^2) \simeq \mathrm{G}L(4,\mathbb{C}) \tag{3.25}$$

by setting, for all $x, y$ in $\mathbb{C}^2$,

$$r(g, g'; 1)(x \otimes y) \;=\; g(x) \otimes g(y)$$

and

$$r(1, 1; \theta)(x \otimes y) \;=\; y \otimes x.$$

At any finite place $w$ of $K$ where $\pi$ is unramified, there is a diagonal matrix $[\alpha_w, \beta_w]$ in $\mathrm{GL}(2, \mathbb{C})$ such that

$$L(s, \pi_w) \;=\; \frac{1}{(1 - \alpha_w q_w^{-s})(1 - \beta_w q_w^{-s})}, \tag{3.26}$$

where $q_w$ is the norm of $w$. If $v$ is any finite place of $F$ which is unramified for $(K/F, \pi)$, i.e., if $v$ is unramified in $K$ and if $\pi$ is unramified at any place $w$ of $K$ above $v$, then one may associate, as in [HLR], a (Langlands) conjugacy class $A_v(\pi)$ in ${}^L(R_{K/F}\mathrm{GL}(2)/K)$. When composed with $r$, one gets

$$r(A_v(\pi)) \;=\; [\alpha_w\alpha_{\theta w}, \alpha_w\beta_{\theta w}, \beta_w\alpha_{\theta w}, \beta_w\alpha_{\theta w}] \tag{3.27}$$

if $v$ *splits* into $(w, \theta w)$ in $K$, and

$$r(A_v(\pi)) \;=\; \begin{pmatrix} \alpha_v & 0 & 0 & 0 \\ 0 & 0 & \alpha_v & 0 \\ 0 & \beta_v & 0 & 0 \\ 0 & 0 & 0 & \beta_v \end{pmatrix}$$

if $v$ remains prime. Since $L(s, \pi_w)$ is $L(s, \sigma_w(\pi))$, we get easily the identity

$$L(s, As_v(\sigma(\pi))) \;=\; L(s, r(A_v(\pi))) \tag{3.28}$$

at any finite place $v$ unramified for $(K/F, \pi)$. This shows the appropriateness of the notation of (3.23). It is also important because the automorphic results we will need later will use the Langlands formalism.

If we admit Theorem D, we then have a unique isobaric automorphic representation $\Pi$ of $\mathrm{GL}(4, \mathbb{A}_F)$ such that

$$L(s, \Pi) \;=\; L(s, \pi, r).$$

In view of the discussion above, it is clear that

$$L(s, \Pi_v) \;=\; L(s, \rho_v) \tag{3.29}$$

at almost all places $v$. By a standard argument comparing the functional equations of $L(s, \Pi)$ and $L(s, \rho)$, we also get such an equality of $L$-factors at *every* place $v$. Since $\rho_K$ is by construction associated to $\pi \boxtimes (\pi \circ \theta)$, we get the identity

$$L(s, \Pi_K) \;=\; L(s, \pi \boxtimes (\pi \circ \theta)). \tag{3.30}$$

Since $\rho_K \simeq (\sigma \otimes (\sigma^\theta)$ is irreducible, the main result of [Ra1] implies that $\pi \boxtimes (\pi \circ \theta)$ is cuspidal. Since $\Pi$ base changes to a cuspidal representation, it must itself be cuspidal by [AC].

This finishes the proof of Theorem A′, and hence Theorem A, modulo Theorem D.

□

## 4. **Reduction to weak lifting, and the cuspidality criterion**

We will begin the proof of Theorem D in this section and finish in section 6.

Fix $K/F$ quadratic with non-trivial automorphism $\theta$ as above, and an arbitrary cuspidal automorphic representation $\pi$ of $\mathrm{GL}(2, \mathbb{A}_K)$. We have to show that there exists an isobaric automorphic representation $\Pi$ of $\mathrm{GL}(4, \mathbb{A}_F)$ such that

$$L(s, \Pi) \,=\, L(s, \pi; r).$$

Suppose $\chi$ is an idele class character of $K$ with restriction $\chi_0$ to $F$. Recall that $\chi_0$ corresonds to the *transfer* from $K$ to $F$ of the character of the Weil group $W_K$ associated to $\chi$ by class field theory. (By abuse of notation we will use the same letter to signify the characters of $\mathbb{A}_K^*/K^*$ and $W_K$.) At any place $v$ of $F$ with divisor $w$ in $K$, we then get, from the definition of the Asai representation,

$$As(\sigma(\pi_w \otimes \chi_w)) \,\simeq\, As(\sigma(\pi_w)) \otimes \chi_{0,v}.$$

Consequently,

$$L(s, \pi \otimes \chi; r) \,=\, L(s, \pi; r \otimes \chi_0). \tag{4.0}$$

One knows, cf. [HLR], that $L(s, \pi; r)$ admits a meromorphic continuation to the whole $s$-plane and satisfies a standard functional equation.

**Proposition 4.1** *Let $\pi$ be a cuspidal automorphic representation of $GL(2, \mathbb{A}_K)$. Suppose we have constructed a weak Asai lifting, i.e., an isobaric automorphic representation $\Pi$ of $GL(4, \mathbb{A}_F)$ satisfying the following identity at almost all places $v$ of $F$:*

$$L(s, \Pi_v) \,=\, L(s, \pi_w; r), \tag{$L_v$}$$

*where $w$ is any place of $K$ above $v$. Then we have $(L_v)$ and $(\varepsilon_v)$ at* **all** *the finite places $v$, and $(L_\infty)$ as well. In addition, the cuspidality criterion of Theorem D holds.*

*Proof.* Let $S$ be the (finite) set of places of $F$ outside which $(L_v)$ holds. Note first that the central character $\Omega$ of $\Pi$ is simply the restriction $\omega_0$ to $F$ of the central character $\omega$ of $\pi$. Indeed at any $v$, $\omega_0$ corresponds to the transfer of the Galois character attached to $\omega$, which also gives the determinant of $As(\sigma_w(\pi))$, with $w$ being a place of $K$ above $v$. Since by definition, $L(s, \pi_w; r)$ equals $L(s, As(\sigma_w(\pi))$, it follows that the idele class characters $\Omega$ and $\omega_0$ agree outside $S$, and hence agree everywhere by a classical result of Hecke.

Now some notations. If $f(s), g(s)$ are two meromorphic functions of $s$ such that their quotient is invertible, we will write $f(s) \,\equiv\, g(s)$. At any place $v$, given a character $\nu$ of $F_v^*$, we can write it as $\nu_0|.|^z$, for a unitary character $\nu_0$ and a complex number $z$. The real part of $z$ is uniquely defined; we will call it the *exponent of* $\nu$, and denote it $e(\nu)$.

Choose a finite order character $\mu$ of $C_F$ with $\mu_\infty = 1$ (which means that $\mu$ is totally even) such that $\mu_v$ is sufficiently ramified at every finite place $u$ in $S$ so as to make the $L$-factors at $u$ of $\Pi\mu_u$, $(\pi, r \otimes \mu_u)$, and their contragredients, all equal 1. This is evident for $L(s, \pi; r \otimes \mu)$ as its local factors are defined here to coincide with the corresponding Galois factors, and it is possible for $L(s, \Pi \otimes \mu)$ by the results of [JPSS1]. Comparing the global functional equations of both $L$-functions, and noting that twisting by $\mu$ does not change anything at infinity, we get

$$\prod_{v|\infty} L(s,\Pi_v)L(1-s,\pi_w^\vee;r)) \equiv \prod_{v|\infty} L(1-s,\Pi_v^\vee)L(s,\pi_w;r). \tag{4.2}$$

For *any* place $v$, archimedean or otherwise, for any $n \geq 1$, and for any cuspidal automorphic representation $\eta$ of $\mathrm{GL}(n,\mathbb{A}_F)$, it is known that $L(s,\eta_v)$ is holomorphic in $\Re(s) > \frac{1}{2} - t$, for some $t = t(\eta, v) > 0$ (see [BaR], Prop. 2.1, part B). Consequently, $L(s,\Pi_v)$ has no pole in common with $L(1-s,\Pi_v^\vee)$.

Thus the poles of $L(s,\Pi_\infty)$ are contained in those of $L(s,\pi_\infty;r)$. For the converse direction, we appeal to the factorization formula

$$L(s,\pi\times(\pi\circ\theta)) = L(s,\pi;r)L(s,\pi;r\otimes\delta), \tag{4.3}$$

where $\delta$ is the quadratic character of $F$ associated to $K/F$. This formula is evident from the definition (3.23). Since the local factors are never zero at any place $v$, the $v$-factor of $L(s,\pi;r)$ can have a pole somewhere only if the $v$-factor of $L(s,\pi\times(\pi\circ\theta))$ also does. On the other hand, by Theorem M of [Ra1], there is a unique isobaric automorphic representation $\pi\boxtimes(\pi\circ\theta)$ of $\mathrm{GL}(4,\mathbb{A}_F)$ whose standard $L$-function coincides with $L(s,\pi\times\pi')$. So, applying [BaR], Proposition 2.1, again, we see that $L(s,\pi_v;r)$ is holomorphic in $\Re(s) > \frac{1}{2} - t$, for some $t > 0$. Hence the poles of $L(s,\pi_\infty;r)$ are distinct from those of $L(1-s;\pi_\infty^\vee;r)$, and so must coincide with those of $L(s,\Pi_\infty)$. Let us use the customary notation $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ and $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. Using the duplication formula expressing $\Gamma_\mathbb{C}(s)$ as $\Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+1)$, and appealing to the fact that $L(s,\Pi_\infty)$ is a standard $L$-factor of $\mathrm{GL}(4,F_\infty)$ and that $L(s,\pi_\infty;r)$ is (by definition) a Galois $L$-factor, we may write

$$L(s,\Pi_\infty) = \prod_{j=1}^n \Gamma_\mathbb{R}(\frac{s+a_j}{2}) \quad \text{and} \quad L(s,\pi_\infty;r) = \prod_{j=1}^n \Gamma_\mathbb{R}(\frac{s+b_j}{2}),$$

for some complex numbers $a_j, b_j, 1 \leq j \leq n = 4[F:\mathbb{Q}]$. (The fact that $n$ is 4 times $[F:\mathbb{Q}]$ will play no role.) We may renumber the $a_j$ and $b_j$ and assume that there exists an integer $m$, with $0 \leq m \leq n$, such that $a_j = b_j$ for all $j > m$ and the sets $\{a_j | j \leq m\}$ and $\{b_j | j \leq m\}$ are totally disjoint. We have nothing to prove if $m = 0$, so assume that $m$ is positive. Now we appeal to the following

**Baby Lemma** Let $m > 0$ be an integer and let $\{a_j | j \leq m\}$, $\{b_j | j \leq m\}$ be subsets of $\mathbb{C}$ with empty intersection. Then the polar divisors of $L_1(s) := \prod_{j=1}^n \Gamma_\mathbb{R}(\frac{s+a_j}{2})$ and $L_2(s) := \prod_{j=1}^n \Gamma_\mathbb{R}(\frac{s+b_j}{2})$ cannot be the same.

*Proof of Baby Lemma.* Clearly the support of the polar divisor of $\Gamma_\mathbb{R}(\frac{s+c}{2})$ is, for any $c \in \mathbb{C}$, the set $\{-c-2k | k \in \mathbb{Z}, k \geq 0\}$. Suppose the poles of $L_1(s)$ and $L_2(s)$ coincide. Then $-a_1$ must be a pole of some $\Gamma_\mathbb{R}(\frac{s+b_j}{2})$. After renumbering the $b_j$, we may then assume that $a_1 = b_1 + 2k_1$ for some positive integer $k_1$. (Since $a_1 \neq b_1$, $k_1$ cannot be 0.) Now $-b_1$ will need to be a pole of some $\Gamma_\mathbb{R}(\frac{s+a_j}{2})$, and $j$ must be $\geq 2$. After renumbering the $a_j$ we may assume that $b_1 = a_2 + 2\ell_1$ for some $\ell_1 > 0$. We may continue thus and write, after suitable renumberings of the $a_j$ and the $b_j$, $a_2 = b_2 + 2k_2$, $b_2 = a_3 + 2\ell_2$, and so on. This leads to the string of inequalities $a_1 < b_1 < a_2 < b_2 < a_3 \cdots < a_m < b_m$. Then $-b_m$ is a pole of $L_2(s)$ and it is not a pole of $L_1(s)$. □

The identity $(L_\infty)$ now folows.

Next fix a finite place $v$ in $S$. Now choose a character $\mu$ which is unramified at $v$, but is highly ramified at every $u$ in $S-\{v\}$. Comparing the functional equations and arguing excatly as in the archimedean case, we get the desired identity $(L_v)$.

Next we prove the identity of epsilon factors. Fix any $v$ in $S$ and note that by [JPSS2], for $\mu_u$ sufficiently ramified at $u \in S-\{v\}$, the epsilon factor of $\Pi_u \otimes \mu_u$ depends only on $\mu_u$ and $\omega_{0,u}$, and the dependence is simple. Similarly, by [DeH], the epsilon factor at $u$ of $(\pi, r \otimes \mu)$ has the same dependence on $\mu_u$ and $\omega_{0,u}$; the reason we can apply [DeH] is that we have defined the local factors of $(\pi, r\otimes\mu)$ as those associated to the corresponding representations of the local Weil groups. The analogous statements hold for the contragredients, and this results in the identity $(\varepsilon_v)$ as we already know that the $L$-factors agree. This finishes the proof of the first part of Proposition 4.1.

It is left to prove the **cuspidality criterion**.

Note that (4.3) implies the identity

$$L(s, \pi \times (\pi\circ\theta)) \,=\, L(s, \Pi_K), \tag{4.4}$$

where $\Pi_K$ denotes the base change of the isobaric representation $\Pi$ to $K$. This forces, by the existence and uniqueness of $\pi \boxtimes (\pi\circ\theta)$ (cf. Theorem M of [Ra1]), we see that

$$\Pi_K \,\simeq\, \pi \boxtimes (\pi\circ\theta). \tag{4.5}$$

First suppose $\Pi$ is cuspidal. If $\Pi_K$ is not cuspidal, then by the theory of base change ([AC]), $\Pi$ must be automorphically induced from a cuspidal automorphic representation $\eta$ of $\mathrm{GL}(2, \mathbb{A}_K)$; write $\Pi = I_K^F(\eta)$. Then we must have

$$\Pi_K \,=\, \eta \boxplus (\eta\circ\theta). \tag{4.6}$$

The idea now is to compute the exterior square $L$-function of $\Pi_K$ in two different ways. We refer to [JS] and [BF] for the relevant facts about these degree 6 $L$-functions. On the one hand, (4.6) gives

$$L^S(s, \Pi_K, \Lambda^2) \,=\, L^S(s, \eta\times(\eta\circ\theta))L^S(s,\omega_\eta)L^S(s,\omega_\eta^\theta), \tag{4.7}$$

where $\omega$ (resp. $\omega_\eta^\theta$) is the central character of $\eta$ (resp. $\eta\circ\theta$). This can be seen easily at the unramified places $w$. Indeed, the above identity is induced by the following:

$$\Lambda^2(\sigma_w(\eta)\otimes\sigma_w(\eta\circ\theta)) \,\simeq\, \sigma_w(\eta)\otimes\sigma_w(\eta\circ\theta) \,\oplus\, \det(\sigma_w(\eta)) \,\oplus\, \det(\sigma_w(\eta\circ\theta)),$$

which is easy to verify. Consequently, $L^S(s,\Pi_K,\Lambda^2)$ is divisible by two abelian $L$-functions, namely $L^S(s,\omega_\eta)$ and $L^S(s,\omega_\eta^\theta)$. On the other hand, we also have the identity

$$\Lambda^2(\sigma_w(\pi)\otimes\sigma_w(\pi\circ\theta)) \,\simeq\, \det(\sigma_w(\pi))\otimes\mathrm{sym}^2(\sigma_w(\pi\circ\theta) \oplus \mathrm{sym}^2(\sigma_w(\pi))\otimes\det(\sigma_w(\pi\circ\theta)),$$

implying the following equality of $L$-functions:

$$L^S(s,\Pi_K,\Lambda^2) \,=\, L^S(s, \mathrm{sym}^2(\pi)\otimes\omega)L^S(s,\mathrm{sym}^2(\pi')\otimes(\omega\circ\theta)). \tag{4.8}$$

Suppose $\pi$ is **non-dihedral**. Then it is known (cf. [GeJ]) that $\mathrm{sym}^2(\pi)$ is cuspidal; so is $\mathrm{sym}^2(\pi\circ\theta)$. Consequently, thanks to (4.8), $L^S(s,\Pi_K,\Lambda^2)$ cannot be divisible by an abelian $L$-function, leading to a contradiction, and so $\Pi_K = \pi\boxtimes(\pi\circ\theta)$ must be cuspidal for non-dihedral $\pi$. In this case, if $\pi\circ\theta$ were isomorphic to $\pi\otimes\chi$ for a character $\chi$, then $L(s, \Pi_K\otimes(\chi\omega)^{-1})$ would have a pole at $s=1$, contradicting the cuspidality of $\Pi_K$. So the cuspidality of $\Pi$ implies, when $\pi$ is non-dihedral, that $\pi \circ \theta$ cannot be a character twist of $\pi$, as asserted in Theorem D.

Conversely, suppose that $\pi$ is non-diheedral and *not* equivalent to any character twist of $\pi \circ \theta$. Then by the cuspidality criterion in Theorem M of [Ra1], we know that $\pi \boxtimes (\pi \circ \theta)$ is a cuspidal automorphic representation of $\mathrm{GL}(4, \mathbb{A}_K)$. But this is just $\Pi_K$ by (4.5). Hence $\Pi$ base changes to a something cuspidal over K, and hence must be cusspidal itself (cf. [AC]). So the cuspidality criterion of Proposition 4.1 is now proven in the *non-dihedral* case.

Now let $\pi$ be **dihedral**, so that $\pi$ is an automorphically induced representation $I_M^K(\chi)$ for an idele class character $\chi$ of a quadratic extension $M$ of $K$. Denote the corresponding 2-dimenssional representation of the Weil group $W_K$ by $\sigma = \mathrm{Ind}_M^K(\chi)$. Since $\pi$ is cuspidal, $\sigma$ s irreducible. In this (dihedral) case there is a 4-dimensional global Asai representation $As(\sigma)$, of $W_F$ such that for any character $\nu$ of $F$, we have

$$L(s, \pi; r) \;=\; L(s, As(\sigma)), \tag{4.9}$$

and by the Tchebotarev density theorem,

$$As(\sigma)|_{W_K} \;\simeq\; \sigma \otimes \sigma^\theta. \tag{4.10}$$

**Lemma 4.11** *Let $\pi$ be dihedral with associated representation $\sigma = Ind_M^K(\mu)$ of $W_K$, where $M/K$ is quadratic and $\mu$ a character of $W_M$. Then $\Pi$ is cuspidal iff $As(\sigma)$ is irreducible.*

*Proof.* Suppose $As(\sigma)$ is reducible. It is known that the $\mathbb{C}$-representations of $W_F$ are completely reducible. So we may write $As(\sigma) = \oplus_j n_j \eta_j$ with eachh $n_j \geq 1$ and $\eta_j$ a proper irreducible summand of $As(\sigma)$, with $\eta_i, \eta_j$ inequivalent if $i \neq j$. Then it is elementary to see that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{W_F}(1, As(\sigma) \otimes As(\sigma)^\vee) \;=\; \sum_j n_j^2. \tag{4.12}$$

On the other hand, since $As(\sigma)$ is a $\mathbb{C}$-representation of $W_F$, we may use Brauer's theorem ([De]) and get a virtual sum decomposition

$$As(\sigma) \otimes As(\sigma)^\vee \;\simeq\; \oplus_{i=1}^r m_i \mathrm{Ind}_{L_i}^F(\lambda_i),$$

where $m_i$ is, for each $i \leq r$, an integer, $L_i$ a finite extension of $F$, and $\lambda_i$ a character of $W_{L_i}$. By the inductivity and additivity of $L$-functions, we see that

$$L(s, As(\sigma) \otimes As(\sigma)^\vee) \;=\; \prod_{i=1}^r L(s, \lambda_i)^{m_i}.$$

Moreover one knows by Hecke that $L(s, \lambda_i)$ is invertible at $s = 1$ unless $\lambda_i$ is the trivial character, in which case it has a pole of order 1. From this one gets

$$-\mathrm{ord}_{s=1} L(s, As(\sigma) \otimes As(\sigma)^\vee) \;=\; \dim_{\mathbb{C}} \mathrm{Hom}_{W_F}(1, As(\sigma) \otimes As(\sigma)^\vee) \tag{4.13}$$

Combining (4.12) and (4.13) we see that $L(s, \Pi \times \Pi^\vee)$ has a pole of order $> 2$ at $s = 1$, which can only happen if $\Pi$ is non-cuspidal ([JS]). The converse assertion holds by reversing the argument.

□

*Proof of the cuspidality criterion* (contd.):

Fix an extension of $\theta$ to the Galois closure $\tilde{M}$ of $M$ over $F$, and denote it again by $\theta$. Denote by $\alpha$ the non-trivial automorphism of $M/K$, and by $\epsilon$ the quadratic character of $W_K$ corresponding to $M/K$. Then $\epsilon^\theta$ is the quadratic character of $W_F$ corresponding to $M^\theta \neq M$. Note that

$$\sigma^\theta \simeq \mathrm{Ind}_{M^\theta}^K(\chi^\theta), \tag{4.14}$$

where $\chi^\theta$ is the character of $W_{M^\theta}$ defined by transporting $\chi$ via $\theta$.

We have to show now that $As(\sigma)$ is irreducible iff $M/F$ is non-Galois *and* the representation

$$\tau := \mathrm{Res}_M^K(\sigma^\theta) \otimes \chi \tag{4.15}$$

of $W_M$ does not extend to $W_F$.

Suppose $M/F$ is Galois. We have by Mackey and the definition of $As(\sigma)$,

$$\mathrm{Res}_K^F(As(\sigma)) \simeq \sigma \otimes \sigma^\theta \simeq \mathrm{Ind}_M^K(\chi\chi^\theta) \oplus \mathrm{Ind}_M^K(\chi\chi^{\theta\alpha}). \tag{4.16}$$

The first summand on the right is evidently $\theta$-invariant, and so extends to a 2-dimensional representation of $W_F$ occurring in $As(\sigma)$. Hence $As(\sigma)$ is reducible in this case.

So we may assume from here on that $M/F$ is non-Galois. We have by the definition of $\tau$,

$$\sigma \otimes \sigma^\theta \simeq \mathrm{Ind}_M^K(\tau). \tag{4.17}$$

Restricting to $W_M$ we get

$$\mathrm{Res}_M^F(As(\sigma)) \simeq \mathrm{Res}_M^K(\sigma \otimes \sigma^\theta) \simeq \mathrm{Res}_M^K(\mathrm{Ind}_M^K(\tau)) \simeq \tau \oplus \tau^\alpha. \tag{4.18}$$

First suppose that $\tau$ extends to $F$. Already the fact that it extends to a representation $\tau^K$, say, of $W_K$ implies that $\sigma \otimes \sigma^\theta$ is reducible. More precisely,

$$\sigma \otimes \sigma^\theta \simeq \tau^K \oplus (\tau^K \otimes \epsilon). \tag{4.19}$$

Now the existence of an extension of $\tau$ all the way to $F$ says that $\tau^K$ extends to a $W_F$-summand of dimension 2 in $As(\sigma)$. Hence $As(\sigma)$ is reducible.

Now we will assume that $As(\sigma)$ is reducible, and prove the more subtle *converse*. Then $\sigma \otimes \sigma^\theta$ is reducible, and by (4.17), either $\tau$ is reducible or $\tau$ extends to $K$.

First consider when $\tau$ *is irreducible*, but extends to a representation $\tau^K$ of $W_K$. Then (4.19) will hold, and moreover, the $\theta$-invariance of $\sigma \otimes \sigma^\theta$ implies one of the following:

(4.20)

(i) $\tau^K$ is $\theta$-invariant, i.e., $\tau$ extends to $F$;

(ii) $(\tau^K)^\theta \simeq \tau^K \otimes \epsilon$.

Suppose (ii) happens without (i). Then

$$(\tau^K)^\theta \simeq \tau^K \otimes \epsilon$$

and this implies by combining (4.17) and (4.19), that $As(\sigma)$ is irreducible, a contradiction! So (i) must hold and we are done if $\tau$ is irreducible.

It is left to consider when $\tau$ is *reducible*. For this case we need the following

**Lemma 4.21** *Let $\tau$ be reducible, with $M/F$ non-Galois. Then there exists a character $\mu$ of $W_M$ such that*

(a) $\sigma^\theta \simeq \mathrm{Ind}_M^K(\mu)$;

(b) $\tau \simeq (\mu \oplus \mu^\alpha) \otimes \chi$;
(c) $\mathrm{Res}^F_K(As(\sigma)) \simeq \mathrm{Ind}^K_M(\mu\chi) \oplus \mathrm{Ind}^K_M(\mu^\alpha\chi)$; *and*
(d) *We have*

$$\chi/\chi^\alpha = \mu/\mu^\alpha = \epsilon^\theta_M.$$

*Proof of Lemma.*
(a) $\tau$ is reducible iff $\mathrm{Res}^K_M(\sigma^\theta)$ is reducible, which happens, thanks to the irreducibility of $\sigma^\theta$, iff $\sigma^\theta$ is $\mathrm{Ind}^K_M(\mu)$ for some character $\mu$ of $W_M$.
(b) By Mackey, $\mathrm{Res}^K_M(\mathrm{Ind}^K_M(\mu)) \simeq \mu \oplus \mu^\alpha$. So by the definition (4.14), $\tau \simeq (\mu \oplus \mu^\alpha) \otimes \chi$.
(c) In view of (4.17), this part follows from (b).
(d) Since $M/F$ is non-Galois, the restriction $\epsilon^\theta_M$ of $\epsilon^\theta$ to $W_M$ is non-trivial. And as $\sigma^\theta$ is induced from $M^\theta$, we have $\sigma^\theta$ is isomorphic to $\sigma^\theta \otimes \epsilon^\theta$. Then (4.14) gives the isomorphism $\tau \simeq \tau \otimes \epsilon^\theta_M$. Using (b), we get $\mu\chi \oplus \mu^\alpha\chi \simeq \mu\chi\epsilon^\theta_M \oplus \mu^\alpha\chi\epsilon^\theta_M$. This forces the desired equality $\mu^\alpha/\mu = \epsilon^\theta_M$, which shows in particular that $\mu^\alpha/\mu$ extends to $K$. On the other hand, since $\sigma^\theta$ is induced from $M$, it follows that $\sigma$ is induced from $M^\theta$, and so we must have $\sigma \otimes \epsilon^\theta \simeq \sigma$. In other words, $\mathrm{Ind}^K_M(\chi\epsilon^\theta_M) \simeq \mathrm{Ind}^K_M(\chi)$. Restricting to $M$, we get the isomorphism $\chi\epsilon^\theta_M \oplus \chi^\alpha\epsilon^\theta_M \simeq \chi \oplus \chi^\alpha$, which gives $\chi^\alpha/\chi = \epsilon^\theta_M$. Done.

□

*End of proof of the cuspidality criterion*:
Using part (d) of the Lemma above, we see that

$$(\mu\chi)^\alpha = (\mu^\alpha/\mu)(\chi^\alpha/\chi)\mu\chi = (\epsilon^\theta_M)^2\mu\chi = \mu\chi.$$

Hence $\mu\chi$ extends to a character $\nu$, unique up to multiplication by $\epsilon$, of $W_K$, and consequently, $\mathrm{Ind}^K_M(\mu\chi) = \nu \oplus \nu\epsilon$. Then $\mu^\alpha\chi = \epsilon^\theta_M\mu\chi$ can be extended to $K$, either as $\epsilon^\theta\nu$ or as $\epsilon\epsilon^\theta\nu$. In any case, part (c) of the Lemma gives

$$\mathrm{Res}^F_K(As(\sigma)) \simeq \nu \oplus \nu\epsilon \oplus \nu\epsilon^\theta \oplus \nu\epsilon\epsilon^\theta.$$

Since $\mathrm{Res}^F_K(As(\sigma))$ is $\theta$-invariant, we need

$$\nu^\theta \in \{\nu, \nu\epsilon, \nu\epsilon^\theta, \nu\epsilon\epsilon^\theta\}.$$

Clearly $\nu^\theta$ cannot be $\nu\epsilon$ or $\nu\epsilon^\theta$ as it will contradict the identity $(\nu^\theta)^\theta = \nu$. So $\nu^\theta$ must be $\nu$ or $\nu\epsilon\epsilon^\theta$. In either case, take the extension $\nu\epsilon\epsilon^\theta$ of $\mu^\alpha\chi$, so that the representation

$$\tau^K := \nu \oplus \nu\epsilon\epsilon^\theta$$

of $W_K$ extends $\tau$ and more importantly, extends to a representation of $W_F$, as we needed to show.

□

## 5. Distinguished representations

Let $K/F$ be a quadratic extension of number fields with $\mathrm{Gal}(K/F) = \{1, \theta\}$. The object of this section is to establish Theorem D for the nice subclass of *distinguished* cusp forms $\pi$ ([HLR]) on GL(2)/K. It is necessary to treat this case separately as certain twists of the Asai $L$-function of $\pi$ will, in such a case, admit poles, complicating the argument using the converse theorem, which we will utilize for $\pi$ of *general type* in the next section.

We will use the following notation. If $\chi$ is an idele class character of $K$, we will write $\chi_0$ for its restriction to $F$. (This corrsponds to taking the *transfer* of the associated Galois character.) Moreover, if $\mu$ is a character of $F$, then we will write $\mu'$ to signify any character of $K$ such that $\mu = \mu'_0$. If $\mu'_1$ is another extension of $\mu$, then there exists a character $\nu$ of $K$ such that

$$\mu'_1 \; = \; \mu'(\nu/(\nu \circ \theta)). \tag{5.1}$$

This is because any character of $K$ whose restriction to $F$ is trivial lies in $\mathrm{Ker}(\theta - 1)$.

Let $\pi$ be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_K)$ with space $\mathcal{V}_\pi$. If $\mu$ is a unitary character of $F$, then $\pi$ is said to be *$\mu$-distinguished* ([HLR]) iff the following *$\mu$-period integral* is non-zero for some function $f$ in $\mathcal{V}_\pi$:

$$\mathcal{P}_\mu(f) := \int_{H(F)Z_H(F_\infty)^+\backslash H(\mathbb{A}_F)} \mu(\mathrm{d}et(h))f(h)dh, \tag{5.2}$$

where $H$ denotes $\mathrm{GL}(2)/F$ with center $Z_H$, and $dh$ is the quotient measure induced by the Haar measure on $H(\mathbb{A}_F)$. It may be useful to note for the uninitiated that when $F = \mathbb{Q}, K$ real quadratic, and $f \in \pi$ a holomorphic newform of weight $(2, 2)$, $\mathcal{P}_\mu(f)$ is the $\mu$-twisted integral of the $(1, 1)$ differential form $(2\pi i)^2 f(z_1, z_2)dz_1 \wedge \overline{dz_2}$ on the associated Hilbert modular surface over (the homology class of) the modular curve; so one is justified in calling this a period integral.

A basic result of [HLR], section 2, asserts that, once we have fixed an extension $\mu'$ of $\mu$, the necessary and sufficient condition for $\pi$ to be $\mu$-distinguished is that there exists a cuspidal automorphic representation $\pi_0$ of $H(\mathbb{A}_F)$ with central character $\nu\delta$ such that

$$\pi_{0,K} \; \simeq \; \pi \otimes \nu'\mu', \tag{5.3}$$

for a suitable extension $\nu'$ of $\nu$.

Fix such a $\mu$-distinguished $\pi$ with $(\pi_0, \nu)$ as above. Since $\pi \boxtimes (\pi \circ \theta)$ is $\theta$-invariant, it descends (by [AC]) to an isobaric automorphic representation of $\mathrm{GL}(4, \mathbb{A}_F)$. We can give an explicit candidate for this descent by setting

$$\Pi \; := \; sym^2(\pi_0) \otimes \delta(\mu\nu)^{-1} \; \boxplus \; \delta\mu^{-1}. \tag{5.4}$$

That the base change $\Pi_K$ is $\pi \boxtimes (\pi \circ \theta)$ is easily deduced from (5.3). There are at least four possible descents, namely by leaving in or removing the character $\delta$ at the places where it appears in (5.4), and this is why we needed to make a specific choice. Note also that the automorphic induction of $\pi$ to $F$ satisfies

$$I_K^F(\pi) \; \simeq \; \pi_0 \boxtimes I_K^F((\mu'\nu')^{-1}). \tag{5.5}$$

It suffices, by Proposition 4.1, to prove that the local factors of $L(s, \Pi)$ and $L(s, \pi; r)$ agree almost everywhere. Let $v$ be a finite place where $\pi$ and $K/F$ are unramified. If $v$ splits in $K$, the desired identity is immediate. So assume $v$ is inert, and denote the unique place of $K$ above it by $w$. Recall that the exterior square of a tensor product $V \otimes W$ is the direct sum of $\mathrm{sym}^2(V) \otimes \Lambda^2(W)$ and $\mathrm{sym}^2(W) \otimes \Lambda^2(V)$. Using this conjunction with (5.5), and by the compatibility of local and global automorphic induction, we have

$$\Lambda^2(\sigma_v(I_K^F((\pi))) \; \simeq \; sym^2(\sigma_v(\pi_0)) \otimes \delta_v(\mu_v\nu_v)^{-1} \; \oplus sym^2(\mathrm{Ind}_{K_w}^{F_v}((\mu'_w\nu'_w)^{-1})) \otimes \nu_v\delta_v. \tag{5.6}$$

We also have

$$\text{(5.7)} \qquad sym^2(\mathrm{Ind}_{K_w}^{F_v}((\mu'_w\nu'_w)^{-1})) \simeq \mathrm{Ind}_{K_w}^{F_v}((\mu'_w\nu'_w)^{-2}) \oplus (\mu_v\nu_v)^{-1}.$$

Combining these two identities with the fact that the induced module on the right of (5.7) is simply the induction of the determinant of $\sigma_w(\pi)$, we get, from the definition of the Asai representation

$$\text{(5.8)} \qquad As_v(\sigma(\pi)) \simeq sym^2(\sigma_v(\pi_0)) \otimes \delta_v(\mu_v\nu_v)^{-1} \oplus \delta_v\mu_v^{-1}.$$

Its $L$-factor, in view of (5.4), coincides with that of $\sigma_v(\Pi)$. Done.

□

## 6. Twisted Asai $L$-functions

Let $K/F$ be a quadratic extension of number fields, and let $\pi$ be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_K)$ of central character $\omega = \omega_\pi$.

Let $m \in \{1,2\}$. Then for any cuspidal automorphic representation $\eta$ of $\mathrm{GL}(m,\mathbb{A}_F)$, one may define the *$\eta$-twisted Asai $L$-function* of $\pi$ by setting

$$\text{(6.1)} \qquad L(s,\pi; r\otimes\eta) = \prod_v L(s, As(\sigma(\pi_w))\otimes\sigma_v(\eta)),$$

where for each $v$ we have chosen a place $w$ of $K$ above it. This $L$-function converges normally in a right half plane and defines an invertible holomorphic function there. Of course, when $m=1$, $\eta$ is simply an idele class character of $F$ with contragredient $\eta^\vee = \eta^{-1}$.

For any idele class character $\chi$ of $K$ with restriction $\chi_0$ to $F$, we get

$$\text{(6.2)} \qquad L(s,\pi\otimes\chi; r\otimes\eta) = L(s,\pi; r\otimes(\eta\otimes\chi_0)).$$

It also follows from the definition that

$$\text{(6.3)} \qquad L(s,\pi\boxtimes(\pi\circ\theta)\otimes\eta_K) = L(s,\pi;r\otimes\eta)L(s,\pi;r\otimes(\eta\otimes\delta)),$$

where again $\delta$ denotes the quadratic character of $F$ defined by $K/F$.

The object of this section is to establish, under some loal hypotheses, the needed analytic properties of these $\eta$-twisted Asai $L$-functions. We need the following

**Proposition 6.4** *Let $F$ be a totally imaginary number field, $K/F$ a quadratic extension with associated character $\delta$ of $W_F$ and non-trivial automorphism $\theta$ of $K/F$. Let $S$ be a non-empty finite set of finite places of $F$ which split in $K$, and $\pi$ a non-distinguished, cuspidal automorphic representation of $GL(2,\mathbb{A}_K)$, which is unramified at any finite place not above $S$. Assume moreover that the square of the central character $\omega_\pi$ is ramified at some place in $S$. Let $\eta$ be a cuspidal automorphic representation of $GL(m,\mathbb{A}_F)$, $m=1,2$, which is unramified at any finite place in $S$. Then we have the following:*

(MC) *$L(s,\pi;r\otimes\eta)$ admits a meromorphic continuation to the whole $s$-plane;*

(FE) *There is a functional equation*

$$L(1-s,\pi^\vee; r\otimes\eta^\vee) = \varepsilon(s,\pi;r\otimes\eta)L(s,\pi;r\otimes\eta);$$

(E) *$L(s,\pi;r\otimes\eta)$ is entire;* *and*

(BV) *$L(s,\pi;r\otimes\eta)$ is bounded in vertical strips of finite width.*

**Remark 6.5**:

(a) When we say that $\pi$ is non-distinguished, we mean that it is not $\mu$-distinguished for *any* character $\mu$ of $F$; so being distinguished (or not) is a property shared by all the character twists.

(b) We will get this without any hypotheses in the next section *after* completing the proof of Theorem D.

*Proof*. Let $\pi, \eta$ be as in the Proposition. Let $\phi = \phi_1 \otimes \phi_2$, with $\phi_1$, resp. $\phi_2$, lying in the space of $\pi$, resp. $\eta$. Denote by $\omega$ the restriction of $\omega_\pi$ to $F$ times the central character $\omega_\eta$ of $\eta$.

*(FE) and (MC)*: There is a closely related $L$-function of the pair $(\pi, \eta)$, which we will denote by $L_1(s, \pi; r \otimes \eta)$, given by an *integral representation*. For $m = 1$, this was done in the work of Harder, Langlands and Rapoport ([HLR]), generalizing the earlier construction of Asai for holomorphic Hilbert modular forms. For $m = 2$, which is the more difficult case, this was done in the work of Piatetski-Shapiro and Rallis ([PS-R]). (Their work was motivated by the earlier work of P. Garrett on the triple product $L$-function attached to triples of cusp forms on GL(2), and there is a formal similarity between such $L$-functions and GL(2)-twisted Asai $L$-functions.) In either case $L_1(s, \pi; r \otimes \eta)$ is defined to be the gcd of a family of global integrals

$$(6.6) \qquad \langle E(f_s), \varphi\rangle_H := \int_{C(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} E(h, f_s)\varphi(h)\omega^{-1}(\det(h))dh.$$

Here $H$ is the reductive group $\mathrm{GL}(2)/F$, resp. $R_{K/F}(\mathrm{GL}(2)/K) \times \mathrm{GL}(2)/F$, when $m = 1$, resp. $m = 2$, with center $C$, and $E(f_s)$ an Eisenstein series on $H(\mathbb{A}_F)$, resp. $\mathrm{GSp}(6, \mathbb{A}_F)$ associated to a *good* section in a representation induced from the Borel subgroup. We refer to the papers [HLR] and [PS-R] for details.

It is known that $L_1(s, \pi; r \otimes \eta)$ satisfies (FE) and (MC), and also that if $T$ is a finite set of places containing the archimedean ones and the places where $\pi$ or $\eta$ is ramified,

$$(6.7) \qquad L^T(s, \pi; r \otimes \eta) \ = \ L_1^T(s, \pi; r \otimes \eta).$$

So we will be done (for (FE) and (MC)) if we show the following

**Lemma 6.8** *Let $F, \pi, \eta$ be as in Proposition 6.4. Then the local factors of $L(s, \pi; r \otimes \eta)$ and $L_1(s, \pi; r \otimes \eta)$ agree at all the places.*

*Proof of Lemma*. Let $v$ be any place of $F$ which splits in $K$, say as $w, \theta v$, and $F_v = K_w = K_{\theta w}$. Then the $v$-factor of $L(s, \pi; r \otimes \eta)$, resp. $L_1(s, \pi; r \otimes \eta)$, is simply the triple product factor $L(s, \pi_w \times \pi_{\theta w} \times \eta_v)$. Similarly for the $\varepsilon$-factors. But it was shown in [Ra1], section 4.4, that

(6.9)

$$L(s, \pi_w \times \pi_{\theta w} \times \eta_v) = L(s, \pi_w \times \pi_{\theta w} \times \eta_v) \quad \text{and} \quad \varepsilon(s, \pi_w \times \pi_{\theta w} \times \eta_v) = \varepsilon(s, \pi_w \times \pi_{\theta w} \times \eta_v).$$

So we get the assertion at any such $v$.

Note that any archimedean $v$ splits in $K$ due to our hypothesis that $F$ is totally imaginary. Moreover, any $v$ in $S$ splits in $K$ by hypothesis, and $\pi$ is unramified at places not above $S$.

So we need only prove the assertion at any *finite* place $v$ such that (i) there is a unique place $w$ of $K$ above $v$, (ii) $\eta_v$ is ramified, and (iii) $\pi_w$ is unramified. Denote by $\lambda(\pi_w)$, resp. $\lambda(\eta_v)$, the (non-negative) *index of non-temperedness* of $\pi_w$, resp. $\eta_v$, as in [Ik1]; it is 0 if the representation is tempered and equals $t > 0$ if it is a complementary series representation defined by the characters $\nu|.|^t, \nu|.|^{-t}$ with $\nu$

unitary. One knows that this index is always less than 1/4 ([GeJ]). (By the recent results of Kim and Shahidi one knows even that it is less than 1/6, but we do not need this.) Put

$$\lambda(\pi_w, \eta_v) \,=\, 2\lambda(\pi_w) + \lambda(\eta_v). \tag{6.10}$$

Suppose $\eta_v$ is non-tempered. Then $\eta_v$ is the twist by a unitary character of an unramified representation, and the truth of the assertion follows from [PS-R], since $\pi_w$ is also unramified.

So we may assume that $\eta_v$ is tempered, so that

$$\lambda(\eta_v) \,=\, 0 \quad \text{and} \quad \lambda(\pi_w, \eta_v) \,<\, 1/2. \tag{6.11}$$

When $\eta$ is a subquotient of the principal series representation, the assertion then follows, because of (6.11), from Lemma 2.2 of [Ik2].

It remains to consider when $\eta_v$ is supercuspidal. Put

$$\gamma(s, \pi_w; r \otimes \eta_v) \,=\, \varepsilon(s, \pi_w; r \otimes \eta_v)\frac{L(1-s, \pi_w^\vee; r \otimes \eta_v^\vee)}{L(s, \pi_w; r \otimes \eta_v)}. \tag{6.12}$$

Similarly define $\gamma_1(s, \pi_w; r \otimes \eta_v)$. Since $\pi_w$ is in the principal series, by applying Prop.5.1 of [Ik1], we get

$$\gamma(s, \pi_w; r \otimes \eta_v) \,=\, \gamma_1(s, \pi_w; r \otimes \eta_v). \tag{6.13}$$

Thanks to (6.11), $L_1(s, \pi_w; r \otimes \eta_v)$, resp. $L(s, \pi_w; r \otimes \eta_v)$, has no pole in common with $L_1(1-s, \pi_w^\vee; r \otimes \eta_v^\vee)$, resp. $L(1-s, \pi_w^\vee; r \otimes \eta_v^\vee)$. (In fact, since $\sigma(\eta_v)$ is irreducible, it can be shown that $L(s, \pi_w; r \otimes \eta_v) = 1$.) Since the $\varepsilon$-factors are invertible, we get the desired equality of $L$ and $\varepsilon$-factors.

□

*(E)*: Let $m = 1$, and suppose $L(s, \pi; r \otimes \eta)$ has a pole for an idele class character $\mu$ of $F$. Then, up to replacing $\pi$ by $\pi \otimes |.|^{s_0}$ for some $s_0$, we may assume the pole to be at $s = 1$. Let $S$ be the finite set of places containing the archimedean and ramified places for $\pi$. Since the local factors have no zeros, the incomplete $L$-function $L^S(s, \pi; r \otimes \eta)$ also has a pole at $s = 1$. It is known that the pole must be simple. Moreover, by Asai's integral representation ([HLR]), the residue at $s = 1$ of this incomplete $L$-function is a non-zero multiple of the $\eta$-period $\mathcal{P}_\eta(f)$ (see (5.2)). This means $\pi$ is distinguished, which is ruled out by our hypothesis. Done.

So let $m = 2$. Suppose $L(s, \pi; r \otimes \eta)$ has a pole. By Lemma 6.8, we may work with $L_1(s, \pi; r \otimes \eta)$. Again, after replacing $\pi$ by $\pi \otimes |.|^{s_0}$ for suitable $s_0$, we may assume that the pole is at $s = 1$. Let $\Omega$ denote the central character of $\pi \otimes \eta$, viewed as a cuspidal automorphic representation of $H(\mathbb{A}_F)$. (Recall that $H = (R_{K/F}\mathrm{GL}(2)/K) \times \mathrm{GL}(2)/F$.) By a result of Ikeda (cf. Proposition 2.3 of [Ik1]), $L_1(s, \pi; r \otimes \eta)$ is entire if $\Omega^2$ is non-trivial.

In our case, by hypothesis, $\eta$, and hence its central character, is unramified at the places in $S$, while the square of the restriction of $\omega_\pi$ to $F$ is ramified at some place above $S$. This implies that $\Omega^2$ is ramified, hence non-trivial, at $S$. Hence we get (E).

*(BV)*: Let $S$ be the (finite) set of places of $F$ containing the archimedean ones and those finite ones ramifiying for $\pi$ or $\eta$. Then the integral representation of

[PS-R] implies the following:

$$L_1(s,\pi;r\otimes\eta)\prod_{v\in S}\frac{\Psi(f_{v,s};W_v)}{L_1(s,\pi_v;r\otimes\eta_v)} \;=\; \langle E(f_s),\varphi\otimes\varphi'\rangle_H, \tag{6.14}$$

where $\varphi$ (resp. $\varphi'$) is a cusp form in the space of $\pi$ (resp. $\eta$), $E(f_s)$ is the Sigel Eisenstein series on GSP(6)$/F$ (see [Ra1], sec.3.4, and [Ik1]) associated to a good section $f_s=\otimes_v f_{v,s}$, $\Psi(f_{v,s};W_v)$ is, for each $v$, a local integral having $L_1(s,\pi_v;r\otimes\eta_v)$ as its gcd for a suitable $f_{v,s}$ and Whittaker function $W)v$,

$$H \;:=\; \{(g,g')\in R_{K/F}\mathrm{G}L(2)/K\times \mathrm{G}L(2)/F \,|\, \mathrm{d}et(g)=\mathrm{d}et(g')\}$$

with center $C$, and

$$\langle E(f_s),\varphi\rangle_H := \int_{C(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} E(h,f_s)\varphi(h)dh = \prod_v \Psi(f_{v,s};W_v).$$

Using Lemma 3.4.5 of [Ra1] and the hypotheses of this Theorem, we get

**Lemma 6.15**. *For each $v$, the function $\frac{\Psi(f_{v,s};W_v)}{L_1(s,\pi_v;r\otimes\eta_v)}$ is entire and of bounded order, for a suitable choice of $f_{v,s}$.*

On the other hand, by Proposition 3.4.6 of [Ra1], we know that $E(f_s)$ is a function of bounded order. (Analogous results have been established in a very general setting in a recent preprint of W. Muller [Mul]). Since $\varphi\otimes\varphi'$ vanishes rapidly at infinity, we deduce, using (6.12), that $L_1(s,\pi;r\otimes\eta)$ is of bounded order in vertical strips of finite width. The same holds then for $L(s,\pi;r\otimes\eta)$ by Lemma 6.8. Furthermore, since this $L$-function has an Euler product, it is bounded for large positive $\Re(s)$, and hence also for large negative $\Re(s)$ by the functional equation. Applying the Phragman-Lindelöf theorem, we then conclude the boundedness in vertical strips of $L(s,\pi;r\otimes\eta)$ as asserted.

Now we are done with the proof of Poposition 6.4.

□

## 7. **Proof of Theorem D**

We begin with the following

**Lemma 7.1** Let $K/F$ be a quadratic extension of number fields with $F$ totally imaginary, and let $\pi$ be a cuspidal automorphic representation of GL$(2,\mathbb{A}_K)$ such that the finite places $w$ of $K$ where $\pi_w$ is ramified are all of degree 1 over $F$. Then there exists an irreducible, admissible, generic representation $\Pi=\otimes_v\Pi_v$ of GL$(4,\mathbb{A}_F)$ such that, for any cuspidal automorphic representation $\eta$ of G$(m,\mathbb{A}_F)$, $m=1,2$, and for any place $v$ of $F$, we have

$$L(s,\Pi_v\times\eta_v) \;=\; L(s,\pi_v;r\otimes\eta_v)$$

and

$$\varepsilon(s,\Pi_v\times\eta_v) \;=\; \varepsilon(s,\pi_v;r\otimes\eta_v).$$

*Proof.* Let $v$ be any place of $F$. Consider first the case when $v$ has a unique divisor $w$ in $K$. By hypothesis, $\pi_w$ is in the principal series i,e, it is an isobaric sum $\mu_1\boxplus\mu_2$. This means $\sigma_w(\pi_w)\simeq\mu_1\oplus\mu_2$, and

$$\Lambda^2(\mathrm{I}nd_{K_w}^{F_v}(\sigma_w(\pi_w)) \;\simeq\; \mathrm{I}nd_{K_w}^{F_v}(\mu_1)\otimes\mathrm{I}nd_{K_w}^{F_v}(\mu_2)\oplus\mu_{1,0}\delta_v\oplus\mu_{2,0}\delta_v. \tag{7.2}$$

Here we have used the fact that the determinant of $\mathrm{Ind}_{K_w}^{F_v}(\mu_j)$ is the product of the restriction $\mu_{j,0}$ of $\mu_j$ to $F_v^*$ times the quadratic character $\delta_v$ associated to $K_w/F_v$. Note also that

$$\mathrm{Ind}_{K_w}^{F_v}(\mu_1) \otimes \mathrm{Ind}_{K_w}^{F_v}(\mu_2) \simeq \mathrm{Ind}_{K_w}^{F_v}(\mu_1\mu_2) \otimes \mathrm{Ind}_{K_w}^{F_v}(\mu_1(\mu_2 \circ \theta)),$$

and

$$\mathrm{Ind}_{K_w}^{F_v}(\det(\sigma_w(\pi_w))) \simeq \mathrm{Ind}_{K_w}^{F_v}(\mu_1\mu_2).$$

Then by the definition (3.17) of the Asai representation, we get

$$As(\sigma_v(\mathrm{Ind}_{K_w}^{F_v}(\pi_w))) \simeq \mathrm{Ind}_{K_w}^{F_v}(\mu_1(\mu_2 \circ \theta)) \oplus \mu_{1,0}\delta_v \oplus \mu_{2,0}\delta_v. \tag{7.3}$$

Consequently, if we set

$$\Pi_v = I_{K_w}^{F_v}(\mu_1(\mu_2 \circ \theta)) \boxplus \mu_{1,0}\delta_v \boxplus \mu_{2,0}\delta_v, \tag{7.3}$$

we will have the properties asserted in the Lemma.

So we may assume that $v$ splits in $K$. Let $w, \theta w$ be the places above $v$. In this case $\mathrm{Ind}_{K_w}^{F_v}(\sigma_w(\pi))$ is just $\sigma_w(\pi) \oplus \sigma_{\theta w}(\pi)$, which implies in turn that

$$As(\sigma_v(\mathrm{Ind}_{K_w}^{F_v}(\pi_w))) \simeq \sigma_w(\pi) \otimes \sigma_{\theta w}(\pi). \tag{7.4}$$

So we may set

$$\Pi_v = \pi_w \boxtimes \pi_{\theta w}, \tag{7.5}$$

where $\boxtimes$ is the one constructed in [Ra1], and this satisfies the asserted properties relative to any $\eta$.

$\square$

**Proposition 7.6** *Let $(K/F, \pi, \Pi)$ be as in Lemma 7.1. Then $\Pi$ is an isobaric automorphic representation of $GL(4, \mathbb{A}_F)$.*

For any cuspidal automorphic representation $\eta$ of $\mathrm{GL}(m, \mathbb{A}_F)$, $m = 1, 2$, we will say, following Piatetski-Shapiro, that the pair $(\Pi, \eta)$ is *nice* if the following hold:

(MC) $L(s, \Pi \times \eta)$ admits a meromorphic continuation to the whole $s$-plane;

(FE) There is a functional equation

$$L(1 - s, \Pi^\vee \times \eta^\vee) = \varepsilon(s, \Pi \times \eta)L(s, \Pi \times \eta);$$

(E) $L(s, \Pi \times \eta)$ is entire; and

(BV) $L(s, \Pi \times \eta)$ is bounded in vertical strips of finite width.

Now we need to appeal to the following crucial

**Theorem 7.7** (Cogdell - Piatetski-Shapiro [CoPS1]) *Let $T$ be a fixed finite set of finite places of $F$. Let $\beta$ be an irreducible unitary, admissible, generic representation of $GL(4, \mathbb{A}_F)$. Suppose that for any cuspidal automorphic representation $\eta$ of $GL(m, \mathbb{A}_F)$, $m = 1, 2$, which is unramified at $T$, the pair $(\beta, \eta)$ is nice. Then $\beta$ is quasi-automorphic, i.e., there is an isobaric automorphic representation $\beta_1$ of $GL(4, A_F)$ such that $\beta_v \simeq \beta_{1,v}$ at almost all $v$.*

*Proof of Proposition 7.6*: We may assume that $\pi$ is not distinguished. Let $\Pi$ be as in Lemma 7.1. Pick any cuspidal automorphic representation $\eta$ of $\mathrm{GL}(m, \mathbb{A}_F)$ for $m \in \{1, 2\}$ which is unramified at the set $T$ of finite places where $\Pi$ is ramified. In view of Theorem 7.7, we have to show that $(\Pi, \eta)$ is nice. But in view of Proposition 6.4, it suffices to show that the $L$ and $\varepsilon$-factors of $(\Pi, \eta)$ agree at every place with the corresponding factors of $(\pi; r \otimes \eta)$. This is what was proved in Lemma 7.1, and so we get the quasi-automorphy of $\Pi$. Let $\Pi_1$ be an isobaric automorphic representation of $\mathrm{GL}(4, \mathbb{A}_F)$ which is almost everywhere equivalent to $\Pi$. Since the central characters of $\Pi$ and $\Pi'$ agree almost everywhere, they must be equal. Now we make the following

**Claim 7.8** *Let $v$ be any place. Then for any irreducible admissible representation $\beta$ of $GL(m, F_v)$, $m = 1, 2$, we have*

$$L(s, \Pi_v \times \beta) \ = \ L(s, \Pi_{1,v} \times \beta)$$

*and*

$$\varepsilon(s, \Pi_v \times \beta) \ = \ \varepsilon(s, \Pi_{1,v} \times \beta).$$

We know this at all the unramified places and also, by [Ra1], at all the places which split in $K$. So it remains only to prove the assertion at the finite set $S$ of non-split finite places where $\Pi_v$ is ramified. Fix a place $u$ in $S$ and a global atomorphic representation $\eta$ of $\mathrm{GL}(m, \mathbb{A}_F)$, $m = 1, 2$, with $\eta_u = \beta$. This is clearly possible for $m = 1$, and hence for $m = 2$ when $\beta$ in the principal series; if $m = 2$ and $\beta$ is square-integrable, use the trace formula to constuct such an $\eta$. Choose a global character $\chi$ which is 1 at $u$ and is *highly ramified* at every $v$ in $S - \{u\}$ (relative to $\pi_v, \eta_v$). Now look at any $v \in S - \{u\}$. Since $\Pi_v$ is by construction attached to the 4-dimensional representation $As(\sigma(\pi_w))$, $w$ being the unique place above $v$, the $L$ and $\varepsilon$-factors of the pair $(\Pi_v, \eta_v \otimes \chi_v)$ are, by the local Langlands correspondence ([HaT], [He]), the same as those of $As(\sigma(\pi_w)) \otimes \sigma(\eta_v) \otimes \chi_v$. Since $\chi_v$ is highly ramified, by [DeH], $L(s, \Pi_v \times \eta_v \otimes \chi_v)$ is 1, and the dependence of $\varepsilon(s, \Pi_v \times \eta_v \otimes \chi_v)$ on $\Pi_v$ is only through the central character $\omega_{\Pi_v}$. And by [JPSS1], the analogous assertions hold for $L(s, \Pi_{1,v} \times \eta_v \otimes \chi_v)$ and $\varepsilon(s, \Pi_{1,v} \times \eta_v \otimes \chi_v)$. From this we get, by comparing the functional equations of $L(s, \Pi \times \eta \otimes \chi)$ (cf. Proposition 6.4) and $L(s, \Pi_1 \times \eta \otimes \chi)$, the asserted identity at $u$. Since $u$ was arbitrary in $S$, the claim is proved.

Finally, Claim 7.8 shows, thanks to a result of Jeff Chen (cf. [Ch], [CoPS2]), that $\Pi_v \simeq \Pi_{1,v}$ at every $v$. So $\Pi$ is isomorphic to the isobaric representation $\Pi_1$. Done with the proof of Proposition 7.6.

□

It remains to prove **Theorem D in the general case**. Let $(K/F, \pi)$ be arbitrary, but with $\pi$ not distinguished. We will need to recall the following *descent criterion*, which is an extension of Proposition 4.2 of [B$\ell$R] to the case of $\mathrm{GL}(n)$ for arbitrary $n$:

**Proposition 7.9** ([Ra1], sec. 3.6) *Fix $n, p \in \mathbb{N}$ with $p$ prime, and a countably infinite set $J$. Let $F$ a number field, $\{F_j \,|\, j \in J\}$ a family of cyclic extensions of $F$ with $[F_j : F] = p$, and for each $j \in J$, let $\Pi_j$ be a cuspidal automorphic representation of $GL(n, \mathbb{A}_{F_j})$. Suppose that, for all $j, r \in J$, the base changes of $\Pi_j, \Pi_r$ to the compositum $F_jF_r$ satisfy*

$$(DC) \qquad (\Pi_j)_{F_jF_r} \ \simeq \ (\Pi_r)_{F_jF_r}.$$

*Then there exists a unique cuspidal automorphic representation $\Pi$ of $GL(n, \mathbb{A}_F)$ such that*

$$(\Pi)_{F_j} \ \simeq \ \Pi_j,$$

*for all but a finite number of $j$ in $J$.*

First we show how we may restrict to totally imaginary base fields. Indeed, enumerating the prime ideals of $F$ as $P_1, P_2, \cdots$, we may find quadratic extensions $F_j/F$, disjoint from $K/F$, such that for each $j \geq 1$, $F_j$ is totally imaginary *and* $P_j$ splits in $F_j$. Put $K_j = KF_j$, which will be a quadratic extension of $F_j$, and let $\pi_j$ denote $\pi_{K_j}$, the base change of $\pi$ to $K_j$. Let $J$ denote the complement in $\mathbb{N}$ of the finite set, possibly empty, of indices $j$ for which either $\pi_j$ is not cuspidal or $\pi$ is ramified at $P_j$. Associate irreducible, admissible, generic representations $\Pi_j$ to $(\pi_j, K_j/F_j)$ as in Lemma 7.1. Then by Proposition 7.6, each $\Pi_j$ is an isobaric automorphic representation. It is easy to see that the cuspidality condition (of Theorem D) will be satisfied by $\pi_j, r)$ for all but possibly a finite number of $j$. Shrink $J$ by excluding the indices for which this falis. Then by Proposition 4.1, $\Pi_j$ will be cuspidal for each $j$ in $J$. Applying Proposition 7.9, we then get the existence of a common descent $\Pi$ on $\mathrm{GL}(4)/F$. By construction, for any $j$ in $J$, the prime $P_j$ splits, say into $Q_j, \overline{Q}_j$ in $F_j$, and consequently,

$$(7.10) \qquad L(s, \Pi_{P_j}) = L(s, (\Pi_j)_{Q_j}) = L(s, (\pi_j)_{Q_j}; r) = L(s, \pi_{P_j}; r).$$

Since this holds at almost all primes $P_j$, $\Pi$ is the desired weak lifting.

Next we reduce to the case when the finite places where $\pi$ is ramified are unramified (for $K/F$) of degree 1 over $F$. Indeed, let $S$ be the (finite) set of finite places $w$ of $K$ where $\pi$ is ramified and $w$ is either ramified or is of degree 2 over $F$, so that every such $w$ sits above a unique place $u(w)$ of $F$. Put $S_0 = \{u(w) | w \in S\}$, and write $P(w)$ the prime ideal of $F$ defined by $u(w)$. Let $\{P_j | j \geq 1\}$ be the set of primes of $F$, and let $J$ denote the complement of the union of $S_0$. For each $j \in J$ choose as above a quadratic extension $F_j$ in which the prime $P_j$ of $F$ splits. We have a ot of freedon in choosing such an $F_j$, and we can take it such that for each $w \in S$, there is a unique place $w(j)$ of $F_j$ such that $F_{j_{w(j)}} \simeq K_w$. Then if we ut $K_j = KF_j$, the base change $\pi_j = \pi_{K_j}$ has the desired ramification property for each $j \in J$. Once we construct $\Pi_j$ over each $F_j$, we can again find a common descent to $F$ using Proposition 7.9. The identity (7.10) will again hold for almost all $j$, and so $\Pi$ will be a weak lifting of $(\pi, r)$.

So we may assume from now on that

(i) $F$ is totally complex,

(ii) $\pi$ is a non-distinguished, cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_K)$, and

(iii) $\pi$ is unramified at the primes of $K$ which are inert or ramified over $F$.

Then the hypotheses of Lemma 7.1 are satisfied, and so we have the existence of the weak lift $\Pi$ on $\mathrm{GL}(4)/F$ by Proposition 7.6. It is also the strong lift by Proposition 4.1.

Theorem D is now proved.

□

## 8. **New cases of Artin's conjecture**

Let $\rho, \rho'$ be continuous $\mathbb{C}$-representations of solvable GO(4)-type. By Theorem A, they are modular, associated to isobaric automorphic representations $\pi$, $\pi'$ of

$\mathrm{GL}(4, \mathbb{A}_F)$. Then

$$L(s, \rho \otimes \rho') = L(s, \pi_f \times \pi'_f). \tag{8.1}$$

The Rankin-Selberg theory of Jacquet, Piatetski-Shapiro and Shalika ([JPSS2], [JS]), and of Shahidi ([Sh1,2]), says that the $L$-function on the right is entire (see [MW] and the references therein). So Corollary B follows immediately.

Now we show how this gives new examples where Artin's conjecture holds. Fix any quadratic extension $E/F$ with non-trivial automorphism $\theta$. Pick three distinct primes $Q_1, Q_2, Q_3$ of $E$ which are inert over $F$. For each $j \leq 3$, let $P_j$ denote the prime of $F$ below $Q_j$, and $\mathbb{F}(j)$ (resp. $\mathbb{F}_0(j)$) the residue field $\mathfrak{O}_E/Q_j$ (resp. $\mathfrak{O}_F/P_j$). Note that $\theta$ induces the non-trivial automorphisms, again denoted by $\theta$, of the $\mathbb{F}(j)$, fixing $\mathbb{F}_0(j)$ pointwise. Choose polynomials $f_j(X) \in \mathbb{F}(j)[X]$, $j \leq 3$, and a polynomial $f(X) \in \mathfrak{O}_K[X]$, as follows:

(i) $f_1$ is irreducible of degree 4, and it does not belong to $\mathbb{F}_0(1)[X]$;
(ii) $f_2$ is the product of an irreducible quadratic polynomial $g_2(X)$ with two distinct linear polynomials;
(iii) $f_3$ is the product of an irreducible cubic polynomial and a linear polynomial;
(iv) $f$ is a quartic polynomial with discriminant $D(f)$, such that $f(X) \equiv f_j(X) (\mathrm{mod}\ Q_j)$ for each $j \leq 3$; and
(v) $D(f)^\theta$ is not a square in $E[\sqrt{D(f)}]$.

(i)-(iv) are easy to achieve. If a particular choice of $f$ satisfying (i)-(iv) fails to satisfy (v), we may then replace it by its sum with a suitable polynomial $h \in \mathfrak{O}_E[X]$ of degree $\leq 4$ with $h \equiv 0 (\mathrm{mod}\ Q_j)$ for each $j$, such that (v) holds.

**Lemma 8.2** *Let $E/F, f, f_j, j \leq 3$ be as above, satisfying (i)-(v). Denote by $K$, resp. $K^\theta$, the splitting field of $f$, resp. $f^\theta$, over $E$. Then $K, K^\theta$ are linearly disjoint over $E$. Moreover, as abstract groups,*

$$\mathrm{Gal}(K/E) \simeq \mathrm{Gal}(K^\theta/E) \simeq S_4.$$

*Proof.* It is standard that the conditions (i)-(iv) imply that $K/E$ is an $S_4$-extension. This is because these properties realize $\mathrm{Gal}(K/E)$ as a transitive subgroup of $S_4$ containing a transposition (coming from $f_2$) and a 3-cycle (coming from $f_3$), and any such group must be all of $S_4$. Since $f_1$ does not belong to $\mathbb{F}_0(1)[X]$, $f$ does not belong to $\mathfrak{O}_F[X]$, and so $K$ does not arise by composing $E$ with an $S_4$-extension of $F$. By construction, $f^\theta$ has the same properties, and so $K^\theta/F$ is also an $S_4$-extension.

It remains to show that the field $L := K \cap K^\theta$, which contains $E$, is just $E$.

Suppose $L \neq E$. Put $M = KK^\theta$ and $G = \mathrm{Gal}(M/E)$. Then

$$G \simeq \mathrm{Gal}(K/L) \times \mathrm{Gal}(K^\theta/L) \subset S_4 \times S_4. \tag{8.3}$$

Since $K, K^\theta$ are Galois over $E$, so is $L$, and thus $\mathrm{Gal}(K/L)$ and $\mathrm{Gal}(K^\theta/L)$ are both proper normal subgroups of $S_4$. It is well known that the only proper normal subgroups are $A_4$ and the Kelin group $V$. Either way, $\mathrm{Gal}(K/L)$ must be a subgroup of the alternating subgroup of $\mathrm{Gal}(K/E) \simeq S_4$, and so $L$ contains $E[\sqrt{D(f)}]$. Similarly, $\mathrm{Gal}(K^\theta/L)$ must sit inside the alternating subgroup of $\mathrm{Gal}(K^\theta/E) \simeq S_4$, and thus $L$ must also contain $E[\sqrt{D(f^\theta)}]$. Then the groups $\mathrm{Gal}(E[\sqrt{D(f)}]/E)$ and $\mathrm{Gal}(E[\sqrt{D(f^\theta)}]/E)$ are both quotients of $\mathrm{Gal}(K/E)$. Since $A_4$ is the only subgroup of $S_4$ of index 2, we must have $E[\sqrt{D(f)}] = E[\sqrt{D(f^\theta)}]$. Then $D(f^\theta)$ will be a square in $E[\sqrt{D(f)}]$, which contradicts (v). Hence the Lemma.

□

There are clearly an infinite number of choices for $f$ satisfying such conditions. We can also vary the inert triple $(Q_1, Q_2, Q_3)$.

Since $S_4$ is a subgroup of $\mathrm{PGL}(2,\mathbb{C})$, we get a projective representation

$$\overline{\sigma} : \mathrm{Gal}(\overline{F}/E) \to \mathrm{PGL}(2,\mathbb{C}), \tag{8.4}$$

such that $\overline{F}^{\mathrm{Ker}(\overline{\sigma})} = K$. Its $\theta$-conjugate $\sigma^\theta$ corresponds to $K^\theta$.

Fix a lifting

$$\sigma : \mathrm{Gal}(\overline{F}/E) \to \mathrm{GL}(2,\mathbb{C})$$

of $\overline{\sigma}$, which is possible by Tate's theorem (see section 2). Denote by $L$ the extension of $F$ corresponding to $\ker(\sigma)$. Clearly, $L \supset K$, and we have a central extension

$$1 \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(L/E) \to \mathrm{Gal}(K/E) \simeq S_4 \to 1. \tag{8.5}$$

Let us now make precise the structure of $\mathrm{Gal}(L/F)$. There are two double covers, up to isomorphism, of $S_4$, denoted $\tilde{S}_4$ and $\hat{S}_4$. $\mathrm{Gal}(L/K)$ will be a cyclic subgroup $C$ of order $2m$, for some $m \geq 1$, lying in the center $Z \simeq \mathbb{C}^*$ of $\mathrm{GL}(2,\mathbb{C})$, such that

$$\mathrm{Gal}(L/E) \simeq C \times_{\{\pm I\}} S_4^*, \tag{8.6}$$

with

$$S_4^* = \tilde{S}_4 \quad \text{or} \quad \hat{S}_4.$$

Here $\times_{\{\pm I\}}$ denotes the direct product with the common subgroup $\{\pm I\}$ amalgamated. One can identify $\tilde{S}_4$ with $\mathrm{GL}(3,\mathbb{F}_3)$, equipped with a natural representation $\pi$ into $\mathrm{GL}(2,\mathbb{C})$. The transpositions $\tau$ in $S_4$ lift to elements $\tilde{\tau}$ of order 2 in $\tilde{S}_4$. Denote by $\tilde{A}_4$ the inverse image of $A_4$ in $\tilde{S}_4$. It is the unique double cover of $A_4$, denoted $2A_4$ in the *Atlas*, identifiable with $\mathrm{SL}(2,\mathbb{F}_3)$. The (matrix representation of the) group $\hat{S}_4$ can be constructed as follows: Multiply all the elements of $\pi(\tilde{S}_4)$ outside (resp. inside) $\pi(\tilde{A}_4)$ by $\sqrt{-1}I$ (resp. $I$). Under this (set-theoretic) map from $\tilde{S}_4$ to $\hat{S}_4$, each $\tilde{\tau}$ goes to an element $\hat{\tau}$ of order 4, and this is what distinguishes $\tilde{S}_4$ from $\hat{S}_4$. For further reference see [Wa], pp.9–10.

We also have the $\theta$-conjugate representation $\sigma^\theta$ of $\mathrm{Gal}(\overline{F}/E)$ into $\mathrm{GL}(2,\mathbb{C})$. The corresponding field $L^\theta$ will contain $K^\theta$ and be linearly disjoint from $L$ over $E$. Clearly, the Galois group $G$ of $LL^\theta$ over $F$ is a non-trivial extension of $\mathbb{Z}/2$ by $\mathrm{Gal}(L/E)\times\mathrm{Gal}(L^\theta/E)$, and the resulting representation

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(4,\mathbb{C}) \tag{8.7}$$

of is evidently irreducible and of GO(4)-type.

Now choose a disjoint quadratic extension $E'/F$ and a quartic polynomial $g$ satisfying analogous properties over $E'$. We can arrange, in an infinite number of ways, for the resulting extension of $F$ to be linearly disjoint from $LL^\theta$. Denoting by $\rho'$ the corresponding representation of $\mathrm{Gal}(\overline{F}/F)$, we see that $\rho \otimes \rho'$ is *irreducible* and satisfies the Artin conjecture by Corollary B.

It remains to check that this is not covered by known cases in lower dimensions. For this it suffices to prove the following

**Proposition 8.8** *Let $\rho, \rho'$ be as above. Then $\rho, \rho'$ and $\rho \otimes \rho'$ are primitive representations of Gal$(\overline{F}/F)$.*

By a *primitive* representation of $\mathrm{Gal}(\overline{F}/F)$ we mean a representation which is not induced by a representation of a proper subgroup.

*Proof.*

To begin, it is well known that the irreducible $\sigma$ is primitive; so is $\sigma^\theta$.

Suppose $\rho$ is not primitive. Write

$$\rho = \mathrm{Ind}_N^F(\tau),$$

for some representation $\tau$ of $\mathrm{Gal}(K/N)$, where $N \neq F$ is an intermediate field. Note that $N$ could not be contained in $E$, for otherwise the restriction of $\rho$ would be reducible, contradicting the fact that

$$\mathrm{Res}_E^F(\rho) \simeq \sigma \otimes \sigma^\theta, \tag{8.9}$$

which is irreducible as $K, K^\theta$ are linearly disjoint over $E$. Since $[E : F] = 2$, this implies that $\mathrm{Gal}(K/E)$ and $\mathrm{Gal}(K/N)$ generate $\mathrm{Gal}(K/F)$. Consequently, by Mackey,

$$\mathrm{Res}_E^F(\rho) \simeq \mathrm{Ind}_R^E(\tau_0), \tag{8.10}$$

whre $R = E \cap N$ and $\tau_0$ is the restriction of $\tau$ to $\mathrm{Gal}(K/R)$. In view of (8.9) and (8.10), it suffices to show that $\sigma \otimes \sigma^\theta$ is primitive. Now we appeal to the following

**Theorem** (Aschbacher [A]) *Let $G_1, G_2$ be finite groups and $\pi_i : G_i \to \mathrm{GL}(V_i)$ be finite-dimensional $\mathbb{C}$-representations which are primitive. Then $\pi_1 \otimes \pi_2$ is a primitive representation of $G$.*

It may be useful to note that when $G_1, G_2$ are solvable, the proof in [A] does not appeal to the classficiation of finite grous.

We have already shown that our $\sigma, \sigma'$ are primitive. Moreover, $\mathrm{Gal}(KK^\theta/E)$ is isomorpic to $\mathrm{Gal}(K/E) \times \mathrm{Gal}(K^\theta/E)$. So we may conclude that $\sigma_1 \otimes \sigma_2$ remains primitive. In fact this can be verified by a direct, though laborious, computation (in our special case), which is what we did originally leading us to pose the general question to Aschbacher, but now that this primitivity question has been solved in general, we can do no better than to refer to [A].

Similarly, $\rho'$ is primitive. Applying [A], Theorem 1, again, we see that $\rho \otimes \rho'$ is also primitive. Done.

□

## 9. The strong Dedekind conjecture: An example with application

The *Dedekind conjecture* asserts that for any finite extension $N/F$ of number fields, the zeta function of $F$ divides the zeta function of $N$, i.e., $\zeta_N(s) = \zeta_F(s)L(s)$ with $L(s)$ entire. If $\tilde{N}$ denotes the Galois closure of $N$ over $F$, then one has

$$\zeta_N(s) = \zeta_F(s)L(s, a_{N/F}), \tag{9.1}$$

where $a_{N/F}$ is defined by

$$a_{N/F} \oplus 1_F = \mathrm{Ind}_N^F(1_N). \tag{9.2}$$

Here $1_K$ denotes, for any $K$, the trivial representation of $\mathrm{Gal}(\overline{K}/K)$. Since $1_F$ does not occur in $a_{N/F}$, the Artin conjecture for $a_{N/F}$ implies the Dedekind conjecture for $N/F$. The Dedekind conjecture is known to be true when $N/F$ is Galois, which follows from the work of Aramata and Brauer, and when $\tilde{N}/F$ is solvable, proved independently by Uchida and Van der Waall. We refer to [Mu-R] for a detailed discussion of known results, references and variants. We should also note that S. Rallis has a very interesting program for studying the ratios $\zeta_N(s)/\zeta_F(s)$ via the adjoint $L$-functions of cusp forms $\pi$ on $\mathrm{GL}([N:F])$. To elaborate a bit, the version of the trace formula due to Jacquet and Zagier ([JZ]) suggests that the divisibility of $\zeta_K(s)$ by $\zeta_F(s)$ for all commutative algebras $K/F$ of dimension $n$ is equivalent to the divisibility of $L(s,\pi\times\pi^\vee)$ by $\zeta_F(s)$ for all unitary cuspidal representations $\pi$ of $\mathrm{GL}(n,\mathbb{A}_F)$ of trivial central character. The adjoint $L$-function of $\pi$ is just the ratio $L(s,\pi\times\pi^\vee)/\zeta_F(s)$; so the divisibility in the known case $n=2$ can be rederived by using the properties of the symmetric square $L$-functions of $\mathrm{GL}(2)/F$ ([GeJ]). Rallis's idea is to study these via certain Eisenstein series on larger groups $G$, and the case $n=3$ has been carried out in his intriguing joint paper [JiR], written with Dihua Jiang, where $G=G_2$.

We will say that $N/F$ satisfies the *strong Dedekind Conjecture* if $a_{N/F}$ is modular, i.e., if there exists an isobaric automorphic representation of $\mathrm{GL}(m,\mathbb{A}_F)$, $m=[N:F]-1$, with the same $L$-function as that of $a_{N/F}$. This is known in the following cases:

(i) $N/F$ is *Galois and solvable*; and

(ii) $[N:F]\le 4$.

The first case is by the work of Arthur and Clozel ([AC]). In the second case one knows something stronger, not known in case (i) (unless $K/F$ is nilpotent), namely that *every irreducible occurring in $a_{N/F}$ is modular.*

When $N$ is a *non-normal cubic* extension of $F$, $a_{N/F}$ is the unique irreducible 2-dimensional representation of $\mathrm{Gal}(\tilde{N}/F)\simeq S_3$; it is dihedral, induced by either of the non-trivial characters of the normal subgroup $A_3=\mathbb{Z}/3$. When $N$ is a *non-normal quartic* extension of $F$, we may assume that $\mathrm{Gal}(\tilde{N}/F)$ is $A_4$ or $S_4$, as otherwise the image is nilpotent. Then $a_{N/F}$ is an irreducible representation induced by a character of a 2-Sylow subgroup $P$. In the $A_4$-case, $P$ is the Klein group $V$, which is normal in $A_4$, and so the modularity can be deduced from [AC] once again. But in the $S_4$-case, $P$ is not normal, and one must appeal to the work of Jacquet, Piatetski-Shapiro and Shalika ([JPSS1]) on GL(3); their method is the converse theorem requiring only twists by characters.

**Theorem 9.3** *Let $F$ be a number field, $\rho$ a solvable $\mathbb{C}$-representation of $Gal(\overline{F}/F)$ of $GO(4)$-type, and $M$ the fixed field of the kernel of $\rho$. Then the strong Dedekind conjecture holds for any intermediate field $N$ (of $M/F$) such that $[N:F]=3^a$, $a\ge 0$. In fact, any irreducible summand of $a_{N/F}$ is modular.*

Before beginning the proof, we will indicate some consequences. Given any cuspidal automorphic representation $\pi$ of $\mathrm{GL}(n,\mathbb{A}_F)$, and a finite extension $N/F$, one can formally define the *base change* $\pi_N$ as an *admissible* representation of $\mathrm{GL}(n,\mathbb{A}_N)$. Here is a way to do it. By the local Langlands correspondence for $\mathrm{GL}(n)$ proved by Harris-Taylor ([HaT]) and Henniart ([He]), we may associate to $\pi_v$, at any place $v$ of $F$, a well defined local base change $\pi_{N_w}$ of $\mathrm{GL}(n,K_w)$, for any place $w$ of $N$ above $v$. To be precise, we take $\pi_{N_w}$ to be the representation associated to the restriction to $W_{K_w}\times\mathrm{SL}(2,\mathbb{C})$ of the $n$-dimensional representation $\sigma(\pi_v)$ of $W_{F_v}\times\mathrm{SL}(2,\mathbb{C})$ defined by $\pi_v$. When $\pi_v$ is unramified, $\pi_{N_w}$ will also be unramified, so that the restricted tensor product of the $\pi_{N_w}$, as $w$ runs over all the places of $N$, makes sense as an admissible representation of $\mathrm{GL}(n,\mathbb{A}_N)$. Let $\pi_N$

denote $\otimes_w \pi_{N_w}$. Of course it is a big open problem to know that $\pi_N$ is automorphic, which is unknown except when $N/F$ is solvable and *normal* ([AC]), and we do not adddress this difficult question here - at all. But it turns out one can still say a little bit about the analytic properties of $L(s, \pi_N)$ for special $N/F$. More precisely, one has the following

**Corollary 9.4** *Let $N/F$ be as in Theorem 9.3. Then for any unitary, cuspidal automorphic representation $\pi$ of $GL(n, \mathbb{A}_F)$, $L(s, \pi_N)$ admits a meromorphic continuation to the whole s-plane with the expected functional equation relating $s$ to $1-s$. Moreover, $L(s, \pi)$ divides $L(s, \pi_N)$.*

Indeed, by Theorem 9.3, $a_{N/F}$ corresponds to an isobaric automorphic representation $\eta$ of $\mathrm{GL}(m, \mathbb{A}_F)$, $m = [N : F] - 1$. It follows that

$$L(s, \pi_N) \;=\; L(s, \pi)L(s, \pi \times \eta). \tag{9.5}$$

The assertions of the corollary then follow immediately by applying the Rankin-Selberg theory of Jacquet, Piatetski-Shapiro and Shalika, and of Shahidi, together with the fact that the local Langlands correspondence for $\mathrm{GL}(n)$ preserves $L$ and $\varepsilon$-factors of pairs.

**Remark 9.6** We became interested in this due to the following problem. Consider any elliptic curve $E$ over $\mathbb{Q}$. Then for any number field $K$, the Birch and Swinnerton-Dyer conjecture says that the rank of $E(K)$ as an abelian group is the order of zero of $L(s, E_K)$, where $E_K$ denotes the base change of $E$ to $K$, assuming $L(s, E_K)$ makes sense at $s = 1$. Since $E(\mathbb{Q}) \subset E(K)$, one expects to have

$$\mathrm{ord}_{s=1} L(s, E_K) \;\geq\; \mathrm{ord}_{s=1} L(s, E). \tag{?}$$

By the monumental work of Wiles ([W]) and Taylor-Wiles ([TW]), followed by that of Breuil-Conrad-Diamond-Taylor ([BCDT]), we know that $E$ is modular, i.e., $L(s, E) = L(s - \frac{1}{2}, \pi_f)$, for a unitary cusp form $\pi = \pi_\infty \otimes \pi_f$ on $\mathrm{GL}(2)/\mathbb{Q}$ of weight 2, i.e., with $\pi_\infty$ in the lowest discrete series. So if $K$ is a 3-primary extension of $\mathbb{Q}$ contained in the field cut out by a continuous, solvable representation $\rho$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\mathrm{GO}(4, \mathbb{C})$, Corollary 9.4 shows that the expectation (?) does hold in that case.

*Proof of Theorem 9.3.* We will use induction on the order of $G := \rho(\mathrm{Gal}(\overline{F}/F))$. There is nothing to prove when $|G| = 1$, so assume that $|G| > 1$ and that the assertion holds for all solvable representations $\rho'$ of GO(4)-type with image of order smaller than $|G|$.

Write $|G| = 3^{r+a} n$, with $(3, n) = 1$, $r \geq 0$. Since $G$ is solvable, there exist subgroups of order $n$, called *Hall subgroups* (relative to the set $S$ of prime divisors of $n$). They form a single conjugacy class, which we will denote by $\mathcal{H}$.

Let $R$ denote the image of $\mathrm{Gal}(\overline{F}/N)$ under $\rho$. We claim that $R$ contains some $H$ in $\mathcal{H}$. Indeed, since $R$ is solvable of order $3^r n$, it contains its own Hall subgroups of order $n$, which must belong to $\mathcal{H}$.

Let $L$ be the fixed field of $H$ so that $K \supset L \supset N \supset F$. By the transitivity of induction, we get

$$a_{L/F} \;\simeq\; \mathrm{Ind}_N^F(\mathrm{Ind}_L^N(1_K)) \ominus 1_F \;\simeq\; \mathrm{Ind}_N^F(a_{L/N}) \oplus a_{N/F}. \tag{9.7}$$

Hence it suffices to prove the assertion for $L/F$.

Suppose $G$ is contained in $\mathrm{SGO}(4, \mathbb{C})$. Then, as we have seen in section 1, it is given by a product $G_1 \times G_2$ with each $G_i$ in $\mathrm{GL}(2, \mathbb{C})$. If $H_i$ denotes the Hall subgroup (relative to $S$) in $G_i$, then $H_1 \times H_2$ is necessarily in $\mathcal{H}$. Since $H$ is conjugate to $H_1 \times H_2$, the corresponding fields have the same zeta function, and we may, without loss, assume that

$$H = H_1 \times H_2, \quad \text{and} \quad L = L_1 L_2, \tag{9.8}$$

where $L_i$ is, for each $i$, the fixed field of $H_i$.

**Lemma 9.9** *Every irreducible subrepresentation $\tau$ of $a_{L_i/F}$, $i \in \{1,2\}$, is of dimension $\leq 2$.*

*Proof of Lemma* Since $G_i$ is a solvable subgroup of $\mathrm{GL}(2,\mathbb{C})$, its image $\overline{G}_i$, say, in $\mathrm{PGL}(2,\mathbb{C})$ is either abelian or dihedral or $A_4$ or $S_4$. It evidently suffices to consider the latter two cases. Note that $G_i$ is an extension of $A_4$ by a *central* subgroup $C$, and the restriction to $C$ of any irreducible $\tau$ occurring in $a_{L_i/F}$ will be, by Schur, of the form $\dim(\tau)\omega_\tau$, for a character $\omega_\tau : C \to \mathbb{C}^*$. Let $\overline{H}_i$ denote the image of the Hall subgroup $H_i$ in $\overline{G}_i$, and let $\overline{L}_i$ be the fixed field of $CH_i$, so that $L_i \supset \overline{L}_i \supset F$.

Suppose first that $\overline{G}_i$ is $A_4$. Then $\overline{H}_i$ is necessarily the Klein group $V \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, which is normal in $A_4$, and the Galois group of $\overline{L}_i/F$ is $A_4/V$. Clearly, $a_{\overline{L}_i/F} \simeq \delta \oplus \delta^2$, where $\delta$ is a generator of the character group of $\mathrm{Gal}(\overline{L}_i/F)$. It follows that any irreducible occurring in $a_{L_i/F}$ is one dimensional, obtained by pasting onto $\delta^j$, $j \in \{1,2\}$, a character $\omega$ of $C$ trivial on $\{\pm I\}$.

Now consider when $\overline{G}_i$ is $S_4$. Now $\overline{H}_i$ is a non-normal subgroup of $S_4$ of index 3. Clearly, $S_4$ is generated by $A_4$ and $\overline{H}_i$, and so by Mackey,

$$\mathrm{Res}^{G_i}_{A_4}(\mathrm{Ind}^{G_i}_{\overline{H}_i}(1)) \simeq \mathrm{Ind}^{A_4}_V(1) \simeq 1 \oplus \delta \oplus \delta^2.$$

Since the cubic character $\delta$ does not extend to $S_4$ (whose abelianization is $\mathbb{Z}/2$), it follows that

$$a_{\overline{L}_i/F} \simeq \mathrm{Ind}^F_k(\delta), \tag{9.10}$$

which is irreducible. Here $k$ denotes the quadratic extension of $F$ corresponding to (the inverse image of) $A_4$. So any irreducible $\tau$ occurring in $a_{L-i/F}$ is obtained by pasting onto this dihedral representation a character $\omega$ of $C$ agreeing on $\{\pm I\}$. The Lemma is now proved.

□

*Proof of Theorem 9.3* (contd.) Thanks to (9.8) we have

$$\mathrm{Ind}^F_L(1_L) \simeq \mathrm{Ind}^F_{L_1}(1_{L_1}) \otimes \mathrm{Ind}^F_{L_2}(1_{L_2}).$$

Hence any irreducible occurring in $a_{L/F}$ is of the form $\tau_1 \otimes \tau_2$, with $\tau_i$ an irreducible occurring in $a_{L_i/F}$. By Langlands, $\tau_1$, resp. $\tau_2$, is modular, associated to a cuspidal automorphic representation $\pi_1$, resp. $\pi_2$, of $\mathrm{GL}(n_i, \mathbb{A}_F)$, with $n_i = \dim(\tau_i) \leq 2$. And by [Ra], there is a cuspidal automorphic representation $\pi_1 \boxtimes \pi_2$ of $\mathrm{GL}(n_1 n_2, \mathbb{A}_F)$ having the same $L$-function as $\tau_1 \otimes \tau_2$. So we are done when $G \subset \mathrm{SGO}(4,\mathbb{C})$.

So we may assume from now on that $G$ is not contained in $\mathrm{SGO}(4,\mathbb{C})$. Then $G$ has a subgroup $G'$, say, of index 2 which is a subgroup of $\mathrm{SGO}(4,\mathbb{C})$. Since $H$ is a Hall subgroup relative to $n$, $H' := H \cap G'$ will necesssarily be a subgroup of $H$ of index 2. Then $G$ is generated by $H$ and $G'$ and so by Mackey,

$$\mathrm{Res}^F_{F'}(\mathrm{Ind}^F_L(1_L)) \simeq \mathrm{Ind}^{F'}_{L'}(1),$$

where $F'$, resp. $L'$, is the quadratic extension of $F$, resp. $L$, corresponding to $G'$, resp. $H'$. Denote by $\theta$ the non-trivial automorphism of $F'/F$.

We have seen that some conjugate of $H'$ by an element $x$ of $G' \subset G$, is of the form $H_1 \times H_2$, with $H_i$, $i \in \{1, 2\}$, being the Hall subgroup of some solvable $G_i$ in $\mathrm{GL}(2, \mathbb{C})$. So we may, after replacing $H$ by its conjugate by $x$, assume that $H'$ is a product group $H_1 \times H_2$. Consequently, for any irreducible $\tau$ occurring in $a_{L/F}$, we have

$$\mathrm{Res}^F_{F'}(\tau) \simeq \tau_1 \otimes \tau_2, \tag{9.11}$$

where each $\tau_i$ is, by Lemma 9.9, irreducible of dimension $\leq 2$. The proof of that Lemma shows even that $\tau_i$ is dihedral if it has dimension 2. But it is important to note that $\tau$ can still be primitive, and this fact provides the content for Theorem 9.3.

Suppose $\tau$ becomes reducible when restricted to $\mathrm{Gal}(\overline{F}/F')$. Then we must have

$$\tau \simeq \mathrm{Ind}^F_{F'}(\tau'), \tag{9.12}$$

for an irreducible $\tau'$ occurring in $a_{L'/F'}$. If $\pi'$ is the cuspidal automorphic representation of $\mathrm{GL}(n', \mathbb{A}_F)$ associated to $\tau'$, with $n' = \dim(\tau') \leq 2$, then $\tau$ is modular, associated to the automorphically induced representation $I^F_{F'}(\pi')$ of $\mathrm{GL}(2n', \mathbb{A}_F)$ constructed by Arthur and Clozel in [AC].

Consequently it suffices to consider when $\tau$ remains irreducible upon restriction to $\mathrm{Gal}(\overline{F}/F')$, being of the form $\tau_1 \otimes \tau_2$ (see (9.10)) above). If either $\tau_1$ or $\tau_2$ is one dimensional, then $\tau$ has dimension $\leq 2$, and since its image is solvable, we are done by taking the Langlands-Tunnell representation $\sigma(\tau)$ on $\mathrm{GL}(2)/F$.

So we may, and we will, assume that $\dim(\tau_i) = 2$ for each $i$. Now we may apply Theorem A$'$ and obtain the desired result. The interesting case is when $\tau_2$ is $\tau_1^\theta$, and one gets

$$\tau \simeq As(\tau_1), \tag{9.13}$$

and the corresponding automorphic representation $\Pi$ of $\mathrm{GL}(4, \mathbb{A}_F)$ is $As(\sigma(\pi_1))$, as constructed in Theorem D.

□

## 10. Solvable Galois representations of GO$(2m+1)$-type

In this section we will prove Proposition C.

Suppose we are given a continuous irreducible representation of $\mathrm{Gal}(\overline{F}/F)$ of $\mathrm{GO}(n)$-type, wth $n$ odd. By applying Lemma 1.2 we may, up to replacing $\rho$ by a one-dimensional twist, which does not affect the conclusion of the Proposition, assume that the image of $\rho$ lies in $\mathrm{O}(n, \mathbb{C})$. Let $K$ be the number field cut out by the kernel of $\rho$ with (finite) Galois group $G$ over $F$, so that $\rho$ can be viewed as a faithful representation of $G$. From the derived series we may extract, by the solvability of $G$, an elementary abelian $p$-group $A$ which is characteristic in $G$, i.e., stable under any automorphism of $G$. Applying Clifford's theorem, we see that

$$\mathrm{Res}^G_A(\rho) \simeq m(\chi_1 \oplus \cdots \oplus \chi_r), \tag{10.1}$$

for some $m, r > 0$ with $mr = n$, and 1-dimesnional representations $\chi_1, \cdots, \chi_r$ of $A$ such that $\chi_i \neq \chi_j$ if $i \neq j$. Moreover, for every $j$ there exists $g_j \in G$ such that

$$\chi_j(a) = \chi_1(g_j a g_j^{-1}) \tag{10.2}$$

for all $a \in A$; hence each $\chi_j$ has the same order, which must be $p$ as $\rho$ is injective.

If $p$ is odd, then no $\chi_j$ is self-dual, while $\rho$ is itself self-dual, giving a contradiction as $n$ is odd. So $p = 2$. Let

$$\rho_1 = m\chi_1 \quad \text{and} \quad G_1 = \mathrm{Stab}_G(\rho_1). \tag{10.3}$$

Then

$$\rho \simeq \mathrm{ind}_{G_1}^{G}(\rho_1) \tag{10.4}$$

by Clifford. We are done if $m = 1$.

So we may assume that $m > 1$. If $r = 1$, $A \simeq \mathbb{Z}/2$, by the faithfulness of $\rho$, and $\rho(A) = \pm I$. But by construction $A$ is contained in the commutator subgroup $(G, G)$, which forces $\det(\rho)$ to be trivial on $A$. On the other hand, since $n$ is odd, $\det\rho(A) = -1$, resulting in a contradiction.

Hence $r$ must be $> 1$ when $m > 1$. Then $G_1$ is a proper subgroup of $G$ and $\rho_1$ is self-dual by virtue of $\chi_1$ being quadratic. Thus $(\rho_1, G_1)$ satisfies the same hypotheses as $(\rho, G)$. Since induction is natural in stages, we are done by infinite descent.

$\square$

## Bibliography


**[AC]:** J. Arthur and L. Clozel, *Simple Algebras, Base Change and the Advanced Theory of the Trace Formula*, Ann. Math. Studies **120** (1989), Princeton, NJ.

**[A]:** M. Aschbacher, On primitive linear representations of finite groups, Journal of Algebra **234**, 627-640 (2000).

**[BaR]:** L. Barthel and D. Ramakrishnan, A non-vanishing result for twists of $L$-functions of $\mathrm{GL}(n)$, Duke Math. Journal **74**, no.3, 681-700 (1994).

**[B$\ell$R]:** D. Blasius and D. Ramakrishnan, Maass forms and Galois representations, in *Galois groups over* $\mathbb{Q}$, ed. by Y. Ihara, K. Ribet and J.-P. Serre, MSRI Publications **16** (1989).

**[BCDT]:** C. Breuil, B. Conrad, F. Diamond and R. Taylor, On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises, to appear in the Journal of the AMS.

**[Bu]:** J.Buhler, *Icosahedral Galois representations*, Lecture Notes in Math. **654**, Springer-Verlag (1978).

**[BF]:** D. Bump and S. Friedberg, The exterior square automorphic $L$-functions on $\mathrm{GL}(n)$, Piatetski-Shapiro Festschrift II, Israel Math. Conference Proceedings **3**, 47–65 (1990).

**[BDST]:** K. Buzzard, M. Dickinson, N.I. Shepherd-Barron and R.L. Taylor, On icosahedral Artin representations, preprint (1999), to appear in the Duke Math Journal

**[BS]:** K.Buzzard and W.Stein, A mod five approach to modularity of icosahedral Galois representations, Preprint (2000), to appear in the Pacific Journal of Math.

**[Ch]:** J.-P. Jeff Chen, *Local factors, Central characters, and Representations of the General Linear Group over Non-archimedean fields*, Doctoral dissertation, Yale University (1996).

**[CoKPSSh]:** J. Cogdell, H. Kim, I. Piatetski-Shapiro and F. Shahidi, On lifting from classical groups to $\mathrm{GL}_N$, preprint (2000).

**[CoPS1]:** J. Cogdell and I. Piatetski-Shapiro, A converse theorem for GL(4), Math. Research Letters **3** (1996), no. 1, 67-76.

**[CoPS2]:** J. Cogdell and I. Piatetski-Shapiro, Converse theorems for $\mathrm{GL}_n$, II, Reine Angew. Math. **507**, 165–188 (1999).

**[C]:** L.Clozel, Base change for GL($n$), Proceedings of the International Congress of Mathematicians, Vol. 1, 791–797, Amer. Math. Soc., Providence, R.I. (1987)

**[Da]:** E.Dade, Accessible characters are monomial, J. Algebra **117**, no.1, 256–266 (1988).

**[DeH]:** P. Deligne and G. Henniart, Sur la variation, par torsion, des constantes locales d'équations fonctionnelles de fonctions $L$, Invent. Math. **64**, no. 1, 89–118 (1981).

**[D]:** J. Dieudonné, *La géométrie des groupes classiques*, troisième edition, Ergebnisse Math., Band **5**, Springer-Verlag (1971).

**[Fr]:** G. Frey, *On Artin's conjecture for odd 2-dimensional representations*, Lecture Notes in Math. **1585**, Springer-Verlag (1994).

**[GeJ]:** S. Gelbart and H. Jacquet, A relation between automorphic representations of GL(2) and GL(3), Ann. Scient. Éc. Norm. Sup. (4) **11** (1979), 471–542.

**[GeSh]:** S. Gelbart and F. Shahidi, Boundedness of automorphic $L$-functions in vertical strips, preprint (2000).

**[HLR]:** G. Harder, R.P. Langlands and M. Rapoport, Algebraische zykeln auf Hilbert-Blumenthal-Flächen, Crelles Journal **366** (1986), 53-120.

**[HaT]:** M. Harris and R. Taylor, On the geometry and cohomology of some simple Shimura varieties, preprint (1999), to appear in the Annals of Math. Studies, Princeton.

**[He]:** G. Henniart, Une preuve simple des conjectures de Langlands pour GL($n$) sur un corps $p$-adique, Invent. Math. **139**, no. 2, 439–455 (2000).

**[Ik1]:** T. Ikeda, On the location of poles of the triple $L$-functions, Compositio Mathematica **83** (1992), 187-237.

**[Ik2]:** T. Ikeda, On the Gamma factor of the triple $L$-function I, Duke Math Journal **97**, no.2 (99), 301-318.

**[JPSS1]:** H. Jacquet, I. Piatetski-Shapiro and J. Shalika, Automorphic forms on GL(3). II. Ann. of Math. (2) **109**, no. 2, 213–258 (1979).

**[JPSS2]:** H. Jacquet, I. Piatetski-Shapiro and J. Shalika, Rankin-Selberg convolutions, Amer. J. Math.**105**, 367-464 (983).

**[JS]:** H. Jacquet and J.A. Shalika, Euler products and the classification of automorphic forms I & II, Amer. J of Math. **103** (1981), 499–558 & 777–815.

**[JZ]:** H. Jacquet and D. Zagier, Eisenstein series and the Selberg trace formula II, Transactions of the AMS **300** (1), 1-48 (1987).

**[JiR]:** D. Jiang and S. Rallis, Fourier coefficients of Eisenstein series of the exceptional group of type $G_2$, Pacific Journal of Mathematics **181**, no.2, 281-314 (1997).

**[K]:** H. Kim, Functoriality for the exterior square of $\mathrm{GL}_4$ and the symmetric fourth power of $\mathrm{GL}_2$, preprint (2000).

**[KSh]:** H. Kim and F. Shahidi, Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and functorial symmetric cube for $\mathrm{GL}_2$, preprint (2000).

**[La1]:** R.P. Langlands, *Base change for GL*(2), Annals of Math. Studies **96**, Princeton (1980).

**[La2]:** R.P. Langlands, On the notion of an automorphic representation. A supplement, in *Automorphic forms, Representations and L-functions*, ed. by A. Borel and W. Casselman, Proc. symp. Pure Math **33**, part 1, 203-207, AMS. Providence (1979).

**[La3]:** R.P. Langlands, Letter to A. Weil (January, 1967), posted on the site http://www.sunsite.ubc.ca/DigitalMathArchive/Langlands/functoriality.html

**[MW]:** C. Moeglin and J.-L. Waldspurger, Poles des fonctions $L$ de paires pour GL($N$), Appendice, Ann. Sci. École Norm. Sup. (4) **22**, 667-674 (1989).

**[Mul]:** W. Müller, On the singularities of residual intertwining operators, preprint (2000).

**[MuR]:** M.R. Murty and A Raghuram, Some variations on the Dedekind conjecture, Journal of the Ramanujan Mathematical Society **15**, No. 4, 225-245 (2000).

**[PS-R]:** I. Piatetski-Shapiro and S. Rallis, Rankin triple $L$-functions, Compositio Mathematica **64** (1987), 31-115.

**[Ra1]:** D. Ramakrishnan, Modularity of the Rankin-Selberg $L$-series, and multiplicity one for SL(2), Annals of Mathematics **152**, 43–108 (2000).

**[Ra2]:** D. Ramakrishnan, Arithmetic of Hilbert-Blumenthal surfaces, CMS Conference Proceedings **7**, 285-370 (1987).

**[Se]:** J.-P. Serre, Modular forms of weight one and Galois representations, in *Algebraic Number Fields*, ed. by A. Fröhlich, Acad. Press (1977), 193-268.

**[Sh1]:** F. Shahidi, On the Ramanujan conjecture and the finiteness of poles for certain $L$-functions, Ann. of Math. (2) **127** (1988), 547–584.

**[Sh2]:** F. Shahidi, A proof of the Langlands conjecture on Plancherel measures; Complementary series for $p$-adic groups, Ann. of Math. **132** (1990), 273-330.

**[T]:** R. Taylor, On icosahedral Artin representations. II, preprint (2000).

**[TW]:** R. Taylor and A. Wiles, Ring-theoretic properties of certain Hecke algebras, Ann. of Math. (2) **141**, no. 3, 553–572 (1995).

**[Tu]:** J. Tunnell, Artin's conjecture for repreesentations of octahedral type, Bulletin AMS **5**, no. 2 (1981), 173-175.

**[Wa]:** D. Wales, Quasiprimitive linear groups with quadratic elements, to appear in the Journal of Algebra.

**[W]:** A. Wiles, Modular elliptic curves and Fermat's last theorem, Ann. of Math. (2) **141**, no. 3, 443–551 (1995).

Dinakar Ramakrishnan
Department of Mathematics
California Institute of Technology, Pasadena, CA 91125.
dinakar@its.caltech.edu

253-37 Caltech, Pasadena, CA 91125
*E-mail address*: dinakar@its.caltech.edu