DETERMINATION OF MODULAR FORMS BY TWISTS OF CRITICAL L-VALUES

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Contents

0 Introduction

Let f be a normalized holomorphic newform (defined on the upper half plane $\mathcal{H}$) of level N, weight $2k$ and trivial character. It is given by a Fourier expansion $f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}, z \in \mathcal{H}$, with $a_1 = 1$. For every primitive Dirichlet character χ of conductor M_χ, we have the associated L-series $L(f, \chi, s) = \sum_{n\geq 1} a_n \chi(n) n^{-s}$, which has an Euler product expansion in $\{s \in \mathbb{C} | \operatorname{Re}(s) > k + \frac{1}{2}\}$, and admits a meromorphic continuation to the whole s-plane with a functional equation relating the values at s and $2k - s$. Let X denote the set of primitive Dirichlet characters with $(M_\chi, N) = 1$. Our first result is that the knowledge of the set of special values $\{L(f, \chi, k) | \chi \in X\}$, up to a constant, is sufficient to determine f. We go further and in fact prove a much stronger result, namely that this assertion holds if X is replaced by the subset X_{quad} of *quadratic* characters (Theorem B, Sec. 3), or by the subset $X(p^\infty)$ consisting of those χ whose conductor and order are powers of a fixed odd prime p not dividing N (Theorem A, Sec. 2).

Our main idea is to analyze (variants of) the following *twisted average* for each integer $m \geq 1$:

$$Y_m(T) = \frac{1}{T} \sum_{\substack{\chi \in X_0 \\ M_\chi \leq T}} L(f, \chi, k)\chi(m)$$

where X_0 denotes a relevant subset of X. If we take m to be 1, we get the usual untwisted average $Y(T)$, whose behavior as $T \to \infty$ has been well understood; see Iwaniec [Iw], Murty [Mu] for $X_0 = X_{\text{quad}}$, and Rohrlich [Ro1] for $X_0 = X$. The subject matter of [Iw] was, to be precise, the average values of $L'(f, \chi, k)$, leading to a non-vanishing result, established earlier and independently by Murty-Murty [MM] and Bump-Friedberg-Hoffstein [BFH], needed to complement Kolyvagin's work on modular elliptic curves. Of relevance to us is also a form of the approximate functional equation, as in the paper of Luo-Rudnick-Sarnak [LRS]. We begin by describing the analogous asymptotic behavior of $Y_m(T)$, relying on the method of Iwaniec for $X = X_{\text{quad}}$.

In the case of p-power twists, our theorem gives a finer result and works at any real point $s = t$. The proof relies on the known properties of Kloosterman sums over $\mathbb{Z}/p^m$. The assertion at the points to the left of center is not a consequence of the functional equation of $L(f, \chi, s)$ as the value of χ at the conductor appears. This necessitates the use of $Y_{m/r}(p^n)$, for all integers m, r. But the key new ingredient in the proof of Theorem A is its *reduction* to Proposition 2.2, and not the verification of the proposition, the details of which are given for the convenience of the reader. An intriguing consequence of the result is the recovery of the fundamental periods $c^{\pm}(f)$ from the limits, as $j \to \infty$, of averages of *algebraic* special values $\{A(f, \chi, m) | \chi \in X(p^j)^\epsilon\}$, at critical integers m, for an appropriate parity ϵ. (See Corollary 2.5 for a precise statement.) Our theorem should also extend to Maass waveforms on GL(2), but we have chosen not to do it in this paper, mainly to avoid lengthening the argument, and because all the applications we have in mind deal only with holomorphic forms.

The proof of our theorem is quite delicate in the case of quadratic twists, where the key step is the expression, for almost all m, of the ratio of the dominant terms of $Y_m(T)$ and $Y(T)$, as a well behaved rational function of the normalized coefficient $\tilde{a}_m = a_m m^{(1-2k)/2}$. To be precise, we show that this function $H_m(t) \in \mathbb{Q}[\sqrt{m}](t)$ has no poles in the interval $[-2, 2]$, and has a positive derivative there. Recall that, for any good prime ℓ, $\tilde{a}_\ell$ lies in this interval by Deligne's proof of the Ramanujan conjecture. Thus $H_m(\tilde{a}_\ell)$ determines $\tilde{a}_\ell$ (for almost all ℓ), allowing us to appeal to the strong multiplicity one theorem. Our method also works for Maass waveforms which are tempered.

We give three applications of our main result. The first one (Theorem C, Sec. 4) shows that the field K_f generated over $\mathbb{Q}$ by the coefficients $\{a_n\}$ can also be generated by the set of ratios of the form

$$A(f,\chi,\chi_0) := g(\chi_0)g(\chi)^{-1}L(f,\chi,k)/L(f,\chi_0,k),$$

where χ_0 is a fixed quadratic character such that $L(f,\chi_0,k) \neq 0$, and χ runs over quadratic characters of the same parity as χ_0. (For any character ν, we denote by $g(\nu)$ the associated Gauss sum.) It was known earlier that these numbers $A(f,\chi,\chi_0)$ were in K_f for all $\chi \in X_{\text{quad}}$. We establish our assertion by combining our main result (Theorem B) with Shimura's reciprocity law ([Sh]) describing the action of Aut$\mathbb{C}$ on $\{A(f,\chi,\chi_0)\}$. At the end of Sec. 4, we raise an analogous question for general critical motives M over $\mathbb{Q}$ with End $(M) \otimes \mathbb{Q}$ a field.

Our second application (Theorem D, Sec. 5) shows that, for any odd prime p not dividing the level N of the newform f, and for any finite order $p-$adic character η of *tame type* i, for any $i \in [0, 2k-2]$, the knowledge (up to a non-zero algebraic multiple) of the associated (one variable) p-adic L-function $L_p(f,\eta,s)$ ([MTT]) determines f uniquely. (One says that η has tame type i if it is of the form $\omega^i\nu$, for a wild character ν, with ω denoting the Teichmüller character.) It should be pointed that the (weaker) fact that the *collection* $\{L_p(f,\eta,s)\}$ determines f has been known for some time, as it has been kindly explained to us by H.Hida (whose method uses the work of Rohrlich [Ro1]). It is useful to note that when p is supersingular for f, the definition of $L_p(f,\eta,s)$ depends on the choice of a root α of the $p-$Hecke polynomial $x^2 - a_px + p^{2k-1}$, and our theorem works for either choice (if $\text{ord}_p(\alpha) < 2k-1$). The proof combines Theorem A with some results on modular symbols ([MTT]) and $p-$adic analytic functions with logarithmic growth ([V]). Specializing to the weight 2 case, we deduce that a modular elliptic curve E over $\mathbb{Q}$ is determined up to $\mathbb{Q}$-isogeny by $L_p(E,s)$, for any p not dividing $2N$; this may be thought of as an analog of the classical isogeny conjecture. In particular, each $L_p(E,s)$ is non-zero, but this is already known by the work of Rohrlich [Ro2]. Our method gives the slightly finer statement that for all but a finite number of positive integers n not divisible by p, we have $L_p(E,n) \neq 0$.

After we communicated our results to H. Stark, he showed by using transcendence arguments ([St]) that a modular elliptic curve E over $\mathbb{Q}$ is in fact determined (up to $\mathbb{Q}$-isogeny) just by $L(E,1)$ *if* it is non-zero. However, his beautiful argument does not seem to work for forms of higher weight as one does not have analogous transcendence results for the periods. Also, given some E with $L(E,1) = 0$, one can find a quadratic character χ such that $L(E,\chi,1) \neq 0$, but this χ will depend on E. It should be stressed that in our approach, it suffices to give the (set of) special values up to a scalar multiple (in $\mathbb{C}$), and so our results are geared more towards giving information about the *algebraic* parts, not about the periods.

The third (and final) application (Theorem E, Sec. 6) answers a (stronger form of the original) question raised by W. Kohnen [K1]. Solving this problem was the original motivation for this paper. Fix $k \geq 1$, and denote, for every odd square free integer

$N \geq 1$, by $S_{k+\frac{1}{2}}(4N)$ the space of cusp forms of weight $k+\frac{1}{2}$, level $4N$ and trivial character, and by $S^+_{k+\frac{1}{2}}(4N)$ the Kohnen subspace. The span of "newforms" in this space is isomorphic, as a Hecke module, to $S_{2k}(N)^{\rm new}$ by the Shimura correspondence ([W2]). (The orthogonal complement of $S^+_{k+\frac{1}{2}}(4N)$ is well understood.) We show that if g_1, g_2 are two newforms in $S^+_{k+\frac{1}{2}}$ with Fourier coefficients $b_1(n), b_2(n)$ respectively, and

$$b_1^2(|D|) = b_2^2(|D|)$$

for almost all *fundamental discriminants* with $(-1)^k D > 0$, then $g_1 = \pm g_2$. In other words, our result says that given a newform g in $S^+_{k+\frac{1}{2}}(4N)$, $k \geq 1$ and N odd, square-free, one cannot simply change the signs of some of the coefficients $b(|D|)$ and obtain a new eigenform. Before this paper, the best known result, due to Kohnen [K1], was that if $b_1(|D|) = b_2(|D|), N = 1$, *and* if g_1 and g_2 have the same eigenvalues relative to the Hecke operator $T^+_{k+\frac{1}{2}}(4)$, then $g_1 = g_2$. The proof of our result uses Theorem B in conjunction with Waldspurger's formula relating the square of $b_j(|D|)$ to (a suitable multiple of) $L(f_j, \chi_D, k)$, where f_j is the eigenform in $S_{2k}(N)$ corresponding to g_j. A crudely stated consequence is that a form in Kohnen's space is determined by the square-free coefficients. It may be helpful to note this space does not contain theta series such as $\sum_n e^{\pi i n^2 z}$, $z \in \mathcal{H}$, which has weight 1/2, and $\sum_n n\chi(n) e^{\pi i n^2 z}$, $\chi(-1) = -1$, which has weight 3/2, but has non-square-free level and non-trivial character.

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1 Preliminaries

Let $N \geq 1$. For any $r \in \frac{1}{2}\mathbb{Z}$, let $\mathbf{S}_r(N)$ denote the space of holomorphic cusp forms of weight r, level N and trivial character. Every $f \in \mathbf{S}_r(N)$ admits a Fourier expansion:

$$f(z) = \sum_{n \geq 1} a_n q^n, \tag{1.1}$$

for $z \in \mathcal{H}$, with $q = e^{2\pi i z}$.

Suppose r is a (positive) integer. Recall that a newform (of weight r) is a Hecke eigenform f in $\mathbf{S}_r(N)$, for some N, orthogonal to the "old" forms coming from lower levels, such that $a_1 = 1$ (cf. [Li]). For such a form f, one has $T_p f = a_p f$, for all primes p not dividing N, where T_p denotes as usual thee p-th Hecke operator. For every primitive Dirichlet character χ of conductor $M = M_\chi$ with $(M, N) = 1$, one has the associated L-function:

$$L(f, \chi, s) = \sum_{n \geq 1} a_n \chi(n) n^{-s}, \tag{1.2}$$

which converges absolutely in $\mathrm{Re}(s) > (r+1)/2$, and admits an analytic continuation to the whole s-plane with a functional equation. Note that the triviality of the character of f forces its weight r to be even. Write $\boxed{r = 2k}$. Some times it will be convenient to use the associated (unitary) cuspidal automorphic representation $\pi = \pi_\infty \otimes \pi_0$ of $GL_2(\mathbb{A}) = GL_2(\mathbb{R}) \times GL_2(\mathbb{A}_0)$ ([Ge]), where $\mathbb{A} = \mathbb{R} \times \mathbb{A}_0$ denotes the adele ring of $\mathbb{Q}$, with $\mathbb{A}_0 \simeq \hat{\mathbb{Z}} \otimes \mathbb{Q}$ being the ring of finite adeles. Viewing Dirichlet characters χ as idele class characters of $\mathbb{Q}$, we have the associated representations $\pi \otimes \chi$ defined by $g \mapsto \pi(g)\chi(\det(g))$, for all $g \in GL_2(\mathbb{A})$. One has

$$L(\pi \otimes \chi, s) = L(\pi_\infty \otimes \chi_\infty, s) L(\pi_0 \otimes \chi_0, s), \tag{1.3}$$

with

$$L(\pi_0 \otimes \chi_0, s) = L(f, \chi, s + k - \frac{1}{2})$$

and

$$L(\pi_\infty \otimes \chi_\infty, s) = \Gamma_{\mathbb{C}}(s + k - \frac{1}{2}),$$

where $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$. The functional equation reads

$$L(\pi \otimes \chi, s) = \chi(N) W(\pi) W(\chi)^2 (NM^2)^{\frac{1}{2} - s} L(\pi \otimes \overline{\chi}, 1 - s), \tag{1.4}$$

where $W(\pi), W(\chi)$ are the root numbers of π, χ respectively. Note that π is self dual, hence isomorphic to the complex conjugate representation $\overline{\pi}$ by unitarity, since f has trivial character, consequently, $W(\pi) = \pm 1$. Moreover, if we define the Gauss sum of χ by the formula

$$g(\chi) = \sum_{a \bmod M} \chi(a) e^{2\pi i a/q}, \tag{1.5}$$

then $W(\chi) = (-i)^\delta g(\chi)/\sqrt{q}$, where δ is 0 (resp. 1) when χ is even (resp. odd).

Auxiliary functions

For any function $\varphi \in C_c^\infty(\mathbb{R}_+^*)_s$ with $\int_0^\infty \varphi(y) \frac{dy}{y} = 1$, set:

$$\kappa(s) = \int_0^\infty \varphi(y) y^s \frac{dy}{y}, \tag{1.6}$$

$$F_1(x) = \frac{1}{2\pi i}\int_{(2)} \kappa(s) x^{-s}\frac{ds}{s},$$

and

$$F_2(x) = \frac{1}{2\pi i}\int_{(2)} \kappa(-s)\frac{L(\pi_\infty, 1-t+s)}{L(\pi_\infty, t-s)} x^{-s}\frac{ds}{s}.$$

Here we follow the (standard) convention that, for any real number a, $\int\limits_{(a)}$ denotes the integral over the vertical line $\Re(s) = a$ in $\mathbb{C}$.

By a standard argument, by appropriately shifting the vertical line of integration (see [LRS], for example), we get

(1.7)

(i) $F_1(x)$ and $F_2(x)$ are both rapidly decreasing as $x \to \infty$,

(ii) $F_1(x) = 1 + O(x^n)$, for all $n \geq 1$, as $x \to 0$, and

(iii) $F_2(x) << 1$, for $x > 0$.

Characters

Throughout this paper, χ will denote a primitive Dirichlet character of conductor $M = M_\chi$.

For every integer $N \geq 1$, we set

(1.8) $$X = X(N) = \{\chi | (M_\chi, N) = 1\},$$

and

$$X_{\text{quad}} = X_{\text{quad}}(N) = \{\chi \in X(N) | \chi^2 = 1\}.$$

If p is a prime not dividing N, we put

(1.9) $$X_{(p)} = X_{(p)}(N) = \{\chi \in X(N) | M_\chi = p^j, \text{ for some } j\},$$

and

$$X_{(p)}^w = \{\chi \in X_{(p)} | \chi \,\text{has}\, p-\text{power order}\}.$$

The elements of the group $X_{(p)}^w$ are called **wild** characters at p. Note that an element of $X_{(p)}$ belongs to $X_{(p)}^w$ iff it is trivial on all the elements of order $p-1$. There is a direct product decomposition

(1.10) $$X_{(p)} \;=\; X_{(p)}^w \times \text{Hom}((\mathbb{Z}/p)^*, \mathbb{C}^*).$$

For every $j \geq 1$, write X_j^w for the subgroup of wild characters of conductor dividing p^j. We will write $\sum^*_{\chi \bmod p^j}$ to denote the summation over the *primitive* wild characters of conductor p^j.

2 p-power twists

Fix an odd prime p, and a Dirichlet character η of conductor R, possibly divisible by p. The object of this section is to prove the following.

Theorem A *Let f, g be newforms in $\mathbf{S}_{2k}(N)$, $\mathbf{S}_{2k'}(N')$ respectively, with $(p, NN') = (R, NN') = 1$, and let t be a real number. Suppose there exist non-zero constants $B, C \in \mathbb{C}$ such that*

$$L(f, \eta\chi, t + k - \frac{1}{2}) = B^j C L(g, \eta\chi, t + k' - \frac{1}{2}),$$

for every χ in X_j^w, for all but a finite number of j. Then $k = k'$, $N = N'$ and $f = g$.

The key point of the proof is to first define an appropriate twisted average depending on p^j, and reduce the theorem to a statement about its limit as $j \to \infty$ (Proposition 2.2 below). The Proposition is later established by appealing to a well known method, and the details are given for the convenience of the reader.

Let m, r denote integers ≥ 1, which are not divisible by p. For every new form $f \in \mathbf{S}_{2k}(N)$ and $t \in (0, 1)$, and for all $j \geq 1$), we set

$$Y_{m/r}(p^j, f; t) = p^{-j} \sideset{}{^*}\sum_{\chi \bmod p^j} \overline{\chi}(m)\chi(r)L(f, \eta\chi, t + k - \frac{1}{2}). \tag{2.1}$$

(See the end of section 1 for a meaning of the $*$ over the summation.) Our aim is to somehow recover the $(m/r)-$th Hecke eigenvalue $a_{m/r}$ (of f) from this sum when $r|m$. We will let $a_{m/r}$ denote 0 if r does not divide m.

Proposition 2.2 *Let $t \geq \frac{1}{2}$. Then we have*

$$\lim_{j\to\infty} Y_{m/r}(p^j, f; t) = \frac{1}{p}\left(1 - \frac{1}{p}\right) \eta(m/r)(m/r)^{-(k-\frac{1}{2}+t)} a_{m/r}.$$

Proposition 2.2 $\Rightarrow$ Theorem A Let f, g be as in the theorem with eigenvalues a_m, b_m respectively. First consider the case when $t \geq \frac{1}{2}$. Applying the Proposition with $m = r = 1$, we get, since $a_1 = b_1 = 1$, $(\lim_{j\to\infty} B^j)C = 1$, which forces $B = C = 1$. Next apply the Proposition with m being any prime ℓ not dividing pNR, and $r = 1$, to obtain the identity $\tilde{a}_\ell = \tilde{b}_\ell$ (normalized Hecke eigenvalues). Then by the strong multiplicity one theorem, we must have $f = g$, $k = r$ and $N = N'$.

It is left to consider when $t < \frac{1}{2}$. In this case, applying the functional equation (1.4), the self-duality of f and the evenness of χ, we get, for every $\chi \in X_{(p)}^w$ of conductor p^j,

$$\overline{\chi}(N)L(f, \eta\chi, 1 - t + k - \frac{1}{2}) \; = \; B^j C_1 \overline{\chi}(N')L(g, \eta\chi, 1 - t + k' - \frac{1}{2}),$$

where $C_1 = C\eta(N/N')(N'/N)^{\frac{1}{2}-t}$. Multiplying both sides by $\chi(Nr)\overline{\chi}(m)$, for any $m \geq 1$, and averaging over (wild) χ of conductor p^j, we get the identity

$$Y_{m/r}(p^j, f; 1-t) \;=\; B^j C_1 Y_{mN'/rN}(p^j, g; 1-t),$$

for all $m, r \geq 1$. We will apply the Proposition, permissible since $1 - t > \frac{1}{2}$, to both sides of this identity (with $j \to \infty$), for suitable choices of (m, r). First we claim that $|B| \leq 1$. Suppose not. Then, applying the Proposition with $m = N$, $r = N'$, we get $a_{N/N'} = (\lim\limits_{j\to\infty} B^j)C_2$, for a non-zero constant C_2, which is impossible if $|B| > 1$. Hence the claim. Next we let $m = r = 1$. If N does not divide N' or if $|B| < 1$, we see that the right hand side (of the identity above) goes to zero, while the left hand side goes to $p^{-1}(1 - \frac{1}{p})$, leading to a contradiction. So we must have $N' = NN''$, for some integer $N'' \geq 1$, $|B| = 1$. In fact. B must be 1, for otherwise the right hand side has no limit as $j \to \infty$. We also deduce from the identity (with $m = r = 1$ and $N = N'N''$) that $1 = C_1\eta(N'')b_{N''}$. Finally, we take m to be any prime ℓ not dividing $pN = pN'N''$ and $r = 1$ to conclude, since $b_{\ell N''} = b_\ell b_{N''}$, that $\tilde{a}_\ell = \tilde{b}_\ell$. Once again, the theorem follows by applying the strong multiplicity one theorem.

Remark 2.3 Let f be a newform in $\mathbf{S}_{2k}(N)$, p a prime not dividing $2N$, and $\beta > 0$ any real number. Denote by ω the Teichmüller character, which is a generator of the character group of $(\mathbb{Z}/p)^*$. By using Proposition 2.2 and some ideas from [Ro2], one can easily establish the following mild variant of a basic theorem of Rohrlich ([Ro2], [Ro3]). Fix any integer i. Then there exist infinitely many primitive, wild characters χ at p such that $L(f, \omega^i\chi, \beta) \neq 0$. Moreover, if β is a critical integer, then all but finitely many of these twisted $L-$values are non-zero.

We would also like to point out an intriguing consequence of Proposition 2.2 concerning the periods of a newform f of weight $2k$ and trivial character. Recall that there exist numbers $c^{\pm}(f) \in \mathbb{C}^*$, well defined up to elements in $\mathfrak{O}_K^*$, which arise by comparing the integral singular and de Rham structures associated to f. For each Dirichlet character χ, define $A(f, \chi)$ by the formula

$$L(f, \chi, k) = A(f, \chi)c^{\epsilon}(f), \tag{2.4}$$

where ϵ is the sign of $(-1)^{k+1}$ (resp. $(-1)^k$) when χ is even (resp. odd). Then one knows that the numbers $A(f, \chi)$ are in $\overline{\mathbb{Q}}$. To elaborate, if $M(f)$ denotes the corresponding rank 2 motive over $\mathbb{Q}$ with coefficients in K ([Sch],[De]), normalized so that $s = 0$ is the critical point in question, then our $c^{\pm}(f)$ is Deligne's $c^{\pm}(M(f)(k))$, which equals, by the self-duality of f, $(2\pi i)^k c^{\mp}(M(f))$.

Corollary 2.5 *Let f be a newform of weight $2k$, level N and trivial character. Then for every odd prime p not dividing N, we have*

$$(i) \qquad c^{(-1)^{k+i}}(f)^{-1} \;=\; \lim_{j\to\infty} |P_j^w|^{-1} \sum_{\chi \in P_j^w} A(f, \omega^i\chi),$$

for all $i \in [0, p-2]$. Moreover, for every odd Dirichlet character ν of conductor prime to Np, and for every m with $a_m \neq 0$, we have

$$(ii) \qquad \frac{c^+(f)}{c^-(f)} = \nu(m) \lim_{j\to\infty} \frac{\sum_{\chi\in P_j^w} A(f, \nu\omega^k\chi)\overline{\chi}(m)}{\sum_{\chi\in P_j^w} A(f, \omega^k\chi)\overline{\chi}(m)}.$$

Recall that P_j^w is the set of primitive wild characters of conductor p^j. The proof is immediate from Proposition 2.2, with η being ω^i (resp. ω^k) for part (i) (resp. part (ii)), and (2.4). Evidently, we can also deduce such results at the other critical integers.

Proof of Proposition 2.2. We will first treat the case, i.e., when $t \in [\frac{1}{2}, 1]$. Since this is outside the range of absolute convergence, we need to make use of the approximate functional equation.

Denote by $\pi = \pi_\infty \otimes \pi_0$ the cuspidal automorphic representation of $GL_2(\mathbb{A})$ defined by f, and let κ, F_1, F_2 be as in (1.6). Let t be a real number in $[\frac{1}{2}, 1)$. Then it is easy to see that (for any $\mu \in X(N)$)

$$\frac{1}{2\pi i} \int\limits_{(2)} \kappa(s) L(\pi_0 \otimes \mu, s+t) y^s \frac{ds}{s} = \sum_{n=1}^{\infty} \frac{\tilde{a}_n \mu(n)}{n^t} F_1\Big(\frac{n}{y}\Big).$$

(See the definition of $\int\limits_{(a)}$ right before (1.6).) Moving the line of integration to $s = -2$, we see that the left hand side equals

$$L(\pi_0 \otimes \mu, t) + \frac{1}{2\pi i} \int\limits_{(-2)} \kappa(s) L(\pi_0 \otimes \mu, s+t) y^s \frac{ds}{s}.$$

On applying the functional equation (1.4), and changing s to $-s$, we obtain (in the usual way) the following approximate functional equation:

(2.6)
$$L(\pi_0 \otimes \mu, t) = \sum_{n=1}^{\infty} \frac{\tilde{a}_n \mu(n)}{n^t} F_1\Big(\frac{n}{y}\Big) + \mu(N) W(\pi) W(\mu)^2 (Np^{2j})^{\frac{1}{2}-t} \sum_{n=1}^{\infty} \frac{\tilde{a}_n \overline{\mu}(n)}{n^{1-t}} F_2\Big(\frac{ny}{NM^2}\Big),$$

where M is the conductor of μ.

Since we are interested in the limit of $Y_{m/r}(p^j, f; t)$ as $j \to \infty$, we may restrict our attention to large j. We will take j to be larger than $\mathrm{ord}_p(R)$, so that $\eta\chi$ will be primitive mod p^j if χ is. We may then use (2.6) with $\mu = \chi\eta$, and write $Y_{m/r}(p^j, f; t)$ as a sum $Y_{1,m/r}(p^j, t) + Y_{2,m/r}(p^j, t)$, with

$$(2.7) \qquad Y_{1,m/r}(p^j, t) = p^{-j} \sum_{\chi \bmod p^j}^{*} \sum_{n=1}^{\infty} \frac{\tilde{a}_n \eta(n) \chi(nrm')}{n^t} F_1\Big(\frac{n}{y}\Big),$$

where m' denotes the inverse of m mod p^j. Each wild character mod p^j is a character of $G(p^j) := \mathrm{Ker}((\mathbb{Z}/p^j)^* \to (\mathbb{Z}/p)^*)$, and the non-primitive ones are precisely those characters which are trivial on the subgroup $K := \ker(G(p^j) \to G(p^{j-1})$. Recall that a character of $(\mathbb{Z}/p^j)^*$ defines a wild character iff it is trivial on the set S_j of integers (mod p^j) of order $(p-1)$. Using the orthogonality of characters, we then see that

$$\sum_{\chi \bmod p^j}^{*} \chi = |G(p^j)|\delta_{S_j} - |G(p^{j-1})|\delta_K, \tag{2.8}$$

where δ_S denotes, for any subset S of $G(p^j)$, the characteristic function of S. Note that $|G(p^j)| = p^{j-1}$ (for $j \geq 1$). So (2.8) allows us to write

$$Y_{1,m/r}(p^j, t) = \frac{1}{p}\Big[\Phi_{m/r}(p^j, t) - p^{-1}\Phi_{m/r}(p^{j-1}, t)\Big], \tag{2.9}$$

where

$$\Phi_{m/r}(p^j, t) \;=\; \sum_{b \in S_j} \sum_{\substack{rn \equiv bm \,(\mathrm{mod} p^j)\\ n \geq 1}} \frac{\tilde{a}_n \eta(n)}{n^t} F_1(\frac{n}{y}).$$

This sum has a distiguished term corresponding to $b = 1$, $n = m/r$ when $r|m$, signifying the unique solution in positive integers n of the congruences $rn \equiv bm$ mod p^j for all j. This allows us to decompose $\Phi_{m/r}(p^j, t)$ as

(2.10)

$$\begin{aligned}\Phi_{m/r}(p^j, t) \;&=\; \left(\frac{\tilde{a}_{m/r}\eta(m/r)}{(m/r)^t}\right) F_1\left(\frac{m}{ry}\right) + \sum_{\substack{rn \equiv bm \text{ (mod } p^j)\\ n \geq 1, n \neq m/r}} \frac{\tilde{a}_n \eta(n)}{n^t} F_1\left(\frac{n}{y}\right) \\ &=\; I + II, \text{ say.}\end{aligned}$$

Again, $\tilde{a}_{m/r}$ denotes the normalized coefficient if $r|m$, and zero otherwise.

Write $y = p^{j\gamma}$ for a positive γ to be chosen below. Then we have by (1.7) (ii), that for $j \to \infty$,

$$I = \frac{\tilde{a}_{m/r}\eta(m/r)}{(m/r)^t} + 0(\frac{1}{y}). \tag{2.11}$$

For $n \geq 1,\ j \geq 1$, put

$$b_{n,j} = \begin{cases} 1, if\ rn = bm + ap^j, a \in \mathbb{Z} \\ 0, \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} II &= \sum_{1 \leq n \leq y} \frac{\tilde{a}_n \eta(n) b_{n,j}}{n^t} F_1\Big(\frac{n}{y}\Big) \\ &+ \sum_{y < n < \infty} \frac{\tilde{a}_n \eta(n) b_{n,j}}{n^t} F_1\Big(\frac{n}{y}\Big) \\ &= II_1 + II_2, \text{ say.} \end{aligned}$$

Recall that by Deligne's proof of the Ramanujan conjecture, we have $|\tilde{a}_n| << n^\epsilon$, for any $\epsilon > 0$. Using this and Abel's lemma, we see that II_2 is bounded in absolute value by $p^{-j} y^{1-t+\epsilon}$. Choose $y = p^{j\gamma}$ such that

$$\frac{1}{2t} \quad < \quad \gamma \quad < \quad \frac{1}{1-t},$$

which makes sense as $t \in [\frac{1}{2}, 1)$. Then $\gamma(1-t) < 1$, and hence $y^{1-t+\epsilon} = p^{j\gamma(1-t)+j\gamma\epsilon}$ is $o(p^j)$ for a small enough ϵ, so that that II_2 goes to 0 as $j \to \infty$.

Similarly,

$$II_1 = \sum_{n \leq y} \frac{\tilde{a}_n \eta(n) b_{n,j}}{n^t} F_1\Big(\frac{n}{y}\Big) << \sum_{n \leq y} \frac{b_{n,j}}{n^{t-\epsilon}} \Big| F_1\Big(\frac{n}{y}\Big) \Big| \quad << \quad p^{-j} y^{1-t+\epsilon}. \tag{2.12}$$

By our choice of γ above, II_1 also goes to zero as j goes to infinity.

Thus we have, by putting together (2.9) through (2.12),

$$\lim_{j \to \infty} Y_{1,m/r}(p^j) = \frac{(p-1)}{p^2} \frac{\tilde{a}_{m/r}}{(m/r)^t}. \tag{2.13}$$

We next analyze

(2.14)

$$Y_{2,m/r}(p^j) = W(\pi) N^{\frac{1}{2}-t} p^{-2jt} \sideset{}{^*}\sum_{\chi \bmod p^j} W(\chi\eta)^2 \sum_{\substack{n \geq 1 \\ (n,p)=1}} \frac{\tilde{a}_n \overline{\chi}(nNrm')}{n^{1-t}} F_2\Big(\frac{ny}{Np^{2j}}\Big).$$

Lemma 2.15 *For any integer b with $(b,p) = 1$, we have*

$$\sideset{}{^*}\sum_{\chi \bmod p^j} \overline{\chi}(b) W(\chi\eta)^2 << p^{j/2}.$$

Proof. We may assume that $(b,p)=1$. Let b' be the inverse of b mod p^j, and denote $\exp(2\pi i t)$ by $e(t)$. Then we have

$$\begin{aligned}\sideset{}{^*}\sum_{\chi \bmod p^j} \overline{\chi}(b)W(\chi\eta)^2 &= \frac{1}{p^j}\sideset{}{^*}\sum_{\chi}\overline{\chi}(b)\Big(\sum_{a \bmod p^j}\chi(a)\eta(a)e\Big(\frac{a}{p^j}\Big)\Big)^2 \\ &= \frac{1}{p^j}\sum_{a_1,a_2 \bmod p^j}\eta(a_1a_2)e\Big(\frac{a_1+a_2}{p^j}\Big)\sideset{}{^*}\sum_{\chi}\chi(a_1a_2b').\end{aligned}$$

Appealing to (2.8), we get

(2.16)

$$\sideset{}{^*}\sum_{\chi \bmod p^j}\overline{\chi}(b)W(\chi)^2 = \frac{(p-1)}{2p}\eta(b)S(1,b;p^j) - \frac{(p-1)}{2p^2}\eta(b)\sum_{i=0}^{p-1}S(1,b+ip^{j-1};p^j),$$

where

$$S(u,v;p^j) = \sum_{\substack{a \bmod p^j \\ (a,p)=1}} e\Big(\frac{ua+ua'}{p^j}\Big).$$

By a theorem of Salié (for $j \geq 2$) (see [S], [W]), one knows that each of these Kloosterman sums is bounded by $2p^{j/2}$. The lemma follows. □

Applying Lemma 2.11 with $b = nNrm'$ in (2.10), and the bound $|\tilde{a}_n| \leq n^{\epsilon}$, we get

$$\begin{aligned}|Y_{2,m/r}(p^j,t)| &\leq N^{\frac{1}{2}-t}p^{-2jt+j/2}\sum_{n=1}^{\infty}\frac{1}{n^{1-t-\epsilon}}\Big|F_2\Big(\frac{ny}{Np^{2j}}\Big)\Big| \\ &= N^{\frac{1}{2}-t}p^{-2jt+j/2}\Big[\sum_{n\leq\frac{Np^{2j}}{y}}\frac{1}{n^{1-t-\epsilon}}\Big|F_2\Big(\frac{ny}{Np^{2j}}\Big)\Big|\Big] + U(p^j,t),\end{aligned}$$

with

$$|U(p^j,t)| \quad << \quad p^{j/2+2j\epsilon}y^{-t-\epsilon}.$$

The partial sum within [ ] can easily be seen to be bounded by a constant times $y^{-t-\epsilon}p^{2j(t+\epsilon)}$. So we get:

(2.17) $$\Big|Y_{2,m/r}(p^j,t)\Big| << p^{j/2+2j\epsilon}y^{-t-\epsilon}.$$

We choose $y = p^{j\gamma}$, with $\frac{1}{2t} < \gamma < 2$. Then, for small enough ϵ,

(2.18) $$\lim_{j\to\infty} Y_{2,m/r}(p^j,t) = 0.$$

For the same reason, $U(p^j, t)$ also goes to zero. Putting all this together with (2.13), we get the Proposition (and Theorem A) for t in $[\frac{1}{2}, 1]$.

For completeness, we will now consider the remaining (easy) situation when $t > 1$. Since the $L-$series is absolutely convergent in this range, we can directly write

$$Y_{m/r}(p^j, t) \;=\; \frac{1}{p}\Big[\Psi_{m/r}(p^j, t) - p^{-1}\Psi_{m/r}(p^{j-1}, t)\Big], \tag{2.19}$$

where

$$\Psi_{m/r}(p^j, t) \;=\; \frac{\tilde{a}_{m/r}\eta(m/r)}{(m/r)^t} \;+\; \sum_{\substack{rn \equiv bm \,(\mathrm{mod}\; p^j) \\ n \geq 1, n \neq m/r}} \frac{\tilde{a}_n \eta(n)}{n^t}. \tag{2.20}$$

Compare (2.7) through (2.10). The second term on the right (of (2.2)) is easily seen to go to zero as j goes to ∞. The rest follows.

Q.E.D.

3 Quadratic twists

In Section 2, we considered twisting by arbitrary (primitive) characters of p-power conductor, while in this section we will consider twists by (primitive) *quadratic* characters $\chi_d = \left(\frac{d}{\cdot}\right)$, with d running over fundamental discriminants. This variant is subtle and useful for applications to forms of half integral weight (see Sec. 6).

Theorem B *Let f, g be normalized newforms in $\mathbf{S}_{2k}(N)$, $\mathbf{S}_{2m}(N')$ respectively. Suppose there is a constant C such that*

$$L(f, \chi_d, k) = CL(g, \chi_d, m)$$

for almost all primitive quadratic charcaters χ_d of conductor d prime to NN'. Then $k = m$, $N = N'$ and $f = g$.

Proof. For $\chi = \chi_d$, the functional equation (1.4) becomes

$$\left(\frac{|d|\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(f, \chi_d, s) = w\chi_d(-N)\left(\frac{|d|\sqrt{N}}{2\pi}\right)^{2k-s} \Gamma(2k-s)L(f, \chi_d, 2k-s), \tag{3.1}$$

where $w = W(\pi) = \pm 1$. Fix a multiple M of N and set

$$\mathcal{D}^w = \Big\{d \in \mathbb{Z} | wd > 0,\; d \equiv \nu^2 (\mathrm{mod}\; 4M), \text{for some } \nu \text{ coprime to } 4M\Big\}. \tag{3.2}$$

Then we have

$$w\chi_d(-N) = 1,\ \forall d \in \mathcal{D}^w. \tag{3.3}$$

Fix a smooth function F, compactly supported on $\mathbb{R}_+$, with positive mean value. Put

$$B = \int_{\mathbb{R}_+} F(t)dt. \tag{3.4}$$

Let ℓ be either 1 or a prime not dividing $4M$. We will use the following variant of the twisted sum partly to be consistent with [Iw].

$$S_\ell(T) = S_{f,\ell}(T) = \sum_{d\in\mathcal{D}^w}{}' L(f,\chi_d,k)F\left(\frac{|d|}{T}\right)\chi_d(\ell), \tag{3.5}$$

where the prime indicates that the summation is only over *square free* numbers d. In [Iw], H. Iwaniec essentially proved an asymptotic formula for $S_{f,1}(T)$. (He actually established a mean value theorem with a very sharp error term for the derivatives of the L-function at the center of the critical strip.) The main new idea here is to consider the sum weighted by $\chi_d(\ell)$. We will apply his method to establish an asymptotic formula for $S_{f,\ell}(T)$.

Proposition 3.6 *We have*

$$S_{f,\ell}(T) = BC_0L_{f,\ell}(k)T + O(T^{\frac{13}{14}+\epsilon}),$$

where

$$C_0 = \frac{3\gamma(4M)}{M\pi^2}\prod_{p|4M}\left(1-\frac{1}{p^2}\right)^{-1},$$

$$\gamma(4M) = \#\{d \pmod{4M},\ d\equiv \nu^2 \pmod{4M}, (\nu,4M)=1\}$$

$$L_{f,\ell}(s) = L_\ell(s) = \sum_{n=kj^2\ell} a_n\left(\prod_{p|n,(p,\,4M)=1}\left(1+\frac{1}{p}\right)^{-1}\right)n^{-s},$$

and for $\ell > 1$,

$$L_\ell(s) = L_1(s)\frac{\sum_{i=0}^\infty a_{\ell^{2i+1}}\ell^{-(2i+1)s}}{\frac{1}{\ell}+\sum_{i=0}^\infty a_{\ell^{2i}}\ell^{-2is}},$$

where $\Re s > k-\frac{1}{4}$, $\ k \mid (4M)^\infty$, $\ (j,\ 4M)=1$.

The function $L_1(s)$ occurs in [Iw] for $k=2$ (as $L(s)$), but the discussion goes through for higher weights as well. Moreover, one has $L_1(k)\neq 0$ (cf. loc. cit.). (Note that there is a misprint on page 373 of [Iw]; $(1+\frac{1}{p})$ should be $(1+\frac{1}{p})^{-1}$.) We also note that $L_\ell(s)$, which is initially defined as a Dirichlet series convergent in a right half plane, can be analytically continued to at least $\{\Re(s) > k - \frac{1}{4}\}$, since $L_1(s)$ can be.

The proof of this proposition requires only minor modifications at different places of the method developed by Iwaniec in [Iw] to analyze the untwisted sum. So we will assume the truth of the proposition for the moment and indicate how to proceed from there to the proof of the theorem. First we need the following crucial

Lemma 3.7 *Let ℓ be a prime with $(\ell, 2N) = 1$. Then $\exists$ a rational function $H_\ell(t) \in \mathbb{R}(t)$, which is regular on $[-2, 2]$ with a non-vanising derivative in that interval, such that*

$$L_\ell(k) = H_\ell(\tilde{a}_\ell)L_1(k).$$

Indeed, by the multiplicativity of the Hecke eigenvalues, we have

$$a_\ell \sum_{m=0}^{\infty} a_\ell^{2m} \ell^{-2mk} = (\ell^k + \ell^{k-1}) \sum_{m=0}^{\infty} a_{\ell^{2m+1}} \ell^{-(2m+1)k}.$$

This gives

$$\frac{\sum_{m\geq 0} a_{\ell^{2m+1}} \ell^{-(2m+1)k}}{\ell^{-1} + \sum_{m\geq 0} a_{\ell^{2m}} \ell^{-2mk}} = \frac{1}{\ell^{\frac{1}{2}} + \ell^{-\frac{1}{2}}} \left(\tilde{a}_\ell - \frac{\tilde{a}_\ell}{1 + \ell \sum_{m\geq 0} a_{\ell^{2m}} \ell^{-2mk}} \right). \tag{3.9}$$

But

$$\sum_{m\geq o} a_{\ell^{2m}} \ell^{-2mk} = \frac{1 + \ell^{-1}}{(1 + \ell^{-1})^2 - \tilde{a}_\ell^2 \ell^{-1}}.$$

Since the left hand expression in (3.9) gives, by Proposition 3.6, the ratio $L_\ell(s)/L_1(s)$ at $s = k$, we get $L_\ell(k) = H_\ell(\tilde{a}_\ell)L_1(k)$ with

$$H_\ell(t) = \frac{\ell^{\frac{5}{2}} t}{(\ell + 1)(\ell^2 + \ell + 1) - \ell t^2}. \tag{3.10}$$

Clearly, this rational function has no singularities in $[-2, 2]$ (for any prime ℓ). Moreover, $H_\ell'(t)$ is easily seen to be > 0 in $[-2, 2]$. Hence the lemma. □

Now let f, g be as in Theorem B, so that $L(f, \chi_d, k) = CL(g, \chi_d, m)$ for $|d|$ large. Choose M to be the least common multiple of N and N'. First note that the constant C cannot be zero as we know that $L(f, \chi_d, k) \neq 0$ for some d. Moreover, we may assume that the signs of the functional equations, $w(f)$ and $w(g)$ say, must be equal. Indeed, if it were otherwise, up to replacing f and g if necessary, we would have

$$\sideset{}{'}\sum_{d\in\mathcal{D}^{w(f)}} L(f, \chi_d, k) F\left(\frac{|d|}{T}\right) = 0,$$

leading to a contradiction. Then by the proposition and lemma, we get

$$L_{f,1}(k) = CL_{g,1}(m) \tag{3.11}$$

and

$$L_{f,1}(k)H_\ell(\tilde{a}_\ell) = CL_{g,1}(m)H_\ell(\tilde{b}_\ell),$$

for every prime ℓ with $(\ell, 2NN') = 1$, where $\tilde{a}_\ell$ and $\tilde{b}_\ell$ denote the normalized ℓ-th Hecke eigenvalues of f, g respectively. Then the rational function H_ℓ takes the same values on $\tilde{a}_\ell$ and $\tilde{b}_\ell$. Moreover, by the Ramanujan conjecture proved by Deligne, we know that $\tilde{a}_\ell, \tilde{b}_\ell$ both lie in the interval $[-2, 2]$. Since by the lemma, H_ℓ is well defined with a positive derivative on $[-2, 2]$, it is injective there and we may conclude that $\tilde{a}_\ell = \tilde{b}_\ell$. Since this holds for all primes ℓ with $(\ell, 2NR) = 1$, we get the assertion of Theorem B by the strong multiplicity one theorem.

It is left to prove the proposition. We will assume familiarity with [Iw] and simply indicate the necessary changes which must be made to obtain our assertion.

We have the general bounds

$$\sum_{m\le M} a_m e(\alpha m) << M^k \log M, \quad \sum_{m\le M} |a_m|^2 << M^{2k}, \tag{3.12}$$

and

$$|a_m| \le m^{\frac{2k-1}{2}}\tau(m).$$

In Lemma I of [Iw], we need only replace M by M^k in the right hand side to cover the general case.

Set

(3.13)

$$V(x) = \frac{1}{2\pi i}\int_{(4/5)} \frac{\Gamma(k+s)}{\Gamma(k)} x^{-s}\frac{ds}{s} = \frac{1}{\Gamma(k)}\int_x^\infty e^{-\xi}\xi^{k-1}d\xi = (1 + x + \cdots + \frac{x^{k-1}}{(k-1)!})e^{-x},$$

and

$$\mathcal{A}(X,\ \chi_d) = \frac{1}{2\pi i}\int_{(4/5)} L(f,\ \chi_d,\ k+s)\frac{\Gamma(k+s)}{\Gamma(k)}\left(\frac{2\pi}{X}\right)^{-s}\frac{ds}{s}$$

$$= \sum_{n=1}^\infty a_n n^{-k}\chi_d(n)V\left(\frac{2\pi n}{X}\right).$$

Then we have

$$L(f,\ \chi_d,\ k) = 2\mathcal{A}(|d|\sqrt{N},\ \chi_d),$$

and

$$L(f,\ \chi_d,\ k) = \mathcal{A}(X,\ \chi_d) + O(|d|X^{-\frac{1}{2}}).$$

The same argument (as in section 5 of [Iw]) leads to

$$\sum_{d\in\mathcal{D},|d|\le Y}{}' |L(f,\ \chi_d,\ k)|^4 \ll Y^{2+\epsilon}.$$

We have

$$S_\ell(T) = 2\sum_{d\in\mathcal{D}}{}' \mathcal{A}(|d|\sqrt{N},\chi_d)F\left(\frac{|d|}{T}\right)\chi_d(\ell) = S+R, \tag{3.14}$$

where

$$\begin{aligned} S &= 2\sum_{a\le A,(a,4M)=1}\mu(a)\sum_{d\in\mathcal{D}}\mathcal{A}(a^2|d|\sqrt{N},\chi_{a^2d})F\left(\frac{a^2|d|}{T}\right)\chi_{a^2d}(\ell)\\ &= 2\sum_{a\le A,(a,4\ell M)=1}\mu(a)\sum_{d\in\mathcal{D}}\mathcal{A}(a^2|d|\sqrt{N},\chi_{a^2d})F\left(\frac{a^2|d|}{T}\right)\chi_d(\ell), \end{aligned}$$

and

$$\begin{aligned} R &= 2\sum_{(b,4M)=1}\Big(\sum_{a|b,a>A}\mu(a)\Big)\sum_{d\in\mathcal{D}}{}'\mathcal{A}(b^2|d|\sqrt{N},\chi_{b^2d})F\left(\frac{b^2|d|}{T}\right)\chi_{b^2d}(\ell)\\ &= 2\sum_{(b,4\ell M)=1}\Big(\sum_{a|b,a>A}\mu(a)\Big)\sum_{d\in\mathcal{D}}{}'\mathcal{A}(b^2|d|\sqrt{N},\chi_{b^2d})F\left(\frac{b^2|d|}{T}\right)\chi_d(\ell). \end{aligned}$$

Just as in section 6 of [Iw], we have

$$R \ll (A^{-\frac{3}{2}}T^{\frac{5}{4}} + A^{-3}T^{\frac{3}{2}})T^{\epsilon}. \tag{3.15}$$

It remains to evaluate S. For $(a,\ 4M)=1$ and $d\in\mathcal{D}$ we have

$$\mathcal{A}(a^2|d|\sqrt{N},\ \chi_{a^2d}) = \sum_{(n,\ a)=1} a_n n^{-k}\chi_d(n)V\Big(\frac{2\pi n}{a^2|d|\sqrt{N}}\Big).$$

Here we write every n uniquely as the product $n = rj^2m$, where r has prime factors in $4M$, jm is coprime to $4M$ and m is squarefree. Hence

$$\begin{aligned} S = 2\sum_{a\le A,(a,4\ell M)=1}\mu(a)&\sum_{n=rj^2m,(n,a)=1}a_n n^{-k}\sum_{q|j}\mu(q)\\ &\times\sum_{dq\in\mathcal{D}}\chi_{dq}(m\ell)F\left(\frac{a^2|d|q}{T}\right)V\Big(\frac{2\pi n}{a^2|d|q\sqrt{N}}\Big). \end{aligned}$$

Now we split the sum S, according to whether ℓ divides m or not,as

$$S = S^1 + S^2,$$

where

$$S^1 = 2 \sum_{a \le A, (a,4\ell M)=1} \mu(a) \sum_{n=rj^2 m\ell, (n,a)=1, (m,\ell)=1} a_n n^{-k} \sum_{q|j\ell} \mu(q)$$

$$\times \sum_{dq \in \mathcal{D}} \chi_{dq}(m) F\left(\frac{a^2|d|q}{T}\right) V\left(\frac{2\pi n}{a^2|d|q\sqrt{N}}\right),$$

$$S^2 = 2 \sum_{a \le A, (a,4\ell M)=1} \mu(a) \sum_{n=rj^2 m, (n,a)=1, (m,\ell)=1} a_n n^{-k} \sum_{q|j} \mu(q)$$

$$\times \sum_{dq \in \mathcal{D}} \chi_{dq}(m\ell) F\left(\frac{a^2|d|q}{T}\right) V\left(\frac{2\pi n}{a^2|d|q\sqrt{N}}\right).$$

We note that the expression for S^1 above is derived by first going back to the initial coprimality condition $(j,d)=1$, and then by using the Mobius function to detect the new condition $(j\ell, d) = 1$.
We then write

$$\chi_d(m) = \overline{\epsilon_m} m^{-1/2} \sum_{2|b|<m} \chi_{Mb}(m) e\left(\frac{\overline{4M} bd}{m}\right),$$

where ϵ_m equals 1 if $m \equiv 1 \pmod 4$, and equals i if $m \equiv -1 \pmod 4$ and $4M\overline{4M} \equiv 1 \pmod m$. Thus

$$S^1 = 2 \sum_{a \le A, (a,4\ell M)=1} \mu(a) \sum_{n=rj^2 m\ell, (n,a)=1, (m,\ell)=1} a_n n^{-k} \overline{\epsilon_m} m^{-1/2} \sum_{q|j\ell} \mu(q)$$

$$\times \sum_{2|b|<m} \chi_{Mbq}(m) \sum_{dq \in \mathcal{D}} F\left(\frac{a^2|d|q}{T}\right) V\left(\frac{2\pi n}{a^2|d|q\sqrt{N}}\right) e\left(\frac{\overline{4M} bd}{m}\right),$$

and

$$S^2 = 2 \sum_{a \le A, (a,4\ell M)=1} \mu(a) \sum_{n=rj^2 m, (n,a)=1, (m,\ell)=1} a_n n^{-k} \overline{\epsilon_{m\ell}} (m\ell)^{-1/2} \sum_{q|j} \mu(q)$$

$$\times \sum_{2|b|<m\ell} \chi_{Mbq}(m\ell) \sum_{dq \in \mathcal{D}} F\left(\frac{a^2|d|q}{T}\right) V\left(\frac{2\pi n}{a^2|d|q\sqrt{N}}\right) e\left(\frac{\overline{4M} bd}{m\ell}\right).$$

Set $\Delta = \min\left(\frac{1}{2}, a^2 q T^{\epsilon-1}\right)$, and then split the sum S^i, $i = 1, 2$, as

$$S^i = S_0^i + S_1^i + S_2^i,$$

defined by the respective conditions (1) $b=0$, (2) $0 < |b\ell^{1-i}| < \Delta m$, and (3) $\Delta m < |b\ell^{1-i}| < m/2$. Note that $S_0^2 = 0$. We will show that S_0^1 gives the main term. Arguing exactly as in [Iw], we infer that

$$|S^2| + |S_1^1| + |S_2^1| \ll A^2 T^{\frac{1}{2}+\epsilon}.$$

It remains to evaluate S_0^1. We have $b=0$, so $\chi_{Mbq}(m) = 0$ unless $m=1$. Thus

$$S_0^1 = 2 \sum_{a \le A, (a,4\ell M)=1} \mu(a) \sum_{n=rj^2\ell, (n,a)=1} a_n n^{-k} \sum_{q|j\ell} \mu(q) \times \sum_{dq\in\mathcal{D}} F\left(\frac{a^2|d|q}{T}\right) V\left(\frac{2\pi n}{a^2|d|q\sqrt{N}}\right)$$

By Euler-Maclaurin formula, the inner sum is

$$\frac{\gamma(4M)T}{4Ma^2q} \int F(t) V\left(\frac{2\pi n}{T\sqrt{N}t}\right) dt \ + \ O(\left(\left(1+\frac{n}{T}\right)^{-e}\right),$$

for all $e \ge 1$. Thus we have (with $\phi(n)$ denoting the Euler $\phi-$function)

$$S_0^1 = \gamma(4M)T \sum_{n=rj^2\ell} \frac{a_n \phi(j\ell)}{2Mj\ell n^k} \sum_{a\le A, (a,4Mj\ell)=1} \frac{\mu(a)}{a^2} \int F(t) V\left(\frac{2\pi n}{T\sqrt{N}t}\right) dt + O(AT^{\frac{1}{2}+\epsilon})$$

$$= CT \int \left(\sum_{n=rj^2\ell} \frac{a_n}{n^k} \prod_{p|\ell j} \left(1+\frac{1}{p}\right)^{-1} V\left(\frac{2\pi n}{t\sqrt{N}T}\right)\right) F(t) dt + O(AT^{\frac{1}{2}+\epsilon} + A^{-1}T^{1+\epsilon}).$$

But the inner sum, by the definition of $V(x)$ and by shifting the line of integration to $(\frac{-1}{5})$, can be evaluated as

$$L_{f,\ell}(k) + O(T^{-\frac{1}{5}}).$$

Therefore, we conclude that

$$S(T) = BCL_\ell(k)\, T + O((T^{\frac{4}{5}} + AT^{\frac{1}{2}} + A^{-1}T + A^2T^{\frac{1}{2}} + A^{-\frac{3}{2}}T^{\frac{5}{4}} + A^{-3}T^{\frac{3}{2}}(T^{\epsilon})).$$

Choosing $A = T^{\frac{3}{14}}$, we obtain Proposition 3.6. Q.E.D.

Remark 3.16 In our statement of Theorem B, we required that $L(f,\chi_d,k) = CL(g,\chi_d,m)$ for all χ_d with $(d,NR)=1$, with at most a finite number of exceptions. But in fact, our proof together with the convexity bound shows that for all d with $|d| << T$, if the exceptional d's are at most $O(T^{\frac{1}{2}-\epsilon})$, then the theorem still holds.

4 Generation of coefficient fields by ratios of L-values

Let f be a normalized newform in $\mathbf{S}_{2k}(N)$, for some $k, N \geq 1$. Put

$$K_f = \mathbb{Q}(\{a_n | (n, N) = 1\}) \tag{4.1}$$

where $a_n = a_n(f)$ is the n-th Hecke eigenvalue of f.

Fix a quadratic Dirichlet character χ_0 such that

$$L(f, \chi_0, k) \neq 0. \tag{4.2}$$

Set, for every $\chi \in X_{\text{quad}}$,

$$A(f, \chi, \chi_0) = \frac{g(\chi)^{-1} L(f, \chi, k)}{g(\chi_0)^{-1} L(f, \chi_0, k)}. \tag{4.3}$$

Then one knows ([Sh]) that these numbers lie in K_f if χ and χ_0 have the same parity.

Theorem C. $K_f = \mathbb{Q}(\{A(f, \chi, \chi_0) | \chi\chi_0$ *even, quadratic*$\})$.

Proof. Let M_f denote the field on the right hand side of the theorem. Since we know that M_f is contained in K_f, we are left to prove the reverse inclusion. For this it suffices to prove the following.

Claim 4.4 *Every $\sigma \in Aut$ $(\mathbb{C}/M_f)$ acts as the identity on K_f..*

Fix any such automorphism σ of $\mathbb{C}$ fixing M_f. By Shimura ([Sh]) we have

$$A(f, \chi, \chi_0)^\sigma = A(f^\sigma, \chi^\sigma, \chi_0^\sigma). \tag{4.5}$$

Since χ, χ_0 are quadratic,

$$\chi^\sigma = \chi, \text{ and } \chi_0^\sigma = \chi_0. \tag{4.6}$$

Since σ fixes M_f, we get

$$A(f, \chi, \chi_0) = A(f^\sigma, \chi, \chi_0). \tag{4.7}$$

which yields

$$L(f, \chi, k) = CL(f^\sigma, \chi, k), \tag{4.8}$$

with

$$C = C(f, \sigma, \chi_0) = \frac{L(f, \chi_0, k)}{L(f^\sigma, \chi_0, k)}.$$

The key point is that C is independent of χ. So we may apply Theorem B with $g = f^\sigma$ and get

$$f = f^\sigma.$$

Since the coefficients of f^σ are the σ-conjugates of those of f, we get $a_n = a_n^\sigma$, $\forall n$. In other words, σ fixes K_f. This proves the claim and the theorem. □

Now fix an odd prime p not dividing N, and consider the field $\mathbb{Q}_\infty = \mathbb{Q}(\mu_{p^\infty})$ generated by (all the) p-power roots of unity. Fix a Dirichlet character χ_0, wild at p, of conductor p^{m_0} such that

$$L(f, \chi_0, k) \neq 0, \tag{4.9}$$

which is possible to do by our Theorem A. For all wild characters χ of conductor p^m, we may define $A(f, \chi, \chi_0, k)$ as before (cf. (4.3)). These numbers lie in the compositum $K_\infty = K_f \mathbb{Q}_\infty$. By using the Shimura reciprocity law ([Sh]) and arguing as in the quadratic case, we then get the following variant of Theorem C.

Theorem C′

$$K_\infty = \mathbb{Q}_\infty(\{A(f, \chi, \chi_0) | \chi \textit{ wild of conductor } p^m\}).$$

Don Blasius has pointed out the following consequence of this theorem. Choose any pair of primes (p, q), and pick wild characters χ_0, μ_0 of $p-$power and $q-$power conductors respectively such that $L(f, \chi_0, k) \neq 0 \neq L(f, \mu_0, k)$. Denote by $L_\infty^{(p)}$ (resp. $L_\infty^{(q)}$) th field generated over $\mathbb{Q}(\mu_{p^\infty})$ (resp. $\mathbb{Q}(\mu_{q^\infty})$) by the numbers $A(f, \chi, \chi_0)$ (resp. $A(f, \mu, \mu_0)$) as χ (resp. μ) runs over even characters of conductor a power of p (resp. q). Then we have

$$K_f \ = \ L_\infty^{(p)} \cap L_\infty^{(q)}. \tag{4.10}$$

We conclude this section with a question about motives $M/\mathbb{Q}$ of pure weight with a critical point $s = m$ for $L(M, s)$. Suppose that $\mathrm{End}(M) \otimes \mathbb{Q}$ is a field, and moreover, when M is self-dual and m central, assume that for some quadratic twist $M \otimes \chi_1$, the sign $W(M \otimes \chi_1)$ is $+1$. We will assume, as expected under the assumption on the sign, that there exists a quadratic character χ_0 such that $L(M \otimes \chi_0, m) \neq 0$. One can then define the numbers $A(M, \chi, \chi_0)$ appropriately, and these numbers are expected by Deligne's conjectures to belong to the field K of coefficients.

Question 4.11 Is K generated by $\{A(M, \chi, \chi_0) | \chi\chi_0$ even quadratic$\}$?

Of course Theorem C gives an affirmative answer for motives M attached to even weight modular forms at the critical center. It may also be useful to note that the answer is in the affirmative for motives $M = [\nu]$ attached to Dirichlet characters ν at any critical point $s = m$, the reason being that the $A([\nu], \chi, \chi_0)$ supply a full set of character values at this point.

5 p-adic L-functions

Let f be a normalized newform in $\mathbf{S}_{2k}(N)$. It is well known that f is determined by its complex L-function $L(f, s)$. One way to deduce this is to invert the Mellin transform expression

$$(2\pi)^{-s}\Gamma(s)L(f,s) = \int_{\mathbb{R}_+^*} f(iy)y^s \frac{dy}{y}, \tag{5.1}$$

determining $f(z)$ on the half line $\{z \in \mathcal{H} | \operatorname{Re}(z) = 0\}$, and hence everywhere since f is holomorphic. A natural question which arises is whether f is also determined by (any of the) $p-$adic cousins $L_p(f, s)$.

Let L_f denote the $\mathbb{Z}$-module in $\mathbb{C}$ generated by the periods of the differential of degree $2k-1$ defined by f over the integral (relative) homology classes defined by modular symbols (see [MTT], Chap. I, Sec. 2). For any (odd) prime p, fix an embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$. Then one has a p-adic L-function $L_p(f, s) \in \mathbb{C}_p \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}L_f$, for $s \in \mathbb{Z}_p$, attached to f by Mazur, Swinnerton-Dyer, Manin, Amice, Vélu, Vishik and Haran (see [MTT], Chap. I, for details). The definition depends on the choice of a root α of the Hecke polynomial $x^2 - a_p x + p^{2k-1}$ with $\operatorname{ord}_p(\alpha) < 2k-1$. Our result below works for either choice. If η denotes a $p-$adic character, i.e., a continuous homomorphism $\mathbb{Z}_p^* \to \mathbb{C}_p^*$, we may consider also the twisted $L-$function $L_p(f, \eta, s)$. There is a Mazur-Mellin transform expression

$$L_p(f, \eta, s) = \int_{\mathbb{Z}_p^*} \eta(x) < x >^s d\mu_f,$$

for a measure $\mu_f = \mu_{f,\alpha}$ on $\mathbb{Z}_p^*$.

We will fix algebraic closures $\overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}}$ of $\mathbb{Q}_p$ and $\mathbb{Q}$ respectively, and also an embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}}_p$. As usual, we will denote by $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ relative to (an extension of) the $p-$adic absolute value on $\mathbb{Q}_p$. We will again denote by $|.|_p$ the extension of the absolute value to $\mathbb{C}_p$.

When a $p-$adic character η has finite order, it can be identified with a primitive Dirichlet character of $p-$power conductor, and can morover be written uniquely as a product $\omega^i\chi$, $0 \leq i \leq p-2$, where χ is a wild character and ω the Teichmüller character; in this case, we say that η is of **tame type** i. The main aim of this section is to establish the following.

Theorem D. *Let f be a normalized newform in $S_{2k}(N)$, p an odd prime with $(p, N) = 1$, and η a finite order $p-$adic character of tame type i (mod $(p-1)$), with $0 \leq i \leq 2k-2$. Then f is determined by $L_p(f, \eta, s)$. More explicitly, let g be a new form in $\mathbf{S}_{2k}(N')$ with $(p, N') = 1$ such that, for an infinite number of positive integers n prime to p, we have*

$$L_p(f, \eta, n) = CL_p(g, \eta, n) \quad \in \mathbb{C}_p \otimes_{\overline{\mathbb{Q}}} W,$$

for a $\overline{\mathbb{Q}}$-subspace W of $\mathbb{C}$ containing L_f, L_g, with $C \in \overline{\mathbb{Q}}$. Then $N = N'$ and $f = g$.

An immediate consequence of the Theorem is the following, which is known for the $p-$adic $L-$functions attached to abelian characters χ of $\mathbb{Q}$ with $\mathbb{Q}(\chi)$ real.

Corollary 5.2. *Let f be a newform of weight $2k$ and trivial character. Then, for any finite order $p-$adic character η of tame type i, $0 \leq i \leq 2k-2$, we have*

$$L_p(f, \eta, n) \neq 0,$$

for all but a finite number of positive integers n prime to p.

The fact that $L_p(f, s)$ is not identically zero has been known for some time by the elegant work of Rohrlich ([Ro2], [Ro3]). To prove this corollary, pick any newform $g \neq f$ of weight $2k$ and trivial character. Suppose $L_p(f, \eta, n) = 0$ for an infinite number of n prime to p. Then the hypothesis of Theorem D holds with $C = 0$, leading to the absurd conclusion $f = g$. Done. Note that, for any quadratic character χ cutting out a real quadratic field of $\mathbb{Q}$, the associated abelian $p-$adic $L-$function $L_p(\chi, s)$has only a finite number of zeros, because it is represented by a (non-zero) Iwasawa power series. The situation is the same for f in the ordinary case (using only that the $L-$function is not identicaly zero). But $L_p(f, \eta, s)$ could have infinitely many zeros in the supersingular case, where one has to use unbounded measures, though they are not too bad as shown by Vishik.

Proof of Theorem D. Recall that every $x \in \mathbb{Z}_p^*$ can be uniquely decomposed as $\langle x \rangle \omega(x)$, with $\omega(x)$ a $(p-1)-$th root of unity and $\langle x \rangle \in 1 + p\mathbb{Z}_p$. For any new form f of weight $2k$, one fixes a root $\alpha = \alpha(f)$ of $X^2 - a_p X + p^{2k-1}$ with minimal p-adic valuation. (α is the unique unit root when p is ordinary for f). By Deligne, α has archimedean absolute value $p^{(2k-1)/2}$ and is non-zero (since p does not divide N).

In view of Theorem A, the assertion of Theorem D is a consequence of the following.

Proposition 5.3. *Let f (resp. g) be a newform in $\mathbf{S}_{2k}(N)$ such that $L_p(f, \eta, n) = CL_p(g, \eta, n)$ for an infinite number of positive integers n prime to p, where C is in $\overline{\mathbb{Q}}$. Then the following identity of complex L-values holds for every primitive character χ mod p^m, for all m large enough:*

$$(*) \qquad \alpha(f)^{-m} L(f, \chi\eta, k) \;=\; C' \alpha(g)^{-m} L(g, \chi\eta, k),$$

for some non-zero constant $C' \in \mathbb{C}$.

In order to prove this Proposition, we need the following

Lemma 5.4. *Let f, g be as in Proposition 5.3. Then we have, for every finite order wild $p-$adic character χ, the following identity:*

$$L_p(f, \chi\eta, s) = CL_p(g, \chi\eta, s),$$

for all $s \in \mathbb{Z}_p$.

Proof of Lemma. By a result of Vishik ([V], Theorem 3.3), we know that the function

$$\lambda \to L_p(f, \eta\lambda)$$

is analytic with logarithmic growth on the group X of continuous homomorphisms of $(\mathbb{Z}_p)^*$ into $\mathbb{C}_p^*$. To be precise, one fixes an integer h such that $\text{ord}_p(\alpha) < h \leq 2k-1$. Then for an anlytic function F to have logarithmic growth (in Vishik's sense) is to obey the condtion

$$\text{sup}_{|u-1|_p<r}||F(u)|| = o(\text{sup}_{|u-1|_p<r}|\log_p^h(u)| \quad \text{as}\, r \to 1^-.$$

To elucidate, let us review the topology on X. It is firstly a product of the finite group $X((\mathbb{Z}/p)^*$ with the group X_0 of wild $p-$adic characters. Secondly, there is a natural identification of X_0 with the $p-$adic disk $T := \{u \in \mathbb{C}_p^* || u-1| < 1\}$ (cf. [V], sec. 2.1), given by sending ν to $\nu(1+p)$. One transports the topology of the disk to X_0.

Now write

$$F(\nu) = L_p(f, \eta\nu) - CL_p(g, \eta\nu),$$

for every $\nu \in X_0$. Then by our hypothesis, there exists an infinite set Y of positive integers n prime to p such that F vanishes on the subset $X_1 := \{\alpha_n | n \in Y\}$. The lemma is equivalent to saying that the analytic function F (with logarithmic growth) is identically zero on X_0. Put

$$T_1 = \{(1+p)^n | n \in Y\} \subset T.$$

We will view F as an analytic function of T vanishing on T_1. Let $|.|_p$ denote the absolute value on $\mathbb{C}_p$ extending the one on $\mathbb{Q}_p$. Now, by the discussion in section 2.5 of [V], we see that for any r, the number of zeros ρ of a non-zero analytic function (with logarithmic growth) such that $|\rho - 1| = r$ is finite. To be precise, Vishik shows that this number is the difference between the successive slopes of a piecewise linear function. (The zeros occur precisely at the break points of this function.) So our lemma will be proved if we show for our F, that for some r, the set of zeros ρ of F such that $|\rho - 1| = r$ is infinite. But for all n in Y, $\rho_n := (1+p)^n$ belongs to T_1 and is hence a zero of F. On the other hand, $|(1+p)^n - 1|_p$ equals $|\sum_{j=1}^n \binom{n}{j} p^j|_p$, which is $1/p$ as n is not divisible by p. Since Y is an infinite set, we get what we want by setting $r = 1/p$. Done.

Proof of Proposition 5.3. Write

$$\eta = \nu w^j,$$

$i \in [0, 2k-2]$ for a wild character ν of finite order. Then, for any wild character χ of conductor p^m, $m \geq 1$, we may consider the character

$$x \to \chi\eta(x) < x >^j = \chi(x)\nu(x)\omega^j(x) < x >^j,$$

which is a **special character** in the sense of [MTT], sec. 13. Then, by applying the Proposition of sec. 14 of [MTT], we get the following identity:

$$L_p(f, \eta\chi, j) = \frac{(p^{(j+1)m} j!}{\alpha^m g(\overline{\chi\nu})(-2\pi i)^j} L(f, \overline{\chi\nu}, j+1),$$

where $g(\mu)$ denotes, for any μ, the Gauss sum attached to μ. The Proposition now follows by applying Lemma 5.4.

Remark 5.5. As mentioned in the introduction, H. Hida has remarked to us that the fact that f is determined by the collection of twisted $p-$adic $L-$functions $\{L_p(f, \chi, s) | \chi$ of finite order $\}$ has been known to him for some time as a consequence of Rohrlich's work giving the non-vanishing of $L(f, \chi, k)$ for some finite order character χ.

6 Forms of half integral weight

Let k, N be integers ≥ 1, with N *odd* and *square-free.* Let $\mathbf{S}^+_{k+\frac{1}{2}}(4N)$ denote the subspace of $\mathbf{S}_{k+\frac{1}{2}}(4N)$ consisting of cusp forms $g(z) = \sum_{n\geq 1} b(n)e^{2\pi i n z}$, $z \in \mathcal{H}$, with $b(n)$ being zero unless $(-1)^k n \equiv 0, 1 \pmod 4$. Then it is known (cf. [W1], [K1] that the subspace $\mathbf{S}^+_{k+\frac{1}{2}}(N)^{\mathrm{new}}$ spanned by "newforms" (in the sense of Kohnen) in $\mathbf{S}^+_{k+\frac{1}{2}}(N)$ is isomorphic to $S_{2k}(N)^{\mathrm{new}}$ (as Hecke modules) under the Shimura correspondence. The object of this section is to establish the following.

Theorem E. *Let g_1, g_2 be Hecke eigenforms in $\mathbf{S}^+_{k+\frac{1}{2}}(4N)^{\mathrm{new}}$, with coefficients $b_1(n), b_2(n)$ respectively. Suppose $b_1(|D|) = \pm b_2(|D|)$, for all (but a finite number of) fundamental discriminants D with $(-1)^k D > 0$. Then $g_1 = \pm g_2$.*

This gives an affirmative answer to (a stronger form of) a question of Kohnen. He asked (in ([K2]) if $g_1 = g_2$ under the assumption $b_1(|D|) = b_2(|D|)$, for all D as above. He himself settled his question in the special case $N = 1$ under the *additional hypothesis* that g_1 and g_2 have the same eigenvalue under the Hecke operator T_4^+. Our approach is totally different and uses Theorem B.

Proof of Theorem. For $i = 1, 2$, let $f_i \in \mathbf{S}_{2k}(N)$ denote the newform associated to g_i by the Shimura correspondence. A well known formula of Waldspurger ([W2]) express $b_i(|D|)^2$ in terms of $L(f_i, \chi_D, k)$. We state it below in the explicit form derived by Kohnen [K1] (see Corollary 1 on p. 242), generalizing earlier work of Kohnen-Zagier for $N = 1$:

Let D be a discriminant such that
(C1) $(-1)^k D > 0$, and
(C2-i) $\left(\frac{D}{\ell}\right) = w_\ell(f_i),\ \forall \ell | N$,
where $w_\ell(f_i)$ denotes the eigenvalue under the Atkin-Lehner involution. Then

$$\frac{|b_i(|D|)|^2}{\langle g_i, g_i \rangle} = 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-\frac{1}{2}} \frac{L(f_i, \chi_D, k)}{\langle f_i, f_i \rangle}, \tag{6.1}$$

where $\nu(N)$ denotes the number of prime divisors of N, and $\langle , \rangle$ denotes the Petersson inner product.

Let us now fix a D_0 prime to N satisfying (C1), (C2-i) above (for both i). Such a D_0 certainly exists. We know from the proof of Proposition 3.6 that there are infinitely many D of this type for which $L(f_1, \chi_D, k) \neq 0$, hence $b_1(|D|) \neq 0$. For any such D, $b_2(|D|)$ is also non-zero, as $b_1(|D|)^2 = b_2(|D|)^2$ by hypothesis. This implies, by the Remark on p.243 of [K1], that

$$w_\ell(f_1) \ = \ w_\ell(f_2) \ = \left(\frac{D}{\ell}\right), \tag{6.2}$$

for all such D.

Put (for $i = 1, 2$)

$$\varphi_i = f_i \otimes \chi_{D_0}. \tag{6.3}$$

Then φ_i is an eigenform in $\mathbf{S}_{2k}(ND_0^2)$, in fact a newform as $(N, D_0) = 1$.

We also set

$$\mathcal{D} = \left\{ d > 0 | \mu(d) \neq 0,\ d \equiv \nu^2 (\text{mod } 4ND_0), \text{for some } \nu \text{ coprime to } 4ND_0 \right\}.$$

Note that for every $D \in \mathcal{D}$, DD_0 satisfies conditions (C1) and (C2-i) above (for each i)..

The hypothesis of Theorem E implies, thanks to (6.1) and (6.2), the identity

$$L(\varphi_1, \chi_D, k) = CL(\varphi_2, \chi_D, k), \tag{6.4}$$

with

$$C = \frac{\langle f_1, f_1 \rangle \ \langle g_2, g_2 \rangle}{\langle f_2, f_2 \rangle \ \langle g_1, g_1 \rangle} \in \mathbb{C}^*,$$

for all D belonging to the set

Applying Theorem B we conclude that $\varphi_1 = \varphi_2$, which implies $f_1 = f_2$ after untwisting by χ_{D_0}.

Appealing to the bijection of the Shimura correspondence, we see that g_1 must be a scalar multiple of g_2. We may write $g_1 = ag_2$, for some $a \in \mathbb{R}$. But then, from (6.1) we deduce that $\langle g_1, g_1 \rangle = \langle g_2, g_2 \rangle$. This forces a^2 to be 1, and so $g_1 = \pm g_2$. QED

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