Base Change of Hecke Characters revisited
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Here we answer a nice question raised by Peter Sarnak as to whether we can give a new proof of any instance of the well known base change result for GL(1) by using trace inequalities.

Fix a finite Galois extension $M/L$ of CM fields of degree $m$, and denote by $E$, resp. $F$, the totally real subfield $M^+$, resp. $L^+$, of $M$, resp. $L$. For any number field $K$, write $A_K = K_\infty \times \mathbb{A}_K^I$ for the adele ring of $K$, $\Gamma_K$ for the absolute Galois group, and $C_K$ the idele class group $\mathbb{A}_K^*/K^*$. For $K \in \{M, L\}$, set, for a fixed tuple $k = (k_v)_{v|\infty}$ with each $k_v$ in $\mathbb{Z}$,

$$X_K(k) := \{ \chi \in \text{Hom}_{\text{cont}}(C_K, S^1) \mid \chi_v(z) = (z/|z|)^{k_v}, \forall v | \infty, \chi|_{C_K^{+,+}} = 1 \},$$

where $C_K^{+,+}$ denotes the image of $\mathbb{A}_K^{*,+}$, the subgroup of $\mathbb{A}_K^{*}$ consisting of totally positive elements. Any $\chi$ in $X_K(k)$ is a unitary Hecke character of weight $k$ of $K$, which is anti-cyclotomic when trivial on $C_K^+$.

Fix a finite set $R$ of primes in $F$, and denote by $R_M$ the set of primes of $M$ above $R$. Let $m = (m_v)_{v \in R}$ be a tuple of positive integers $m_v$. Put

$$q(R_M, m) := \prod_{v \in R} \prod_{w | v} q_{w}^{m_w-1}(q_w - 1) \geq \prod_{v \in R} q_v^{m_v([M_L:L_v]-1)}.$$ 

The object here is to construct the base change of $\chi \in X_L(k)$ of conductor dividing $\prod_{v \in R} P_v^{m_v}$ to $M$ without knowing a priori the answer as being $\chi \circ N_{M/L}$.

We will choose, as we may, $(R, m)$ large enough so that the following holds:

$$(0) \quad q(R_M, m) \geq C := (6\pi)^{|M:L|}[M:L] \left( \frac{d_Ld_E}{d_md_F} \right)^{1/2} \frac{L(1, \nu_{L/F})}{L(1, \nu_{M/E})},$$

where $d_K$ denotes, for any number field $K$, the absolute value of the discriminant of $K$, and $\nu_{L/F}$, resp. $\nu_{M/K}$ is the quadratic character of $F$, resp. $E$, attached to $L$, resp. $M$.

For the choice of measures we will take below,

$$C = (6\pi)^{|M:L|}[M:L] \frac{\text{vol}(L^* \mathbb{A}_{F, +} \backslash \mathbb{A}_L^*)}{\text{vol}(M^* \mathbb{A}_{E, +} \backslash \mathbb{A}_M^*)}.$$ 

**Theorem** Let $M/L$ be a finite Galois extension of CM fields. Fix a weight $k = (k_v)$. Then there exists a set $Y$ of positive density, consisting of degree one primes $w$ in $M$ such that for every $\chi$ in $X_L(k)$ $\exists \chi' \in$....
\(X_M(k')\) for \(k' = (k'_w)\) with \(k'_w \in \{0, \pm k_w\}\) for every \(w \mid \infty\), such that for all but finitely many \(w \in Y\), \(\chi'_w = \chi_v\) if \(v\) is the place of \(L\) lying below \(w\).

This is what the trace inequality argument below will give, and when \(k \neq 0\), a modification of the argument will even allow us to take \(Y\) to be of density one. In any case, to go further and obtain the full base change, we may appeal to a theorem of Hecke, which will assert that \(\chi'\) is necessarily of the form \((\chi \circ N_{M/L})\nu\) for a finite order character \(\nu\) of \(M\) as \(\chi'\) and \(\chi \circ N_{M/L}\) agree on a set of primes of positive density; in particular, \(k' = k\). Thus the desired base change arises by setting \(\chi_M = \chi' \nu^{-1}\).

To approach this without knowing about \(\chi \circ N_{M/L}\), note that by Serre every \(\chi\), resp. \(\chi'\), is attached, for \(\ell\) a prime, to an \(\ell\)-adic character \(\chi_\ell\), resp. \(\chi'_\ell\) of \(\Gamma_L\), resp. \(\Gamma_M\) with the same \(L\)-function as \(\chi\), resp. \(\chi'\). Then for every \(w\) in the set \(Y\) of degree one primes given by Theorem, the restrictions of \(\chi_\ell\) to \(\Gamma_{L_w}\) and \(\chi'_\ell\) to \(\Gamma_{M_w}\) are the same modulo the identification of \(M_w\) with \(L_w\) (when \(w \mid v\)). This forces the quotient \(\nu_\ell := \chi'_\ell/\chi_\ell|_{\Gamma_M}\) and \(\chi'_\ell\) to have trivial restrictions to \(\Gamma_{M_w}\) for \(w\) is a set of positive density. This forces their quotient \(\nu_\ell\) to have finite order. If we accept the existence of a finite order idele class character \(\nu\) attached to \(\nu_\ell\), we again see that \(\chi_M := \chi' \nu^{-1}\) is the desired base change, with

\[
L(s, \chi_M) = L(s, \chi_\ell|_{\Gamma_M}).
\]

**Proof of Theorem:**

Denote by \(\Sigma^1_M\) the (density one) set of finite places \(w\) of \(M\) which are unramified and of degree one over \(Q\). Let \(S\) be any finite set of places in \(\Sigma^1_M\) with the property that no pair of elements of \(S\) has the same norm (in \(Q\)). The key is to prove the following:

**Proposition A**  For every \(\chi \in X_L(k)\) and for every \(S\) as above which does not meet the (finite) set \(R(\chi)\) of primes where \(\chi\) is ramified, there is a character \(\chi_{M,S}^{M,S} \in X_M(k')\), for some \(k' \in \{0, \pm k\}\), such that for every \(w \in S\), \(\chi_{M,S}^{M,S} = \chi_v\) if the place \(v\) of \(L\) lies below \(w\).

**Proof of Proposition A**

Fix a character \(\chi_0\) in \(X_L(k)\). If \(\chi_0\) is the trivial character, then we can take \(\chi_{0,S}^{M,S}\) to be trivial as well, so we may, and we will, assume that \(\chi_0\) is non-trivial. At each finite place \(v\), let \(m_v = m_v(\chi_0)\) denote the exponent of the conductor \(U_v^{(m)}\) of \(\chi_0\), where \(U_v^{(n)}\) is \(O_v^*\) if \(n = 0\) and \(1 + P_v^n\) if \(n > 0\). (Here \(P_v\) denotes the prime ideal at \(v\).) Put \(R = R(\chi_0)\), which is the finite set of \(v\) where \(m_v > 0\). Write \(R_M\) for the set of primes of \(M\) above \(R\).
For any number field \( k \), assign the product measure \( d^x x = \otimes_v d^x x_v \) on the idele group \( \mathbb{A}_k^* \) such that at any finite place \( v \), \( d^x x_v \) gives volume 1 to \( \mathcal{O}_{k,v}^* \), at real \( v \) the measure \( d^x x_v = dx_v/|x_v| \) (with \( dx_v \) being Lebesgue), and for \( v \) complex, \( d^x x_v = \frac{d}{d\theta} \) (in polar coordinates \( x_v = re^{i\theta} \), \( r \in \mathbb{R}_+, \theta \in [0, 2\pi] \). We also get an induced measure on the relevant quotients of \( \mathbb{A}_K^* \).

At any finite place \( v \) of \( L \), call a compactly supported, locally constant function \( f_v \) on \( L_v^* \) strongly positive if it is of the form \( \sum_{j=1}^r c_j \xi_{j,v} \) such that for every \( j \leq r \), \( c_j \geq 0 \), and \( \xi_{j,v} \) takes values in \( \mathbb{R}_+ \). (Here, as usual, \( \xi_{j,v} \) denotes the function \( x \mapsto \xi_{j,v}(x) \).) Thus such an \( f_v \) is of positive type and takes non-negative values.

Write \( S_0 \) for the set of places of \( L \) below \( S \), and define a (factorizable) function \( f = \otimes_v f_v \) on \( \mathbb{A}_L^* \) such that

(a) \( f_v \) is the characteristic function of \( U_v^{(m_v)} \), if \( v \) is a finite place outside \( S_0 \).

(b) If \( v \) is archimedean, let \( f_v(re^{i\theta}) \) equal 1, resp. \( 3 + 2 \cos(k\theta) \), if \( k = 0 \), resp. \( k \neq 0 \);

(c) If \( v \in S_0 \), it is in the (closed) positive cone spanned by functions of the form \( \xi \ast \xi' \), with \( \xi = \sum_{n=1}^r a_n ch_{w,v} \), \( a_n \geq 0 \).

Here \( ch \) denotes the characteristic function, and \( w_v \) the uniformizer at \( v \).

Note that each \( f_v \), and hence \( f \), takes non-negative values on \( \mathbb{A}_L^* \). Moreover, \( f_v \) is invariant under \( U_v = U_v^{(m_v)} \) for each \( v \in S_0 \), and \( f \) is invariant under \( \prod_{v \notin R} U_v \times \prod_{v \in R} U_v^{(m_v)} \). If \( v \) is archimedean, \( f_v \) is by construction invariant under \( \mathbb{R}_+^* \).

Associate to \( f \) a function \( \phi = \otimes_w \phi_w \) on \( \mathbb{A}_M^* \) by defining \( \phi_w \) to be the characteristic function of \( U_w^{(m_w)} \) if \( w \notin S \), and the function \( f_v \) if \( w \in S \), for \( w \mid v \). (This makes sense as \( M_w = L \) for \( w \in S \).)

For \( K \in \{ L, M \} \), put

\[
\mathcal{V}_K := L^2(\mathbb{A}_K^{*+} \backslash \mathbb{A}_K^{*}),
\]

which is a unitary representation under the translation action \( r_K \) of \( \mathbb{A}_K^* \). Then, by the compactness of \( K^* \mathbb{A}_K^{*+} \backslash \mathbb{A}_K^{*} \), we have the decomposition as the Hilbert sum

\[
\mathcal{V}_K \simeq \bigoplus_{n \in \mathbb{Z}[K:Q]} \bigoplus_{\chi \in \chi_K(n)} H_{\chi},
\]

where each \( H_{\chi} \) is the \( \chi \)-isotypic component, which is one-dimensional.
If we denote by $\text{tr}_K(h)$ for the trace of any smooth, compactly supported function $h$ on $\mathbb{A}_K^*$, then for $h$ factorizable,

$$\text{tr}_K(h) = \sum_n \sum_\chi \text{tr} \chi(h) = \sum_n \sum_\chi \prod_v \text{tr} \chi_v(h_v),$$

with

$$\text{tr} \chi_v(h_v) = \hat{h}_v(\chi_v) := \int_{K_v^*} \chi_v(t_v)h_v(t_v)d^x t_v.$$

On the right hand side expression of (1), the product inside is over all the places $v$ of $K$, and since for all but a finite number of places $\chi_v$ is unramified and $h_v$ is the characteristic function of $\mathcal{O}_v$, $\text{tr} \chi_v(h_v)$ equals, with our choice of measures, 1 for almost all $v$, and so the product is finite. Note that at any $v$, if $h_v$ is of the form $\xi_v \ast \xi_v'$, then

$$\text{tr} \chi_v(h_v) = |\text{tr} \chi_v(\xi_v)|^2 \geq 0,$$

which is well known, but let us indicate why. Since $h_v(x) = \int_{L_v^*} \xi_v(xy) \overline{\xi}_v(y)dy$, $\text{tr} \chi_v(h_v)$ equals

$$\int_{L_v^*} \overline{\xi}_v(y)dy \int_{L_v^*} \xi_v(xy)\chi_v(x)d^x x = \text{tr} \chi_v(\xi_v) \int_{L_v^*} \overline{\xi}_v(y)\chi_v(y^{-1})dy,$$

where we have made the change of variables $x \mapsto xy^{-1}$ to get the expression on the right. Now (2) follows as $\chi_v$ is unitary.

Now let $v$ be an archimedean place. Then if $h_v = 3 + 2\cos(k_v \theta_v)$, we have for $\chi_v(re^{i\theta}) = e^{in_v \theta}$,

$$\text{tr} (\chi_v(h_v)) = \int_0^{2\pi} (3e^{in_v \theta} + e^{i(n_v-k_v)\theta} + e^{i(n_v+k_v)\theta}) d\theta = 2\pi \langle n_v, 1 \rangle,$$

where $\langle n_v, 1 \rangle$ is 0, resp. 3, resp. 1, if $n_v \not\in \{0, \pm k_v\}$, resp. $n_v = 0$, resp. $n_v = \pm k_v$. In any case, it is always $\geq 0$ when $h_v$ is of this special form.

Now let $K$ be $L$, with $h$ being our choice of $f = \otimes_v f_v$. Then

$$\text{tr} r_L(f) = \sum_n \sum_{\chi \in \mathcal{X}_L(n)} b(n) \left( \prod_{v \in \mathcal{R}} [\mathcal{O}_v^* : U_v^{(m_v)}] \right) \prod_{v \in \mathcal{S}} |\text{tr} \chi_v(\xi_v)|^2 \geq 0,$$

where $m(\chi) \leq m$ means $m(\chi_v) \leq m_v$ at each finite $v$, the first sum is over $\{n \mid n_v \in \{0, \pm k_v\}, \forall v\}$, and $b(n) = (b(n_w))_{w \mid \infty}$ is s.t. $b(n_w)$ is 6$\pi$, resp. 2$\pi$ when $n_w$ is 0, resp. $\pm k_w$.

Indeed, suppose that for some $\chi \in \mathcal{X}_L$, $m(\chi_v) > m_v$ at some finite place $v$, so that $\chi_v$ is non-trivial on $U_v^{m_v}$. Then, since $f_v$ is invariant
under $U_{v}$:
\[
\text{tr} \chi_{v}(f_{v}) = \sum_{j \in \mathbb{Z}} \chi(\varpi^{j}) \int_{U_{v}^{m_{v}}} f_{v}(\varpi^{j} u) \chi(v(u)) du,
\]
with the inside integral being
\[
\sum_{\varpi \in U_{v}/U_{v}^{(m_{v})}} f_{v}(\varpi^{j} \varpi) \chi(v(u)) du,
\]
which is zero by the orthogonality of characters of the compact group $U_{v}^{(m_{v})}$.

It is important that the sums over $n$ and $\chi$ in the expression (4) are both finite, which is because there are only a finite number of possible infinity types and the conductor of $\chi$ is restricted.

Similarly, for $K = M$ and $h$ being our $\phi$ attached to $f$ (and $S$), (5)
\[
\text{tr} r_{M}(\phi) = \sum_{w} \sum_{\chi' \in X_{M}(w')} \left[ \prod_{w \in B_{M}} [O_{w}^{*} : U_{w}^{(m_{v}(w))} - 1] \prod_{v \in S} |\text{tr} \chi'_{\sigma(v)}(\xi_{v})|^{2},
\]
which is again non-negative, with $v(w)$ denoting the place of $L$ below $w$. As with $\text{tr} r_{L}(f)$, the only tuples $n' = (n'_{w})_{w} \in \mathbb{L}$ which contribute are those with $n'_{w} \in \{0, \pm k_{v}(w)\}$, which restricts the $\chi'$ to lie in a finite set (since its conductor is also restricted).

Now define, for $K \in \{L, M\}$, the $A_{K}^{*}$-stable subspace $V_{K}^{0}$ of $V_{K}$ by
\[
V_{K} = V_{K}^{0} \oplus \mathbb{C},
\]
so that the characters $\chi$ appearing in $V_{K}^{0}$ are all the non-trivial ones (which are 1 on $K^{*}A_{K}^{*,+}$). Let $r_{K}^{0}$ denote the restriction of $r_{K}$ to $V_{K}^{0}$.

Comparing the geometric sides of the two trace formulae below, we will obtain

**Proposition B** Let $(M/L, \chi, R, S, f, \phi)$ be as above, with $\chi \neq 1$. Then
(a) $\text{tr}(r_{L}(f)) \leq C' \text{tr}(r_{M}^{0}(\phi))$, where
\[
C' = (6\pi)^{|L:Q| - |M:Q|} \left( \prod_{v \in R} \frac{[O_{v}^{*} : U_{v}^{(m_{v})}]}{\prod_{w \mid v} [O_{w}^{*} : U_{w}^{(m_{w})}]} \right) \frac{\text{vol}(L^{*}A_{F,+,+}^{*} \setminus A_{L}^{*})}{\text{vol}(M^{*}A_{E,+,+}^{*} \setminus A_{M}^{*})} > 0.
\]
(b) Suppose (0) holds, with $C$ given by the expression there. Then
\[
\text{tr}(r_{K}^{0}(f)) \leq C \text{tr}(r_{M}^{0}(\phi)).
\]
Proof of Proposition B  Since the group is commutative, the con-
jugacy classes are singletons. Noting that $A_{F,+}^* \cap L^* = F_+^*$, we get

$$\text{tr}(r_L(f)) = \text{vol}(L^* A_{F,+}^* \setminus A_L^*) \sum_{\alpha \in F_+^* \setminus L^*} f(\alpha) \geq 0,$$

using the fact that $f$ takes non-negative values.

We may, by replacing each $\alpha$ by its multiple by $\prod_{v \in S} p_v^{\ell_v}$ for suitable
$\ell_v \in \mathbb{Z}$, $p_v = N(v)$, we may choose a representative for $\alpha$ in $L^*$ which
is a unit at every $v \in S$. We have

$$f(\alpha) = \prod_v f_v(\alpha),$$

where at all the finite places $v$ outside $S$, as $f_v$ is the characteristic
function of $U_v^{(m_v)}$, $\alpha_v$ needs to be a unit for $f_v(\alpha)$ to be non-zero. Furthermore, since every $v$ in $S$ is of degree one over $\mathbb{Q}$, say with norm
$p_v$, we may also modify $\alpha$ by such that the modified $\alpha$ has valuation
$0$ at each $v \in S$. (Since the places $v$ in $S$ have distinct norms, the
modification at any $v \in S$ does not affect the situation at another
place (in $S$). Thus, for $f(\alpha)$ to be non-zero, it is necessary for $\alpha$ to be
a global unit with

$$\alpha \in U_v^{m_v}, \forall v < \infty.$$

Similarly, we have, over $M$, the geometric expression

$$\text{tr}(r_M(\phi)) = \text{vol}(M^* A_{E,+}^* \setminus A_M^*) \sum_{\beta \in \mathbb{Q}^* \setminus M^*} \phi(\beta) \geq 0.$$

Since we have a natural injection $F^* \setminus L^* \hookrightarrow E^* \setminus M^*$, we get a lower bound

$$\text{tr}(r_M(\phi)) \geq \text{vol}(M^* A_{E,+}^* \setminus A_M^*) \sum_{\alpha \in F_+^* \setminus L^*} \phi(\alpha) \geq 0,$$

and we call the right hand side the $L$-hereditary part of the trace of
$\phi$.

Lemma C  For every $\alpha$ in $L^*$ lying in $U_v^{(m_v)}$ for all $v < \infty$,

$$f(\alpha) \leq \phi(\alpha).$$

Proof of Lemma C  It suffices to check that for each place $v$ of $L$,

$$f_v(\alpha) \leq \prod_{w \mid v} \phi_w(\alpha).$$

When $v \not\in R \cup S$, each side is 1. When $v \in R$, the left hand side is
1 if $\alpha \in U_v^{(m_v)}$ and 0 otherwise, and in the former case, $\alpha$ is also in
$U_w^{(m_w)}$ for each $w$, and by our choice of $\phi_w$, the value there is 1. So
now consider when $v$ is in $S$. If $w$ is above $v$ but is not $\sigma(v)$, then $\phi_w$ is the characteristic function of the unit group at $w$, and so $\phi_w(\alpha) = 1$ for such $w$. Finally, $\phi_{\sigma(v)} = f_v$, and so takes on the same value at $\alpha$ as $f_v(\alpha)$. Hence the Lemma.

**Back to the proof of Proposition B**

Thanks to Lemma C, we now have, applying (7):

$$\text{(8)} \quad \text{tr}(r_M(\phi)) \geq \text{vol}(M^*\mathbb{A}_{E,+}^* \backslash \mathbb{A}_E^*) \sum_{\alpha \in F_+^* \backslash L^*} f(\alpha) \geq 0,$$

Comparing this with (6), we then obtain

$$\text{(9)} \quad \text{tr}(r_L(f)) \leq B \text{tr}(r_M(\phi)), \quad \text{with } B = \frac{\text{vol}(L^*F_+^* \backslash \mathbb{A}_L^*)}{\text{vol}(M^*\mathbb{A}_{E,+}^* \backslash \mathbb{A}_E^*)},$$

which proves part (a) of Proposition B.

On the other hand, we have by (4),

$$\text{(10)} \quad \text{tr}(r_L(f)) = \text{tr}(r_L^0(f)) + (6\pi)^{|L:Q|} \left( \prod_{v \in R} \text{vol}(U_{mv}^v) \right) \prod_{v \in S} \hat{f}_v(1_v).$$

Similarly by (5), using $\phi_{\sigma(v)} = f_v$,

$$\text{(11)} \quad \text{tr}(r_M(\phi)) = \text{tr}(r_M^0(\phi)) + (6\pi)^{|M:Q|} \left( \prod_{v \in R} \prod_{w \mid v} \text{vol}(U_{mv}^v) \right) \prod_{v \in S} \hat{f}_v(1_v).$$

Consequently, using (9),

$$\text{(12)} \quad \text{tr}(r_L^0(f)) + (6\pi)^{|L:Q|} \prod_{v \in S} \hat{f}_v(1_v) \leq B \text{tr}(r_M^0(\phi)) + (6\pi)^{|M:Q|}B \left( \prod_{v \in R} \frac{\prod_{w \mid v} \text{vol}(U_{mv}^v)}{\text{vol}(U_{mv}^v)} \right) \prod_{v \in S} \hat{f}_v(1_v)$$

Note that by our normalization of measures which gives volume 1 to $U_v$ and $U_w$,

$$\prod_{v \in R} \frac{\prod_{w \mid v} \text{vol}(U_{mv}^v)}{\text{vol}(U_{mv}^v)} = q(R_M, m).$$

The following can be deduced, as remarked by H. Jacquet, for our choice of measures, by imitating the computations in the book, *Basic Number theory*, by A. Weil.

**Lemma D** \text{For } K \in \{L, M\}, \text{ we have}

$$\text{vol}(K^*\mathbb{A}_{K,+}^* \backslash \mathbb{A}_K^*) = \left( \frac{d_K}{d_{K^+}} \right)^{1/2} L(1, \chi_K/K^+).$$
Thus (0) implies that
\[
(6\pi)^{|M:Q|} B \left( \prod_{v \in R} \frac{\prod_{w|v} \text{vol}(U^m_w)}{\text{vol}(U^m_v)} \right) \prod_{v \in S} \hat{f}_v(1_v) \leq (6\pi)^{|L:Q|} \prod_{v \in S} \hat{f}_v(1_v),
\]
which, when used in conjunction with (12), yields part (b) of Proposition B.

(In fact, we didn’t really need Lemma D, as (0) could have been phrased directly in terms of the toric volumes.)

**Back to the proof of Proposition A**

We now have a positive constant $C$, independent of $S, f$, such that
\[
C \text{tr} r_M^0(\phi) - \text{tr} r_L^0(f) \geq 0.
\]

Recall that we started with a character $\chi_0$ in $X_L(k)$ of conductor $\prod_{w \in R} P_w^{m_w}$, so $\text{tr} r_L^0(f) > 0$. Spectrally expanding, we get, using the strong positivity of $\phi$, $f$ and $\phi$,
\[
\text{tr} \chi_0(f) \leq C \sum_{\chi'} \text{tr} \chi'(\phi),
\]
where $\chi'$ runs over Hecke characters in $\bigoplus_{k'} \chi_M^0(k')$, with each $k'_w \in \{0, \pm k_w\}$ if $w | v$, and with conductor $\prod_{w \in R} P_w^{m'_w}$ with $m'_w \leq m_w$. In any case the number of $\chi'$ which intervene are finite in number.

Note that $\text{tr} \chi_v(f_v)$ is 1 when $v$ is not in the set $S_0$ below $S$, and similarly, $\text{tr} \chi'_v(\phi_w) = 1$ when $w \not\in S$. At any archimedean place, the trace is $2\pi$ or $6\pi$ from (3) (as we have reduced to considering only those $n_v$, resp. $n'_w$, which are in $\{0, \pm k_v\}$). Putting these together, we obtain, using $\phi_{\sigma(v)} = f_v$ for $w = \sigma(v) \in S$,
\[
(13) \quad \prod_{v \in S_0} \hat{f}_v(\chi_v) \leq A \sum_{\chi'} \prod_{v \in S_0} \hat{f}_v(\chi'_v(\chi'_v(v))), \quad A = (2\pi)^{|M:Q| - [L:Q]} 3^{[M:Q]} C > 0.
\]
Since our $f_v$ is, for each $v \in S_0$, of the form $\xi_v * \xi'_v$, with $\xi_v = \sum m b_m ch_{w^m U_v}$ with $b_m \geq 0$ (and 0 for all but a finite number of $m \in \mathbb{Z}$), we get for any unramified character $\lambda$ of $F^*$, using $\text{tr} r_L^0(f) = 1$,
\[
(14) \quad \hat{f}_v(\lambda) = |\xi_v(\lambda)|^2 = |\sum m b_m q_v^{-m} \lambda(\varpi_v)^m|^2.
\]
We have freedom in choosing the $b_m$, as long as they are non-negative and are non-zero only for a finite number of $m$.

We have to show that for some $\chi', \chi'(\varpi_v)$ equals $\chi(\varpi_v)$ at every $v$ in $S_0$. Suppose not. Then there is a non-empty subset $S_1$ of $S_0$ such that
\[
(15) \quad \chi'(\varpi_v) \neq \chi(\varpi_v), \forall v \in S_1.
\]
And (13) becomes
\[ \prod_{v \in S_1} |\hat{\xi}_v(\chi_v)|^2 \leq A \sum_{\chi'} \prod_{v \in S_1} |\hat{\xi}_v(\chi'_v(v))|^2. \]
We get a contradiction once we prove the following

**Lemma E**  For every \( v \in S_1 \), and for any pair of positive real numbers \( \alpha, \beta \) with \( \alpha > \beta \), there exists \( \xi_v \) such that
\[ |\hat{\xi}_v(\chi_v)| > \alpha, \text{ and } \hat{\xi}_v(\chi'_v) < \beta, \forall \chi'. \]

Fix any \( v \in S_1 \), and enumerate the \( \chi' \) in the relevant finite set as \( \chi_{1}, \ldots, \chi' \). Since \( \chi_v, \chi'_v \) are unitary, we may write \( \chi_v(\overline{w}_v) = q^{it_0}_v \) and \( \chi'_{j,v}(\overline{w}_v) = q^{it_j}_v, 1 \leq j \leq r \). Clearly, \( t_0, t_j \) are uniquely defined only up to translation by \( 2\pi (\log q_v)Z \). Write \( \theta_n = t_n (\log q_v)/2\pi \), for every \( n = 0, 1, \ldots, r \). By hypothesis, \( \theta_j \neq \theta_0 \) for any \( j \geq 1 \). Moreover, by our choice of \( S \), no \( c_i \) takes the value \( -1 \) on \( \overline{w}_v \), so if we think of \( \theta_n \) as lying in \([\pi, \pi]\), it never takes the value \( 0 \) for any \( n \geq 1 \). If \( \xi_v = \sum_m b_m \chi^{mU}_v \), with \( b_m \geq 0 \), writing \( \chi_v \) as \( \chi_{0,v} \) and putting \( c_m = b_m q^{-m}_v \), we then have,
\[ \hat{\xi}_v(\chi'_{n,v}) = g(\theta_n) := 2 \sum_{m \geq 0} c_m e^{i\theta_n}, \forall n = 0, 1, \ldots, r. \]
Such a finite Fourier series \( g(\theta) \) necessarily takes its maximum value at \( \theta = 0 \), as \( c_m \geq 0 \). The Lemma is easy to check when \( \theta_0 = 0 \), so we may assume that there is an \( \varepsilon > 0 \) such that \( \theta_n \) lies in the set \( J := (\pi, \pi] - (\varepsilon, \varepsilon) \), for every \( n = 0, 1, \ldots, r \). Choose an even, smooth, positive function \( h \) on \([\pi, \pi]\) such that its absolute value satisfies the requisite inequalities at the \( \theta_j, 0 \leq j \leq r \). We can also make \( h \) a positive definite function by choosing it to be sufficiently large at \( 0 \). Then the coefficients \( c_m \) are non-negative for the Fourier series representing \( h \), which converges in \( C^\infty \)-norm, implying uniform convergence of the series and the derivatives. So we may uniformly approximate \( h \) by a positive sum of characters. Since the inequalities satisfied by it form an open condition, the approximating positive character sum also satisfies the same inequalities at the \( \theta'_j \) and at \( \theta_0 \). \( \square \)

**The passage from \( S \) to a set of positive density**

Fix \( k = (k_v) \), \( R, m = (m_v) \) as above. Let \( Y \) be the set of primes \( w \) of \( M \) which are of degree one over \( \mathbb{Q} \) and where no character \( \chi' \) of \( M \) of level dividing \( a := \prod_{v \in R} \prod_{w|v} P_{w}^{m_v} \) of weight \( k' = (k'_w) \) with \( k'_w \in \{0, \pm k_v(w)\} \) has the value \( 1 \) on \( \overline{w}_v \).

**Lemma F**  \( Y \) has positive density. If \( k \neq 0 \), it has density one.
Proof of Lemma E  First look at $\chi'$ of weight 0, when it defines a ray class character of conductor dividing $a$, and if the corresponding ray class group is a product $C_1 \times \cdots \times C_n$, with each $C_j$ cyclic of generator $x_j$, Tchebotarev assures of the existence of a set $Z_\nu$, say, of primes $P_\nu$ of positive density $\delta$ whose Frobenius elements are $(x_1, \ldots, x_n)$. ($\delta$ is at least $1/h_M(a)$.)

Now suppose $\chi'$ is of weight $k' \neq 0$. We claim that the set $Z$ of primes $P_\nu$ where $\chi'(\overline{w}_\nu) = 1$ has density zero. Indeed, if $Z$ has positive density, as $\chi'$ and the trivial character agree on $Z$, their ratio will need to be of finite order, hence of weight 0, contradicting the fact that $k' \neq 0$. The Lemma now follows because there are only a finite number of relevant $k'$ and hence $\chi'$ (because of bounded ramification).

Let $Y_0$ denote the set of rational primes below $Y$, and write elements of $Y_0$ as $p_1, p_2, \ldots, p_n, \ldots$ with $p_i < p_j$ if $i < j$. Then enumerate the primes of $M$ lying in $Y$ as $\{v_{i,j} \mid N(w_{i,j}) = p_i\}$. Given a finite set $S_0$ of $Y_0$ and a section $\sigma : S_0 \to Y$, i.e., a choice of a place $w = \sigma(p)$ for every $p \in S_0$, it defines uniquely a finite set $S$ of the type we have considered above, and every such $S$ arises this way. Let $\chi'(S) (= \chi'(S_0, \sigma))$ be the character of $M$ attached above to $\chi$.

Lemma G  There exists a (unique) character $\chi_M$ from among the family of characters $\{\chi'(S_0, \sigma)\}$ indexed by $(S_0, \sigma)$, such that for all place $w$ in $Y$ but a finite number, we have

$$\chi_M, w = \chi_v(w).$$

Proof of Lemma G

Let $N$ denote the cardinality of the finite set of anti-cyclotomic Hecke characters $\chi'$ of $M$ of conductor dividing $a$ and weight $k' = (k'_w)$ such that for each archimedean $w$ above $v$, $k'_w \in \{0, \pm k_v\}$. Put

$$\nu := \log N / \log (1 + (1/(d - 1))$$

Suppose the Proposition is false. Then there exists some integer $m > \nu$ such that, for any $\pi_{E, v}^s$, $(\ast)$ fails to hold at $\nu + 1$ (or more) places $v_j$ with $j \leq m$. Fix such an $m$, and define an equivalence relation on the finite set $S_m$ of restrictions of sections, again denoted by $s$, to $\{v_1, \ldots, v_m\}$, by setting

$$s \sim s' \Leftrightarrow \pi_{E, s(v_j)}^s \simeq \pi_{E, s'(v_j)}^s.$$

Then the set $S_m/ \sim$ of equivalence classes $[s]$ in $S_m$ has cardinality bounded above by $N$. Of course, $|S_m| = d^m$. Then, by the pigeon-hole principle, at least one class $[s]$, call it $O_m$, must have at least $d^m / N$ members in $S_m$. 

Note that if (*) fails to hold at $r$ places in $S_m$, then the cardinality of $O_m$ can be at most $(d-1)^r d^{m-r} = \left(\frac{d-1}{d}\right)^r d^m$. Since $O_m \geq \frac{d^m}{N}$, it follows that

$$\frac{1}{N} \geq \left(\frac{d-1}{d}\right)^r,$$

or equivalently,

$$r \leq \frac{\log N}{\log (1 + (1/(d-1)))} = \nu.$$

Since $r \geq \nu + 1$, we get the desired contradiction.

This completes the proof of the main Theorem.

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