Lecture 9

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(The odd numbered lectures are given by D. Ramakrishnan, and the even ones by R. Tanner.)

0.1 Comparison of m with m_{BH} and m_{MM}

Let m denote the slope at $s = 0$ obtained in Lecture 7. Put

$$m_{BH} := \frac{k_1 V}{k_2 + k_3}, \tag{6.1a}$$

which is the slope in the Briggs-Haldane model, and

$$m_{MM} := \frac{k_1 V}{k_2}, \tag{6.1b}$$

the slope coming from the Michaelis-Menton model.

Proposition *We have*

$$m_{BH} \leq m \leq m_{MM}.$$

0.2 A quadratic approximation for small s

We explicated the linear term of our method in the previous lectures and compared it to the Briggs-Haldane and Michaelis-Menton approaches. Our approach can go much further and give an approximation of any order desired (for small s). In the next section we will in fact give an exact infinite series expansion, from which approximations of any order can be deduced. Now let us delineate the quadratic case. We will preserve the earlier notations involving v_n, c_n, etc., denoting the values of v, c, etc., at the n-th stage of the recursion. Let us write

$$v_n = m_n s + q_n s^2 + O(s^3), \tag{6.2}$$

where m_n is the slope at the n-th stage, and q_n the coefficient controlling the quadratic term. As usual, $O(s^r)$ denotes, for any $r > 0$, a sum of terms of order at least s^r. It follows by differentiation that

$$[v]'_n = m_n s' + 2q_n ss' + s'O(s^2) + O(s^3). \tag{6.3}$$

Since $v = k_3 c$, we get from (6.2),

$$k_2 c_n = \frac{k_2}{k_3} m_n s + \frac{k_2}{k_3} q_n s^2 + O(s^3), \tag{6.4a}$$

and

$$k_1 s c_n = \frac{k_1}{k_3} m_n s^2 + O(s^3). \tag{6.4b}$$

These give, thanks to (1) and (5), at stage n,

$$s', = \left(\frac{k_2}{k_3} m_n - \frac{k_1}{k_3} V\right) s + \frac{(k_1 m_n + k_2 q_n)}{k_3} s^2 O(s^3). \tag{6.5}$$

In particular, s' is $O(s)$, and

$$ss' = \left(\frac{k_2}{k_3} m_n - \frac{k_1}{k_3} V\right) s^2 + O(s^3).$$

So (6.3) simplifies as

$$k_3 c'_n = m_n s' + \left(\frac{k_2}{k_3} m_n - \frac{k_1}{k_3} V\right) s^2 + O(s^3). \tag{6.6}$$

The recursion is defined in such a way that

$$v_{n+1} = -s'_n - c'_n, \tag{6.7}$$

where s_n denotes the value of s' at stage n. Thus, by (6.6),

$$v_{n+1} = -\left(1 + \frac{m_n}{k_3}\right) s'_n - \left(\frac{k_2}{k_3} m_n - \frac{k_1}{k_3} V\right) s^2 + O(s^3). \tag{6.8}$$

Plugging in (6.5), we then get
(6.9)

$$v_{n+1} = -\left(1 + \frac{m_n}{k_3}\right)\left(\frac{k_2}{k_3} m_n - \frac{k_1}{k_3} V\right) s - \left((1 + \frac{m_n}{k_3})(\frac{(k_1 m_n + k_2 q_n)}{k_3}) - (\frac{k_2}{k_3} m_n - \frac{k_1}{k_3} V)\right).$$

Since this is $m_{n+1}s + q_{n+1}s^2 + O(s^3)$, we obtain

$$q_{n+1} = -\left(1 + \frac{m_n}{k_3}\right)\left(\frac{k_1 m_n + k_2 q_n}{k_3}\right). \tag{6.10}$$

Now we let n go to infinity, and obtain, for $q = \lim_{n\to\infty} q_n$, the relation

$$q = -\left(1 + \frac{m}{k_3}\right)\left(\frac{k_1 m + k_2 q}{k_3}\right).$$

This yields

$$q = -\frac{(1 + \frac{m}{k_3})\frac{k_1 m}{k_3}}{1 + (1 + \frac{m}{k_3})\frac{k_2}{k_3}}. \tag{6.11}$$

Note that this is independent of e_0.

Putting $v = \lim_{n\to\infty} v_n$, this gives the expression

$$v = ms - \left(\frac{(k_3 + m)k_1 m}{k_3^2 + (k_3 + m)k_2}\right) s^2 + O(s^3), \tag{6.12}$$

where m is given by (5.12b). Consequently,

$$\frac{dv}{ds} = m - \left(\frac{2(k_3 + m)k_1 m}{k_3^2 + (k_3 + m)k_2}\right) s + O(s^2). \tag{6.13}$$

If s_0 is small, then so is S_p, in which case this quadratic approximation has some validity near it, allowing to deduce that S_p is close to $\frac{k_3^2 + (k_3+m)k_2}{2(k_3+m)k_1}$.