Lecture 11

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(The odd numbered lectures are given by D. Ramakrishnan, and the even ones by R. Tanner.)

0.1 A quadratic approximation at $s = s_0$

This section is not delicate as the previous ones, since the s-derivatives of $v\frac{dp}{ds}$ are all well defined and easily calculated at s_0 (unlike at $s = 0$). Nevertheless, the formulae below are useful in the following section. As before, we will write s' for $\frac{ds}{dt}$, $c' = \frac{dc}{dt}$, etc.

Lemma *t the point $s = s_0$, the following values hold:*

(a) $s' = -k_2 s_0 e_0, \quad c' = k_1 s_0 e_0,$ *and* $e' = k_1 s_0 e_0.$

(b) $\frac{dv}{ds} = -k_3.$

(c) $\frac{d^2v}{ds^2} = -\frac{k_3^2}{k_1 s_0 e_0}.$

Consequently, the quadratic Taylor approximation to v near $s = s_0$ is given by

$$v = -k_3(s - s_0) - \frac{k_3^2}{2k_1 s_0 e_0}(s - s_0)^2 + O((s - s_0)^3).$$

Proof. (a): This follows directly from the basic differential equations by evaluation at s_0.

(b): We saw in the proof of Lemma 2.1 that

$$\frac{dc}{ds} = -1 - k_3\frac{c}{s'}.$$

Since $v = k_3 c$ and $c = e_0 - e$ is zero at s_0, we get $\frac{dv}{ds} = -k_3$.

(c): Differentiating relative to t,

$$\frac{d}{dt}\left(\frac{dc}{ds}\right) = -k_3\frac{s'c' - s''c}{(s')^2}.$$

0.2 Approximations to s_p

Now that we have expansions for v at 0 and at s_0, we can find a series of approximations $s_{p,n}$ to s_p, which will be good for small s_0, by equating the n-th order terms of the respective expansions.

Proposition

(a) $s_{p,1} = \frac{k_3 s_0}{m+k_3}$;

(b) $s_{p,2}$ *satisfies a quadratic equation:*

$$AX^2 + BX + C = 0,$$

with

$$A = \left(\frac{k_3^2}{2k_1 s_0 e_0} - \frac{(k_3+m)k_1 m}{k_3^2 + k_3 + mk_2} \right),$$

$$B = \left(\frac{k_3}{2k_1 e_0} - m - k_3 \right),$$

and

$$C = \left(\frac{k_3^2}{2k_1 e_0} - k_3 s_0 \right) \left(\frac{k_3}{2k_1 s_0 e_0} - 1 \right).$$

Note that $s_{p,1}$ corresponds to the s-coordinate of the point obtained by intersecting the tangent lines to the (v, s)-curve at $s = 0$ and $s = s_0$. On the other hand, $s_{p,2}$ denotes the s-coordinate of the meeting of the quadratic approximations to the (v, s)-curve at 0 and s_0, which provides a better approximation.