Lecture 6 **Dinakar Ramakrishnan**

Stability

This is a very important and not so transparent a concept. It arises from the need to know if the solutions one comes up with for differential equations are stable, especially the special solutions called the equilibrium solutions. What one wants to know is this:

If we start with a solution close to a desired solution, will we stay close to this solution as things evolve in time?

This is of practical importance, as it is next to impossible in real situations to manufacture exact solutions, and when one deviates a little initially, one needs to know that what evolves later will stick close to the plan.

Suppose we're given a 1st order ODE:

$$\frac{dy}{dt} = f(t, y). \qquad (*)$$

When we succeed in solving this equation, we find that there are infinitely many solutions involving some constant c, due to the ambiguity arising from indefinite integration. Very important solutions are those which satisfy $\frac{dy}{dt} = 0$; they are called the *equilibrium solutions.*

Often, but not always, we start with an initial condition say $y = y_0$ at $t = 0$, which allows us to solve for c and find one solution. At times, though, it is profitable to look at the structure of all the solutions, without fixing an initial state.

There are many solutions if we do not fix an initial condition!

What one wants to do:

i) Look for *stable* solutions;

ii) Check if the equilibrium solutions are *asymptotically stable.*

Intuitive Definition Suppose $y_1(t)$ is a solution of $(*)$. We say that it's *stable* iff any other solution $y_2(t)$ which starts out being close to $y_1(t)$ at $t = 0$ remains close to $y_1(t)$ for all $t > 0$.

To make this precise, we need to quantify what it means to be close:

A precise definition of stability:

Start with a fixed solution $y_1(t)$ of $(*)$. It is a stable solution if for every $\epsilon > 0$, $\exists\, \delta > 0$ such that, for every other solution $y_2(t)$ satisfying

$$|y_1(0) - y_2(0)| < \delta,$$

one has

$$|y_2(t) - y_1(t)| < \epsilon, \ \ \forall t > 0.$$

Example:

$$(\#) \qquad \frac{dy}{dt} = ry$$

The only possible —it equilibrium solution is $y = 0$ when $r \neq 0$. When $r = 0$, every solution, necessarily of the form $y = \textit{constant}$, is an equilibrium solution,

When $y \neq 0$,

$$\frac{1}{y}\frac{dy}{dt} = r$$

$$\begin{aligned} \int \frac{d}{dt}(\log|y|)dt &= r\int dt \\ &= rt + c \end{aligned}$$

General solution:

$$y \ = \ Be^{rt},$$

where B is any constant, with $B = 0$ corresponding to the equilibrium solution $y = 0$.

First consider when $\mathbf{r < 0}$. The solution Be^{rt}, for any fixed B, approaches the equilibrium solution $y = 0$ as $t \to \infty$. We claim the following:

Every solution of $(\#)$ *is stable for* $r < 0$.

Indeed, fix a solution $y_B(t) = Be^{rt}$, and consider any other solution $y_C(t) = Ce^{rt}$, and an arbitrary positive number ϵ. We have to be able to choose a $\delta > 0$ such that whenever $|B - C| < \delta$, we have $|y_B(t) - y_C(t)| < \epsilon$. Observe that since $r < 0$, e^{rt} is a decreasing function, so $0 \leq e^{rt} \leq 1$, $\forall\, t \geq 0$. So we may just choose $\delta = \epsilon$, since

$$|B - C| < \epsilon \implies |Be^{rt} - Ce^{rt}| = |B - C|e^{rt} \leq |B - C| < \epsilon,$$

for all $t > 0$. Hence the claim.

Next consider when $r > 0$. We claim that *no solution* $y_B(t) = Be^{rt}$ *is stable in this case*. This is so because e^{rt} is an increasing function and in fact becomes unbounded as t becomes arbitrarily large. So, given any $\epsilon > 0$, for any $C \neq B$, $|Be^{rt} - Ce^{rt}|$ goes to ∞ as $t \to \infty$, and so becomes larger than ϵ, regardless of how small $|B - C|$ is. The claim follows.

Finally, let $r = 0$. Then every solution is of the form $y(t) = c$, and it is a equilibrium solution. Fix a c, and consider a nearby solution $y_b(t) = b$. Then, $|y_b(0) - y(0)|$ is the same as $|y_b(t) - y(t)|$ for any t, implying that if b is close to c at $t = 0$, then it remains close for all t. (We can take $\delta = \epsilon$ here, if one wants to be explicit.) So the equilibrium solutions, which are all the solutions, are stable when $r = 0$.

Definition An equilibrium solution y_{eq}, say, is *asymptotically stable* if any other solution near it at $t = 0$ becomes asymptotic to y_{eq} in the limit as t goes to ∞.

Note that in the above example, the equilibrium solutions is asymptotically stable if $r < 0$, but not when $r \geq 0$. When $r = 0$, all the solutions are equilibrium solutions and all are stable, but not asymptotically stable, as the solutions are constant.

Example:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y, \;\; r, K > 0$$

This represents a better model for population growth than the simple minded one we looked at before without the quadratic correction term.

Equilibrium points: $y = 0$ and $y = K$

Suppose we look at the equilibrium solution $y = 0$. We want to look at all the solutions $y(t)$ which are close to 0 at $t = 0$ and see how they evolve. For this note that for y very close to 0, y^2 negligible compared to y, and the given differential equation is approximated by its *linearization*:

$$\frac{dy}{dt} = ry$$

We have already seen that the general solution of this linear ODE is given by $y = Be^{rt}$, with B a constant. We will now admit *a general fact* (which we may discuss in the last week of the term if time permits), which says that the evolution of the solution $y(t)$ of the non-linear ODE (with $y(0)$ close to

0) remains close to the evolution of a corresponding solution of the linearized ODE, and has the same asymptotic as $t \to \infty$. This means that we can check the asymptotic stability by looking at how $y(t)$ behaves as $t \to \infty$ if it were to satisfy the associated linear ODE. If $y(0) = B$ with $|B|$ small, $\lim_{t\to\infty} y(t)$ is then given by $\lim_{t\to\infty} Be^{rt}$, which is unbounded. Hence this equilibrium solution, namely $y = 0$ is not asymptotically stable.

Next look at the other equilibrium solution $y = K$. Let us change variables and put $u = y - K$, so that the (non-linear) ODE becomes

$$\frac{du}{dt} = \left(\frac{-r}{K}\right) u(u + K),$$

and in this changed coordinates, the equilibrium solution $y = K$ becomes $u = 0$. Again, we can approximate any solution $u(t)$ with $u(0)$ close to 0 by a corresponding solution of the linearized ODE:

$$\frac{du}{dt} = -ru,$$

whose general solution is $u(t) = Be^{-rt}$, with B a constant; putting $B = 0$ gives the equilibrium solution $u = 0$. So for $|B|$ small, we need to check that $\lim_{t\to\infty} Be^{-rt}$ equals 0, which is true since $r > 0$. So $u = 0$ is an asymptotically stable solution of the non-linear equation (in u), which in turn implies the same for the equilibrium solution $y = K$ of the non-linear ODE we started with.

Difference equations

We have been looking at *ODE*'s of the form $\frac{dy}{dt} = f(t, y)$, where $y(t)$: a differentiable function of x, often, but not always, with a fixed initial condition: $y_0 = y(0)$. Now we will consider its **discrete analog**:

$$y_{n+1} - y_n = f(n, y_n),$$

where y_n is a sequence, with initial value y_0. Here the independent variable is $n \geq 0$ in $\mathbb{Z}$, and instead of the continuously varying dependent variable $y(t)$, we have the discretely varying sequence $y_0, y_1, y_2, y_3, \ldots, y_n, \ldots$. Moreover, the analog of the derivative $\frac{dy}{dt}$ is the difference quotient

$$\frac{y_{n+1} - y_n}{(n+1) - n} = y_{n+1} - y_n.$$

One often rewrites the difference equation above as

$$y_{n+1} = g(n, y_n),$$

where $g(n, y_n) = f(n, y_n) + y_n$.

The *equilibrium solution* is where the difference quotient $\frac{y_{n+1}-y_n}{(n+1)-n}$ is zero, i.e., where $y_{n+1} = y_n$.

Instead of starting the sequence at the usual $n = 0$ point, we could start with any integer and go to infinity. In fact we can also replace the indexing sequence n by any other strictly increasing sequence $\{t_n\}$.

Examples

1) $y_{n+1} = (-1)^{n+1} y_n$:

   The sequence $\{y_n\}$ is $y_0, y_1 = -y_0, y_2 = -y_1, \ldots$, oscillating oscillates between y_0 and $-y_0$, so there is no limit as $n \to \infty$. Moreover, the only possible equilibrium solution is $y_n = 0$.

2) $y_{n+1} = \frac{n+1}{2n+1} y_n$, $y_0 = 1$:

   Put $L = \lim_{n\to\infty} y_n$ if it exists. Then, letting n go to ∞ on both sides of the difference equation, we get

$$L = \left(\lim_{n\to\infty} \frac{n+1}{2n+1} \right) L.$$

   Since $\frac{n+1}{2n+1} = \frac{1}{2}\left(\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\right)$ goes to $\frac{1}{2}$ as $n \to \infty$, we get $L = \frac{L}{2}$, which implies that $L = 0$ if L exists. One usually says that there is no limit even if $L = \pm\infty$ (which is different from the sequence $\{\sin(n)\}$ not having any limit). Here each y_n is positive and also $y_{n+1} < y_n$, so the sequence can't go to ∞ (or $-\infty$). Explicitly,

$$y_1 = y_0 = 1, y_2 = \frac{2}{3}, y_3 = \left(\frac{3}{5}\right)\left(\frac{2}{3}\right) = \frac{2}{5}, y_4 = \left(\frac{4}{7}\right)\left(\frac{2}{5}\right) = \frac{8}{35}, \ldots$$

   Since $\frac{n+1}{2n+1}$ decreases from 2/3 monotonically to 1/2 in the limit, we have for any $n \geq 1$, $0 < y_{n+1} < \left(\frac{2}{3}\right) y_n < \left(\frac{2}{3}\right)^n$, and since $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n =$ (as $\frac{2}{3} < 1$), the squeeze theorem for limits (see the online Notes for Ma1a) implies that $\{y_n\}$ has the limit 0 as expected.

Lecture 7

More remarks on stability (concerning first order ODEs)

$$\frac{dy}{dt} = f(t, y)$$

Typically, t represents time, and we usually look only at the asymptotics as $t \to \infty$. However, sometimes it also makes sense to look at $t \to -\infty$.

Recall: A solution $y_1(t)$ is *stable* iff any other solution $y_2(t)$ close to $y_1(t)$ at $t = 0$ (or any other starting pt $t = t_0$) remains close to $y_1(t)$ for all $t > 0$. More precisely, given any $\epsilon > 0$, we need to be able to find a $\delta > 0$ such that

$$|y_1(0) - y_2(0)| < \delta \implies |y_1(t) - y_2(t)| < \epsilon, \ \forall t > 0.$$

Definition *An equilibrium solution $y_{eq}(t)$ is asymptotically stable iff for any solution $y_2(t)$ close to $y_{eq}(t)$ at $t = 0$, we have*

$$\lim_{t\to\infty} |y_{eq}(t) - y_2(t)| = 0.$$

Facts:

1) If $f(t, y)$ is continuous, then asymptotically stable implies stable; the converse is not true.
   Counterexample: $\frac{dy}{dt} = 0$, all of whose the solutions are of the form $y(t) = c$, c a constant, and there is a unique equilibrium solution y_{eq} corresponding to $c = 0$. Since the solutions are constants, if y is close to y_{eq} at $t = 0$, it will remain close, in fact at the same distance, for all $t > 0$, implying that y_{eq} is stable. (In fact, every solution is stable.) However, for any $c \neq 0$, the solution $y = c$ will not approach y_{eq} as $t \to \infty$. Hence y_{eq} is not asymptotically stable.

2) There is **a useful criterion to check for the asymptotic stability of an equilibrium solution** y_{eq} of $\frac{dy}{dt} = f(t, y)$ *when*, and this is a serious condition to check, $f(t, y) = \varphi(y)$, i.e., f *does not involve* t, only y. In this case, if we put $\varphi'(y) = \frac{d\varphi}{dy}$, then

   $y_{eq}(t)$ is *asymptotically stable* if $\varphi'(y_{eq}) < 0$; and

   $y_{eq}(t)$ is unstable if $\varphi'(y_{eq}) > 0$.

One can draw no conclusion if $\varphi'(y_{eq}) = 0$. This criterion is sometimes called the *linear stability test*, not because $\varphi(y)$ needs to be linear, but because whether or not y_{eq} is asymptotically stable depends only on the linear approximation of $\varphi(y)$ at $y = y_{eq}$. Recall (from Ma1a) that by Taylor's theorem, $\varphi(y)$ is approximated to first order near any $y = b$ by $\varphi(b) + \varphi'(b)(y - b)$, and when $b = y_{eq}$, $\varphi(y_{eq}) = 0$ (since it is an equilibrium solution of the ODE), and so $\varphi'(y_{eq})$ determines the first order Taylor approximation, and the stability criterion requires only that, unless $\varphi'(y_{eq}) = 0$, in which case one gets no information.

Example: $\frac{dy}{dt} = y(1 - y)$

Equilibrium points: $y = 1, y = 0$

Put $\varphi(y) = y(1 - y)$, which does not involve t and is moreover differentiable (being a polynomial in y), so we may apply the criterion above for checking asymptotic stability. Clearly, $\varphi'(y)(= \frac{d\varphi}{dy})$ is $1 - 2y$, which is positive at $y = 0$ and is negative at $y = 1$.

Conclusion: The equilibrium solution $y = 0$ is unstable, while $y = 1$ is asymptotically stable.

Difference Equations:

Continuous realm	*Discrete realm*
t: continous parameter $t \to \infty$	n: positive integer $n \to \infty$
y: function of t	$\{y_n\}$: sequence
$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$ $\frac{d^2y}{dt^2}$ $\frac{d^my}{dt^m}$ $\sum_{j=0}^{n} a_j \frac{d^jy}{dt^j} = \varphi(t)$	$\frac{y_{n+1}-y_n}{n+1-n} = y_{n+1} - y_n$ $y_{n+2} - y_n$ $y_{n+m} - y_n$ $\sum_{j=0}^{m} a_j(y_{n+j} - y_n) = \varphi(n)$ $\Leftrightarrow \sum_{j=0}^{m} b_j y_{n+j} = 0$ $b_j = a_j$ for $j > 0$, $b_0 = -\sum_{j=1}^{m} a_j$
First order: $\frac{dy}{dt} = g(t, y)$	 $y_{n+1} - y_n = g(n, y_n)$ $\Leftrightarrow y_{n+1} = f(n, y_n) = g(n_1 y_n) + y_n$
Equilibrium solutions: when $\frac{dy}{dt} = g(t, y) = 0$	 when $y_{n+1} = f(n, y_n) = y_n$ $f(n, y_n) = g(n, y_n) + y_n$ i.e., $g(n, y_n) = 0$ and $y_{n+1} = y_n$

Definition An equilbrium solution y_n, i.e., a solution of $y_{n+1} = f(n, y_n)$ with $y_n = y_{n+1}$, is *asymptotically stable* iff for any close solution $z_n = y_n + v_n$, where v_n is a small perturbation,

$$\lim_{n \to \infty} |y_n - z_n| = 0.$$

An important special case:

$$y_{n+1} = f(y_n)$$

Here f does not involve n, i.e., $f(n, y_n) = f(y_n)$. Such an f is called a **transformation**, and any equilibrium solution is just a **fixed point**: $y_n = f(y_n)$.

Example: $y_{n+1} = f(y_n) = y_n^2$ (quadratic not linear).

Equilibrium solutions: $y_{n+1} = y_n^2 = y_n$, i.e., $y_n = 0$ or $y_n = 1$.
$\mathbf{y_n = 0}$: We argue as in the continuous case (first order ODE). If z_n is close to $y_n = 0$, then $z_n^2 \approx 0$ (very small). The difference equation $y_{n+1} = y_n^2$ is close to the linear equation $y_{n+1} = 0$. (One drops the quadratic term in the approximation, but *not* the linear term.)

Conclusion: z_{n+j} remains close to 0 for all j if it is so for $j = 0$. So $y_n = 0$ is asymptotically stable.
The case of $\mathbf{y_n = 1}$ is left as an exercise.

Note that in this example, $y_n = (y_0)^{2^n}$, so

$$\lim_{n\to\infty} y_n = \begin{cases} 0, \text{if } |y_0| < 1 \\ 1, \text{if } y_0 = 1 \\ \text{undefined otherwise} \end{cases}$$

Lecture 8

Euler's Approximate solution

Whenever we try to find an approximate solution to $y' = f(t, y)$, we inadvertently move to a discrete situation. A prime example is furnished by Euler's method for finding approximate solutions to first order ODE's of the form

$$y' = f(t, y), \; y_0 = y(0).$$

Recall that even if we have difficulty solving it, we can always get a qualitative picture by considering the slope field $f(t_j, y_j)$ drawn on a grid $\{(t_j, y_j)\}$ of points in the (t, y)-plane.

If $f(t, y)$ is continuous, then this gives a good picture, and one can visualize different possible flow lines of solutions by seeing how the arrows point and evolve with time.

$$f(t_0, y_0) \;=\; \text{slope at } \; t_0$$

Even if we know the starting point, there are many possible paths by which $y(t)$ could evolve.

Euler's Idea:

Pick some small number $h > 0$, and consider

$$\begin{array}{ccccccccccccc} t_0 & < & t_1 & < & t_2 & < & \dots & < & t_n & < & t_{n+1} & < & \dots \\ \shortparallel & & \shortparallel & & \shortparallel & & & & & & & & \\ 0 & & h & & 2h & & & & & & & & \end{array}$$

$t_n h$, and $f(t_j, y_j)$ gives the slope at each (t_j, y_j).

Start with the line L_0 at $y = y_0$ given by the equation

$$y = \underbrace{y_0}_{y(t_0)} + \underbrace{(\text{slope at } t_0)}_{f(t_0, y_0)}(t - t_0)$$

This gives a linear approximation to the true solution $y = \varphi(t)$ when $t - t_0$ is very small.

At t_1, let y_1 denote the y-coordinate of L_0 at t_1, i.e., $y_1 = y_0 + f(t_0, y_0)t$ (assuming $t_0 = 0$). Then y_1 is different from $\varphi(t_1)$, but it will be close if h is small. Draw the line L_1, say, starting at (t_1, u_1), of slope $f(t_1, y_1)$.

The next point in Euler's approximation is (t_2, y_2) with

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1).$$

There draw a line L_2 starting at (t_2, y_2) of slope $f(t_2, y_2)$. We proceed this way till we reach a point of interest, say t_n. This gives a **piecewise linear approximation** to the true solution curve $y = \varphi(t)$ (which we don't know).

The approximation gets better as $h = t_{n+1} - t_n$ is made smaller. Thus we get a sequence of numbers $y_0, y_1, y_2, \ldots$ satisfying the difference equation

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$$

More on difference equations

Given

$$y_{n+1} = f(n, y_n),$$

we want to know the following:

(i) The limit $L := \lim_{n\to\infty} y_n$, if it exists

(ii) The equilibrium points

(iii) Asymptotic stability

Examples:

(1)

$$y_{n+1} = \frac{1}{1 + \frac{1}{y_n}}$$

Suppose L exists, then

$$L = \lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n+1}.$$

As $n \to \infty$, the difference equation tends to the limiting equation

$$L = \frac{1}{1 + \frac{1}{L}}$$
$$L = \frac{L}{L+1} \Rightarrow L(L+1) = L \Rightarrow L = 0.$$

Start with y_0; if $y_0 > 0$, then all y_n's are > 0.

Then the limit L should be ≥ 0. Explicitly,

$$y_0 = 1, y_1 = \tfrac{1}{1+1} = \tfrac{1}{2}, y_2 = \tfrac{1}{1+\frac{1}{1/2}} = \tfrac{1}{3}, y_3 = \tfrac{1}{4}, \ldots$$

By induction, $y_n = \frac{1}{n}$, for all $n > 0$. Thus $L = \lim_{n\to\infty} \frac{1}{n} = 0$, as expected.

The equilibrium points are obtained by solving

$$y_{n+1} = \frac{1}{1+y_n^{-1}} = y_n,$$

i.e., $1 = y_n + 1$, implying that the only equilibrium solution is

$$y_n = 0.$$

This is asymptotically stable since $L = \lim_{n\to\infty} y_n = 0$.

(2) Solve the initial value problem (with $n \geq 4$)

$$y_{n+1} = \left(\frac{n-3}{n+7}\right) y_n, \;\; y_4 = 1, \qquad L = \lim_{n\to\infty} y_n$$

$$\downarrow$$

$$L = \underbrace{\lim_{n\to\infty} \left(\frac{n-3}{n+7}\right)}_{=1} \cdot L$$

$$L = L,$$

which is not helpful. On the other hand, since $y_4 = 1$, we have for all $n \geq 5$,

$$0 < y_n = \frac{(n-3)!}{(n+7)(n+6)\dots(12)(11)} = \frac{10!}{(n+7)(n+6)\dots(n-2)} < \frac{10!}{(n-2)^{10}},$$

implying, by the squeeze theorem (of Ma1a), that $L = 0$, since

$$\lim_{n\to\infty} \frac{10!}{(n-2)^{10}} = 10! \lim_{n\to\infty} \frac{1}{(n-2)^{10}} = 0.$$

(3)

$$y_{n+1} = r y_n (1 - y_n), \; r > 0 \qquad (^{**})$$

Equilibrium points: $y_n = r y_n (1 - y_n) \implies y_n = 0, \;\; y_n = \frac{r-1}{r}$.

Is $y_n = 0$ asymptotically stable?

Here are two ways to proceed:

i) Solve (∗∗) explicitly and look at the asymptotics; or

ii) Use linearization!

Let's do the latter here.

We want to start with a solution v_n ("perturbation") close to 0.

When a real number u is small, then

$$c_0 + c_1 u + c_2 u^2 + \cdots + c_m u^m$$

can be well approximated by the linear term $c_0 + c_1 u$.

So the linear approximation to $ry_n(1 - y_n)$ equals just ry_n. Since v_n is very close to 0, we can approximate the equation $v_{n+1} = rv_n(1 - v_n)$ by its linearization $v_{n+1} = rv_n$.

Basic Fact: The asymptotic of v_n (as $n \to \infty$) can be evaluated by a corresponding solution of the linearized difference equation $v_{n+1} = vn^r$, which is easy to solve:

$$L := \lim_{n\to\infty} v_n = \underbrace{\left(\lim_{n\to\infty} r^n\right)}_{\downarrow} v_0 \qquad \begin{array}{|l} v_1 = rv_0 \end{array}$$

0 if $r < 1$	$v_2 = r^2 v_0$
1 if $r = 1$	$v_3 \ldots$
undefined otherwise	$v_n = r^n v_0$

Hence $L = 0$ if $r < 1$. (Recall that we have started with a positive r.)

So the equilibrium solution $y_n = 0$ is asymptotically stable for $r < 1$, and not so for $r \geq 1$. (When $r = 1$, it is stable since $L = v_0$, but not asymptotically stable for $v_0 \neq 0$.)

Lecture 9

Existence and uniqueness of First order ODEs

The object of this section is to justify our implicitly assuming earlier that solutions exist for first order ODE's of the form $y' = f(t, y)$, at least when f is nice enough, say continuously differentiable. What is striking about the Theorem below is that it also furnishes an explicit method to find (in fact) a unique solution through an iterative procedure.

Theorem (Picard) *Suppose we are given* $\frac{du}{dx} = g(x, u)$, *with* $x_0 = 0, u_0 = 0$ *such that* g *and* $\frac{\partial g}{\partial u}$ *are continuous on the rectangular region in the plane given by* $|x| \leq a$, $|u| \leq b$, *with* $a, b > 0$. *Then for some positive* $h < a$, *the ODE has a unique solution* $u = \varphi(x)$, *valid for all* x *in* $|x| \leq h$.

Picard's iteration method to find a solution (with the given hypothesis on g, g_u):

Idea: Construct a sequence of functions

$$\varphi_0(x), \varphi_1(x)_1 \ldots \varphi_n(x), \ldots$$

such that the desired solution is obtained as the limit

$$u = \varphi(x) \ = \lim_{n\to\infty} \varphi_n(x)$$

The assumptions on g, g_u are needed to make sure that the sequence $\{\varphi_n(x)\}$ has a limit for all x in $|x| < h$, for some $0 < h < a$

Picard's idea is to think of $\varphi_n(x)$ as

$$\varphi_0(x) + (\varphi_1(x) - \varphi_0(x)) + \cdots + (\varphi_{n-1}(x) - \varphi_{n-2}(x)) + (\varphi_n(x) - \varphi_{n-1}(x)),$$

i.e.,

$$\varphi_n(x) \ = \ \varphi_0(x) + \underbrace{\sum_{m=1}^{n} (\varphi_m(x) - \varphi_{m-1}(x))}_{\text{should be small}}$$

$$\Rightarrow \lim_{n\to\infty} \varphi_n(x) \ = \ \varphi_0(x) + \underbrace{\sum_{m=1}^{\infty} (\varphi_m(x) - \varphi_{m-1}(x))}_{\text{when this make sense}}$$

This limit, called $\varphi(x)$, exists if $\varphi_m(x) - \varphi_{m-1}(x)$ becomes sufficiently small as m becomes large.

Here is the explicit procedure of Picard:

Start with any given $\varphi_0(x)$, which satisfies the initial condition $\varphi_0(0) = 0$.

At the end we want a solution $u = \varphi(x)$ st

$$\varphi'(x) = g(\varphi(x), x)$$

with $\varphi(0) = 0$

Once we start with a $\varphi_0(x)$ with $\varphi_0(0) = 0$, we may solve for $\varphi_1(x)$ from

$$\varphi_1(x) = \int_0^x g(\varphi_0(t), t)dt,$$

which should be a better approximation. Next, put $\varphi_2(x) = \int_0^x g(\varphi_1(t), t)dt$, and so on.

So for any $n \geq 1$,

$$\varphi_n(x) = \int_0^x g(\varphi_{n-1}(t), t)dt.$$

Finally, set $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$.

Remark: Independent of the $\varphi_0(x)$ one starts with, one always gets the same answer in the limit. This is a remarkable property.

Example:

(1) $\frac{du}{dx} = u, \quad \text{with } u_0 = 0, x_0 = 0$

Recall the general solution: $u = Be^x, B = u_0$. The unique solution satisfying the initial condition is $u = 0$. This seems like we are belaboring over a trivial case. But what we are trying to do is to check Picard's assertion (in this example) that the final solution $\varphi(x)$ is independent of the starting function $\varphi_0(x)$ (as long as φ_0 satisfies the initial condition):

If we start with $\varphi_0(x) = 0$ (identically) then $\varphi_n(x) = 0, \ \forall n$ so $\varphi(x) = 0$ (by Picard).

Suppose we start with another φ_0, say $\varphi_0(x) = x$; then $\varphi_0(x) = 0$ as needed. Here

$$g(x, u) = u, \ g(x, \varphi_0(x)) = \varphi_0(x) = x.$$

Moreover,

$$\varphi_1(x) = \int_0^x \varphi_0(t)dt = \int_0^x +dt = \frac{1}{2}x^2$$
$$\varphi_2(x) = \int_0^x \left(\frac{1}{2}t^2\right) dt = \frac{1}{3!}x^3$$
$$\varphi_n(x) = \frac{x^n}{n!} \to 0 \text{ as } n \to \infty, \quad \text{if } |x| < 1.$$

Conclusion: The limit function $\varphi(x)$ is just the 0 function for all x, when we start with $\varphi_0(x) = x$.

Suppose we start instead with $\varphi_0(x) = \sin x$, which also satisfies the initial condition at $x = 0$. We obtain

$$\varphi_1(x) = \int_0^x \sin t dt = 1 - \cos x$$
$$\varphi_2(x) = \int_0^x (1 - \cos t)dt = x - \sin x$$
$$\varphi_3(x) = \int_0^x (t - \sin t)dt = \frac{1}{2} - (1 - \cos x) = -1 + \frac{x^2}{2} + \cos x$$
$$\varphi_4(x) = -x + \frac{x^2}{3!} + \sin x$$

The appearance of the alternating sin and cos functions make it seem like there is no limit, but there is one. Indeed, the odd φ_n's go in the limit to

$$\pm \lim_{n\to\infty} \left[\left(1 - \frac{x^2}{2!} + \frac{x^2}{4!} - \dots\right) - \cos x\right] = 0,$$

because the infinite series $1 - \frac{x^2}{2!} + \frac{x^2}{4!} - \dots$ is the Taylor expansion of $\cos x$ (cf. the online Notes for Ma1a).

Similarly, the even φ_n's go in the limit to

$$\pm \lim_{n\to\infty} \left[\left(x - \frac{x^3}{3!} + \frac{x^s}{s!}\right) - \sin x\right] = 0.$$

Nice!