CHAPTER 7

DIV, GRAD, AND CURL

1. The operator ∇ and the gradient:

Recall that the gradient of a differentiable scalar field φ on an open set $\mathcal{D}$ in $\mathbb{R}^n$ is given by the formula:

$$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \ldots, \frac{\partial\varphi}{\partial x_n}\right). \tag{1}$$

It is often convenient to define formally the differential operator in vector form as:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right). \tag{2}$$

Then we may view the gradient of φ, as the notation $\nabla\varphi$ suggests, as the result of multiplying the vector ∇ by the scalar field φ. Note that the order of multiplication matters, i.e., $\frac{\partial\varphi}{\partial x_j}$ is **not** $\varphi\frac{\partial}{\partial x_j}$.

Let us now review a couple of facts about the gradient. For any $j \leq n$, $\frac{\partial\varphi}{\partial x_j}$ is identically zero on $\mathcal{D}$ iff $\varphi(x_1, x_2, \ldots, x_n)$ is independent of x_j. Consequently,

$$\nabla\varphi = 0 \text{ on } \mathcal{D} \quad \Leftrightarrow \quad \varphi = \text{constant}. \tag{3}$$

Moreover, for any scalar c, we have:

$$\nabla\varphi \text{ is normal to the level set } L_c(\varphi). \tag{4}$$

Thus $\nabla\varphi$ gives the direction of steepest change of φ.

2. Divergence

Let $F : \mathcal{D} \to \mathbb{R}^n$, $\mathcal{D} \subset \mathbb{R}^n$, be a differentiable vector field. (Note that *both* spaces are n-dimensional.) Let $F_1, F_2, \ldots, F_n$ be the component (scalar) fields of f. The **divergence of F** is defined to be

$$\operatorname{div}(F) = \nabla \cdot F = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}. \tag{5}$$

This can be reexpressed symbolically in terms of the dot product as

$$\nabla \cdot F = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \cdot (F_1, \ldots, F_n). \tag{6}$$

Note that $\operatorname{div}(F)$ is a scalar field.

Given any $n \times n$ matrix $A = (a_{ij})$, its **trace** is defined to be:

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Then it is easy to see that, if DF denotes the **Jacobian matrix** of F, i.e., the $n \times n$-matrix $(\partial F_i/\partial x_j)$, $1 \leq i, j \leq n$, then

$$\nabla \cdot F = \operatorname{tr}(DF). \tag{7}$$

Let φ be a twice differentiable scalar field. Then its **Laplacian** is defined to be

$$\nabla^2 \varphi = \nabla \cdot (\nabla \varphi). \tag{8}$$

It follows from (1),(5),(6) that

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \cdots + \frac{\partial^2 \varphi}{\partial x_n^2}. \tag{9}$$

One says that φ is **harmonic** iff $\nabla^2 \varphi = 0$. Note that we can formally consider the dot product

$$\nabla \cdot \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \cdot (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{10}$$

Then we have

$$\nabla^2 \varphi = (\nabla \cdot \nabla) \varphi. \tag{11}$$

Examples of harmonic functions:

(i) $\mathcal{D} = \mathbb{R}^2$; $\varphi(x, y) = e^x \cos y$.

Then $\frac{\partial\varphi}{\partial x} = e^x \cos y$, $\frac{\partial\varphi}{\partial y} = -e^x \sin y$,

and $\frac{\partial^2\varphi}{\partial x^2} = e^x \cos y$, $\frac{\partial^2\varphi}{\partial y^2} = -e^x \cos y$. So, $\nabla^2\varphi = 0$.

(ii) $\mathcal{D} = \mathbb{R}^2 - \{0\}$. $\varphi(x, y) = \log(x^2 + y^2) = 2\log(r)$.

Then $\frac{\partial\varphi}{\partial x} = \frac{2x}{x^2+y^2}$, $\frac{\partial\varphi}{\partial y} = \frac{2y}{x^2+y^2}$, $\frac{\partial^2\varphi}{\partial x^2} = \frac{2(x^2+y^2)-2x(2x)}{(x^2+y^2)^2} = \frac{-2(x^2-y^2)}{(x^2+y^2)^2}$, and $\frac{\partial^2\varphi}{\partial y^2} = \frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$. So, $\nabla^2\varphi = 0$.

(iii) $\mathcal{D} = \mathbb{R}^n - \{0\}$. $\varphi(x_1, x_2, ..., x_n) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\alpha/2} = r^\alpha$ for some fixed $\alpha \in \mathbb{R}$.

Then $\frac{\partial\varphi}{\partial x_i} = \alpha r^{\alpha-1}\frac{x_i}{r} = \alpha r^{\alpha-2}x_i$, and

$\frac{\partial^2\varphi}{\partial x_i^2} = \alpha(\alpha-2)r^{\alpha-4}x_i \cdot x_i + \alpha r^{\alpha-2} \cdot 1$.

Hence $\nabla^2\phi = \sum_{i=1}^{n}\left(\alpha(\alpha-2)r^{\alpha-4}x_i^2 + \alpha r^{\alpha-2}\right) = \alpha(\alpha-2+n)r^{\alpha-2}$.

So ϕ is harmonic for $\alpha = 0$ or $\alpha = 2 - n$ ($\alpha = -1$ for $n = 3$).

3. Cross product in $\mathbb{R}^3$

The three-dimensional space is very special in that it admits a **vector product**, often called the **cross product**. Let **i**,**j**,**k** denote the standard basis of $\mathbb{R}^3$. Then, for all pairs of vectors $v = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $v' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$, the cross product is defined by

$$(12)\quad v \times v' = det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ x' & y' & z' \end{pmatrix} \quad = (yz' - y'z)\mathbf{i} - (xz' - x'z)\mathbf{j} + (xy' - x'y)\mathbf{k}.$$

Lemma 1. ***(a)*** *$v \times v' = -v' \times v$ (anti-commutativity)*

(b) $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$, $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$

(c) $v \cdot (v \times v') = v' \cdot (v \times v') = 0$.

Corollary: $v \times v = 0$.

Proof of Lemma (a) $v' \times v$ is obtained by interchanging the second and third rows of the matrix whose determinant gives $v \times v'$. Thus $v' \times v = -v \times v'$.

(b) $\mathbf{i} \times \mathbf{j} = det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, which is **k** as asserted. The other two identities are similar.

(c) $v \cdot (v \times v') = x(yz' - y'z) - y(xz' - x'z) + z(xy' - x'y) = 0$. Similarly for $v' \cdot (v \times v')$.

Geometrically, $v \times v'$ can, thanks to the Lemma, be interpreted as follows. Consider the plane P in $\mathbb{R}^3$ defined by v,v'. Then $v \times v'$ will lie along the normal to this plane at the origin, and its orientation is given as follows. Imagine a corkscrew perpendicular to P with its tip at the origin, such that it turns clockwise when we rotate the line Ov towards Ov' in the plane P. Then $v \times v'$ will point in the direction in which the corkscrew moves perpendicular to P.

Finally the length $||v \times v'||$ is equal to the area of the parallelogram spanned by v and v'. Indeed this area is equal to the volume of the parallelepiped spanned by v, v' and a unit vector $u = (u_x, u_y, u_z)$ orthogonal to v and v'. We can take $u = v \times v'/||v \times v'||$ and the (signed) volume equals

$$\det \begin{pmatrix} u_x & u_y & u_z \\ x & y & z \\ x' & y' & z' \end{pmatrix} = u_x(yz' - y'z) - u_y(xz' - x'z) + u_z(xy' - x'y)$$
$$= ||v \times v'|| \cdot (u_x^2 + u_y^2 + u_z^2) = ||v \times v'||.$$

4. Curl of vector fields in $\mathbb{R}^3$

Let $F : \mathcal{D} \to \mathbb{R}^3$, $\mathcal{D} \subset \mathbb{R}^3$ be a differentiable vector field. Denote by P,Q,R its coordinate scalar fields, so that $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then the **curl of F** is defined to be:

$$\text{curl}(F) = \nabla \times F = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}. \tag{13}$$

Note that it makes sense to denote it $\nabla \times F$, as it is formally the cross product of ∇ with f. Explicitly we have

$$\nabla \times F = (\partial R/\partial y - \partial Q/\partial z)\,\mathbf{i} - (\partial R/\partial x - \partial P/\partial z)\,\mathbf{j} + (\partial Q/\partial x - \partial P/\partial y)\,\mathbf{k}$$

If the vector field F represents the **f**low of a fluid, then the **curl** measures *how the flow rotates the vectors*, whence its name.

Proposition 1. *Let h (resp. F) be a $\mathcal{C}^2$ scalar (resp. vector) field. Then*

(a): $\nabla \times (\nabla h) = 0$.
(b): $\nabla \cdot (\nabla \times F) = 0$.

Proof: (a) By definition of gradient and curl,

$$\nabla \times (\nabla h) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix}$$

$$= \left(\frac{\partial^2 h}{\partial y \partial z} - \frac{\partial^2 h}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 h}{\partial z \partial x} - \frac{\partial^2 h}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 h}{\partial y \partial x}\right)\mathbf{k}.$$

Since h is $\mathcal{C}^2$, its second mixed partial derivatives are independent of the order in which the partial derivatives are computed. Thus, $\nabla \times (\nabla f h) = 0$.

(b) By the definition of divergence and curl,

$$\nabla \cdot (\nabla \times F) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \left(\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z}\right) + \left(-\frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 P}{\partial y \partial z}\right) + \left(\frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y}\right).$$

Again, since F is $\mathcal{C}^2$, $\frac{\partial^2 R}{\partial x \partial y} = \frac{\partial^2 R}{\partial y \partial x}$, etc., and we get the assertion.

□

Warning: There exist **twice differentiable** scalar (resp. vector) fields h (resp. F), which are **not** $\mathcal{C}^2$, for which (a) (resp. (b)) does **not** hold.

When the vector field F represents fluid flow, it is often called **irrotational** when its curl is 0. If this flow describes the movement of water in a stream, for example, to be *irrotational* means that a small boat being pulled by the flow will not rotate about its axis. We will see later that the condition $\nabla \times F = 0$ occurs naturally in a purely mathematical setting as well.

Examples: (i) Let $\mathcal{D} = \mathbb{R}^3 - \{0\}$ and $F(x, y, z) = \frac{y}{(x^2+y^2)}\mathbf{i} - \frac{x}{(x^2+y^2)}\mathbf{j}$. Show that F is irrotational. Indeed, by the definition of curl,

$$\nabla \times F = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{(x^2+y^2)} & \frac{-x}{(x^2+y^2)} & 0 \end{pmatrix}$$

$$= \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\mathbf{i} + \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2}\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right)\mathbf{k}$$

$$= \left[\frac{-(x^2+y^2)+2x^2}{(x^2+y^2)^2} - \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right]\mathbf{k} = 0.$$

(ii) Let m be any integer $\neq 3$, $\mathcal{D} = \mathbb{R}^3 - \{0\}$, and $F(x, y, z) = \frac{1}{r^m}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, where $r = \sqrt{x^2 + y^2 + z^2}$. Show that F is not the curl of another vector field. Indeed, suppose $F = \nabla \times G$. Then, since F is $\mathcal{C}^1$, G will be $\mathcal{C}^2$, and by the Proposition proved above, $\nabla \cdot F = \nabla \cdot (\nabla \times G)$ would be zero. But,

$$\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r^m}, \frac{y}{r^m}, \frac{z}{r^m}\right)$$

$$= \frac{r^m - 2x^2(\frac{m}{2})r^{m-2}}{r^{2m}} + \frac{r^m - 2y^2(\frac{m}{2})r^{m-2}}{r^{2m}} + \frac{r^m - 2z^2(\frac{m}{2})r^{m-2}}{r^{2m}}$$

$$= \frac{1}{r^{2m}}\left(3r^m - m(x^2+y^2+z^2)r^{m-2}\right) = \frac{1}{r^m}(3-m).$$

This is non-zero as $m \neq 3$. So F is **not** a curl.

Warning: It may be true that the divergence of F is zero, but F is still not a curl. In fact this happens in example (ii) above if we allow $m = 3$. We cannot treat this case, however, without establishing Stoke's theorem.

5. An interpretation of Green's theorem via the curl

Recall that Green's theorem for a plane region Φ with boundary a piecewise $\mathcal{C}^1$ Jordan curve C says that, given any $\mathcal{C}^1$ vector field $G = (P, Q)$ on an open set $\mathcal{D}$ containing Φ, we have:

$$\iint_\Phi \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_C P\,dx + Q\,dy. \tag{14}$$

We will now interpret the term $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. To do that, we think of the plane as sitting in $\mathbb{R}^3$ as $\{z = 0\}$, and define a $\mathcal{C}^1$ vector field F on $\tilde{D} := \{(x,y,z) \in \mathbb{R}^3 | (x,y) \in \mathcal{D}\}$ by setting $F(x,y,z) = G(x,y) = P\mathbf{i} + Q\mathbf{j}$. We can interpret this as taking values in $\mathbb{R}^3$ by thinking of its value as $P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$.Then $\nabla \times F = det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{pmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$, because $\frac{\partial P}{\partial z} = \frac{\partial Q}{\partial z} = 0$. Thus we get:

$$(\nabla \times F) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}. \tag{15}$$

And Green's theorem becomes:

Theorem 1. $\iint_\Phi (\nabla \times F) \cdot \boldsymbol{k}\, dx\, dy = \oint_C P\,dx + Q\,dy$

6. A criterion for being conservative via the curl

A consequence of the reformulation above of Green's theorem using the curl is the following:

Proposition 1. *Let $G : \mathcal{D} \to \mathbb{R}^2$, $\mathcal{D} \subset \mathbb{R}^2$ open and* **simply connected***, $G = (P, Q)$, be a $\mathcal{C}^1$ vector field. Set $F(x,y,z) = G(x,y)$, for all $(x,y,z) \in \mathbb{R}^3$ with $(x,y) \in \mathcal{D}$. Suppose $\nabla \times F = 0$. Then G is conservative on D.*

Proof: Since $\nabla \times F = 0$, the reformulation in section 5 of Green's theorem implies that $\oint_C P\,dx + Q\,dy = 0$ for all Jordan curves C contained in $\mathcal{D}$. **QED**

Example: $\mathcal{D} = \mathbb{R}^2 - \{(x,0) \in \mathbb{R}^2 \mid x \leq 0\}$, $G(x,y) = \frac{y}{x^2+y^2}\mathbf{i} - \frac{x}{x^2+y^2}\mathbf{j}$. Determine if G is conservative on $\mathcal{D}$:

Again, define $F(x,y,z)$ to be $G(x,y)$ for all (x,y,z) in $\mathbb{R}^3$ such that $(x,y) \in \mathcal{D}$. Since G is evidently $\mathcal{C}^1$, F will be $\mathcal{C}^1$ as well. By the Proposition above, it will suffice to check if F is irrotational, i.e., $\nabla \times F = 0$, on $\mathcal{D} \times \mathbb{R}$. This was already shown in Example (i) of section 4 of this chapter. So G is conservative.