

The Fluctuation-imposed Limit for Temperature Measurement

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In experimental sciences, random processes often place a fundamental limit on the achievable measurement resolution. A well known example is the Johnson noise voltage across a resistor. In this paper, we describe observations of the spontaneous transfer of heat in two equilibrium systems: one consisting of a thermometer linked to a reservoir, the other consisting of two thermometers connected to each other and linked to a reservoir for the purpose of temperature stabilization. In the second system, we find anti-correlations between the temperature fluctuations of the two thermometers at intermediate frequencies. We also find that the low frequency temperature noise density of the thermometers, in units of K/\sqrt{Hz} , is given by $\sqrt{4Rk_B T^2}$, where R is the thermal resistance of the link between the thermometer and the reservoir. This implies that for noise reduction purposes, R is the only available engineering parameter to adjust. In a recent thermometer design, we have reduced R to achieve a low frequency temperature noise density of $5 \times 10^{-11} K/\sqrt{Hz}$ at 2.17 K.

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1. INTRODUCTION

In thermodynamic terms, a *thermometer* is modeled as a subsystem in contact with a heat reservoir. The *reservoir*, the object whose temperature is to be measured, is assumed to be isothermal and to have infinite heat capacity. The thermometer incorporates an easily measured quantity, like the magnetization of a salt, whose mean value is a function of temperature and can thus be used to infer the reservoir temperature. From statistical mechanics, it is well known that thermodynamic properties undergo spontaneous fluctuations about their mean values. In the case of a magnetic thermometer, this phenomenon leads to magnetization fluctuations over a frequency band from dc to the inverse of a characteristic time constant. If these are large compared to the resolution of the thermometer, they appear as noise in the temperature measurement. Another source of fluctuation noise in a thermometer is the temperature variation caused by energy transfer through the thermal link to a reservoir. The first qualitative observation of this effect was in an experiment that measured the intensity of light emitted from a hot filament.¹ In the last decade, very high resolution magnetic thermometers^{2,3} have been developed for studies of the superfluid phase transition of ⁴He. The best of these thermometers exhibit low frequency noise on the order of 10^{-9} to 10^{-10} K/ $\sqrt{\text{Hz}}$. Estimates indicate that the magnetization noise is negligible at this level because of the wide bandwidth involved. In several recent papers^{4,3}, the noise spectra were found to agree with predictions of the fluctuation-dissipation theorem (FDT), indicating that temperature measurement has, in at least these cases, reached the limit set by energy fluctuations coupled through a thermal link. In the following, we describe the results of two experiments that further clarify the nature of this limiting source of noise, and discuss a method for reducing the effect.

The high resolution thermometer (HRT) used in both experiments employed a paramagnetic salt, copper ammonium bromide [Cu(NH₄)₂Br₄·2H₂O] (CAB), as the temperature sensing element. The change in the magnetization of the salt induced by variations of its temperature is measured in a constant applied magnetic field using a superconducting quantum interference device (SQUID). A superconducting tube is used to trap a very stable DC field (~20 mTesla) in the salt for the measurement. The SQUID output voltage is calibrated against a germanium resistance thermometer. HRTs of this kind have been routinely used in studies of critical phenomena⁵ near the superfluid transition of ⁴He, which occurs at a temperature of 2.1768 K under its saturated vapor pressure.

2. SINGLE THERMOMETER EXPERIMENT

In the first experiment, we examined the noise of an HRT as a function of temperature. This work extended the previously mentioned results⁴, that found the noise spectra to agree with the FDT, over a much wider temperature range. We performed our measurements with a thermometer consisting of a single salt pill which was thermally linked to a superfluid helium reservoir by a copper rod. The experimental design and thermal control system were very similar to those used before⁴. The reservoir was separated from a surrounding liquid helium bath by a four-stage thermal isolation system. The final stage was connected to a shield surrounding the superfluid reservoir and the thermometer being studied. The shield stage temperature was read by another HRT and regulated to within 10^{-8} K. The other stages were controlled to within 10^{-5} K using germanium resistance thermometers. The measurements described below were taken after the isolation system had been adjusted so that the cooling rate of the reservoir was less than 10^{-13} K/sec. We refer the reader to Ref. (4) for a more detailed description of the experimental setup as well as the methods of data collection and analysis.

Figure 1a shows histograms of the thermometer noise for two reservoir temperatures plotted as a function of ΔT , the temperature deviation from the mean. Gaussian functions were fitted to the data to find the widths of the probability distributions. The solid lines show the fitted curves. The filled circles in figure 1b show the measured RMS thermometry noise at different temperatures. From statistical mechanics we find that the probability distribution for thermal fluctuations will be Gaussian with a width defined by⁶:

$$\langle(\Delta T)^2\rangle = k_B T^2 / C, \quad (1)$$

where C is the heat capacity of a subsystem, which in our case is the magnetic salt pill and the materials which diffusively couple to it within the measurement bandwidth. The heat capacity of CAB was determined by interpolation of the measurements of Meidema et al⁷, which gives the specific heat as a function of temperature and magnetic field. The interpolation uncertainty is $\pm 10\%$. The heat capacity value for our salt is 3.1 mJ/K at 2.18 K and 20 mTesla. The estimated thermal fluctuation noise derived from eq. 1 is illustrated by the shaded band in figure 1b, with no adjustable parameters. The excellent fit to the Gaussian functions and the close match between theoretical and measured values of their widths give additional support to the claim that the noise of the system originates from spontaneous thermal fluctuations. In addition, the magnitudes of other potential sources of noise are considerably lower than the resolution of our measurement. While our signal was on the order of 10^{-10} K, the RMS noise due to thermal fluctuations in the reservoir can be calculated to

be on the order of 1×10^{-12} K and that induced by our thermal control system to be approximately 2×10^{-13} K. Magnetic moment fluctuations in our salt occur on faster time scales than the thermal fluctuations, so although they have an RMS noise on the order of 2×10^{-10} K, their noise density is only 4.5×10^{-12} K/ $\sqrt{\text{Hz}}$ in the relevant bandwidth. SQUID noise has a density of 7×10^{-12} K/ $\sqrt{\text{Hz}}$ over the range of interest. Our data was filtered at 20 Hz, so the contribution of these two sources to our calculated RMS noise is negligible. We refer the reader to Ref. (4) for a more thorough discussion of these values.

3. DOUBLE THERMOMETER EXPERIMENT

To obtain more information on the nature of the fluctuation phenomenon we constructed another thermometer consisting of two CAB salt pills of similar size. The two pills were connected together with a thermal link, and the center of this connection was attached with a weaker link to a thermal reservoir (Fig 2). The beryllium copper (BeCu) holder for the two pills must also be treated as a parallel thermal link to the reservoir. The temperature changes of each pill were measured independently using separate SQUID magnetometers. This design allowed us to monitor the flow of energy from one salt pill to the other. If the two-salt system was entirely isolated, conservation of energy would imply that when the temperature of one pill rises, the temperature of the other has to fall. However, if heat was exchanged with the reservoir, the temperatures of the two salts would change in the same direction. A study of the correlation in the noise of the two thermometers would therefore determine if fluctuations were a result of energy transfer or were due to a phenomenon internal to the individual salt pills.

The rest of the experimental apparatus and procedure were similar to those used in the single thermometer experiment. After the thermal reservoir was stabilized at 1.89 K, the temperatures of the two salt pills were independently recorded. The Fourier transforms of the thermometer outputs (T_1 and T_2) were then computed and normalized such that $\tilde{T}_1(f) \bullet \tilde{T}_1^*(f)$ gives the power spectral density (PSD) of T_1 , where $\tilde{T}_1(f)$ is the normalized Fourier transform of T_1 . To ensure that the normalization was done correctly, we computed the integral $\int_0^\infty \tilde{T}_1(f) \bullet \tilde{T}_1^*(f) df$ and showed that it is equal to the RMS noise squared, $\langle (\Delta T)^2 \rangle$, as required by Parseval's theorem⁸. The open circles in figure 3 show the real part of the cross-correlation function, $Re[\tilde{T}_1(f) \bullet \tilde{T}_2^*(f)]$, computed from the data. Negative values in this plot indicate anti-correlation. At lower frequencies the temperatures of the two

HRTs are correlated while at higher frequencies they become anti-correlated. This can be explained qualitatively by the fact that the RC time constant for the thermal link to the heat reservoir was longer than for that between the HRTs, resulting in low frequency energy fluctuations between the two-salt pill system and the reservoir, and higher frequency fluctuations between the two salt pills. We also measured the cross-correlation between one of the salt pills in the double thermometer and a single thermometer attached to the same reservoir at a different location. The two signals were completely uncorrelated over the measured frequency range, consistent with the above interpretation.

4. ANALYSIS

The model used to analyze our data treats the thermal fluctuations in each pill as a combination of a correlated part (T_+) and an anti-correlated part (T_-). The correlated part is due to energy fluctuations between the pills and the heat reservoir, while the anti-correlated part is the result of energy fluctuations between the two pills. The inset of figure (3) is a lumped model approximation of the experiment, where R_{12} includes the thermal resistance of the copper wires linking the two pills as well as the link between them through the BeCu holder, and R_o represents all the parallel links to the reservoir and is assumed to be connected to the middle of R_{12} for simplicity. This thermal model gives the following result based on the FDT^{9,6} and conservation of energy:

$$\tilde{T}_1(f) \cdot \tilde{T}_2^*(f) = \left(\frac{4k_B T^2}{C_1 + C_2} \right) \cdot \left[\frac{\tau_0}{(1 + 4\pi^2 \tau_0^2 f^2)} - \frac{\tau_{eff}}{(1 + 4\pi^2 \tau_{eff}^2 f^2)} \right] \quad (2)$$

where $C_{1,2}$ are the heat capacities of salts 1 and 2, $\tau_0 = (R_0 + R_{12}/4) \cdot (C_1 + C_2)$, $C_{eff} = C_1 C_2 / (C_1 + C_2)$, and $\tau_{eff} = R_{12} C_{eff}$. An outline of the derivation of eq. 2 is given in the Appendix. Integrating eq. (2) over the entire frequency domain results in zero because the area under the curve of the correlated part is equal to the area under the anti-correlated part. A fit of this function to the data gives $(C_1 + C_2) = 15$ mJ/K, $\tau_0 = 2.3$ sec, $\tau_{eff} = 0.5$ sec. From the heat capacity data of Meidema et al.⁷, we estimated that $(C_1 + C_2) = 10 \pm 2$ mJ/K, in reasonable agreement with the fit. The close agreement between the model and our cross-correlation data is further confirmation that spontaneous thermal fluctuations are now the limiting factor in ultra-high resolution thermometry. The existence of an anti-correlation region, in the intermediate frequency range, argues strongly that the source of the observed noise is energy transfer through the thermal links.

5. IMPROVED HRT DESIGN

The relationship $\langle(\Delta T)^2\rangle = k_B T^2 / C$ suggests that the temperature resolution may be improved by increasing the heat capacity of the thermometer. However, increasing C also increases the time constant, τ , of the thermometer. Because it is only possible to get one statistically independent reading per time interval τ , longer τ means that fewer data points are available with which to determine the mean of the salt pill magnetization, from which the reservoir temperature is inferred. Since the error in the average varies as $1/\sqrt{N}$ where N is the number of data points, we expect that the thermometry noise, averaged over a fixed time interval, is proportional to $\sqrt{\tau} \cdot \Delta T \sim \sqrt{\tau} / \sqrt{C} = \sqrt{R}$. By decreasing R , more thermodynamic averaging over the fluctuating energy states takes place, yielding higher resolution thermometry. Quantitatively, the FDT gives the noise power density as $\tilde{T}(f) \cdot \tilde{T}^*(f) = 4R k_B T^2$ at low frequencies, where the thermometer has maximum sensitivity. Therefore, R is the only relevant engineering parameter we can adjust to reduce noise originating from the spontaneous thermal fluctuations. Variation of the salt material or the magnetic field is not expected to affect the thermometry noise density at low frequencies.

In a recent paper, Swanson *et al*³ reported that when τ was reduced by a factor of nearly 10 to 0.12 seconds, the thermometer resolution was improved by a factor of about 3 to $10^{-10} K / \sqrt{Hz}$, in agreement with the above analysis. They reduced R on their HRT by using annealed high purity copper wires which link the salt pill to the reservoir. Their value of R can be estimated from the relation $\tau = RC$, and the heat capacity of the salt ($\sim 3\text{mJ/K}$) to be 40 K/Watt. Our new HRT design, shown in the inset of figure 4, reduces R even further. The thermal link to the salt was accomplished by machining numerous chimney-shaped heat fins into a high purity copper rod. The cutting was accomplished by wire electrical discharge machining. The space between the heat fins was filled with a salt material by slow crystallization in a quartz tube filled with the molten salt. Because of the reactive nature of CAB, we used gadolinium trichloride, GdCl_3 , which can be grown directly on to the copper. A previous study¹⁰ indicates that GdCl_3 and CAB have comparable thermal noise levels when used in two thermometers of a similar design. The new thermal link was designed not only to decrease the thermal impedance, but to minimize the noise pickup due to spontaneous fluctuating electrical currents in the metal.

The noise spectrum of the above thermometer at 2.17 K is shown in figure 4. We find that at low frequencies, the noise density is $5 \times 10^{-11} K / \sqrt{Hz}$. A value of R can roughly be estimated from the volume of the salt (0.24 cm^3), the observed value of τ (0.37 sec.), and the zero field heat

capacity data of Hovi *et al*¹¹. We find that $R = 9 \pm 4$ K/Watt using the value of 10 J/mole K for the heat capacity of the salt at 20 mTesla. This value of R is approximately 4 times lower than that of the HRT reported earlier³, resulting in a factor of 2 reduction in the low frequency noise. We expect that still lower noise may be achieved by further minimizing R . In some applications, it is possible to immerse the salt pill directly into superfluid helium, where the thermal conductivity is limited only by the boundary resistance. In this case, the fluctuation effects are difficult to model with lumped circuit analysis, but they are expected to be small. They may be sufficiently reduced that the noise due to intrinsic thermal fluctuations will fall below the SQUID noise level. In this limit, the noise density is dependent on the sensitivity of the salt. With the sensitivity of $24 \Phi_0/\mu\text{K}$ at 2.17 K observed in our GdCl_3 thermometer, we expect this limit to be 1.2×10^{-12} K/ $\sqrt{\text{Hz}}$ and 2×10^{-13} K/ $\sqrt{\text{Hz}}$ for commercially available RF and DC SQUIDS, respectively, at frequencies higher than the $1/f$ noise of the SQUID.

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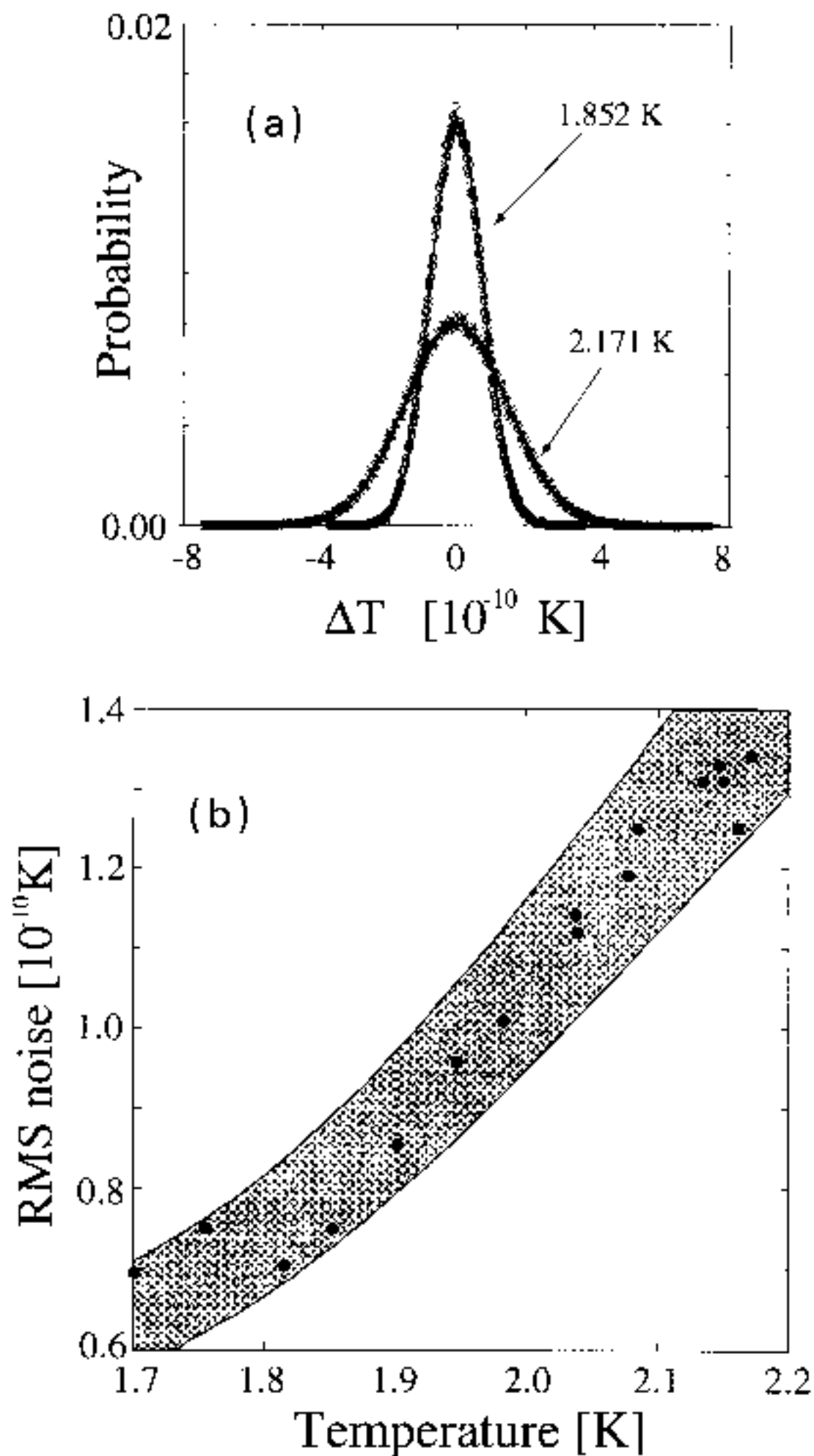


Fig. 1. (a) Histograms of the outputs of an HRT at two different reservoir temperatures. The x-axis represents the temperature deviation from the mean. Solid lines represent Gaussian fits. (b) Temperature dependence of the RMS thermometer noise.

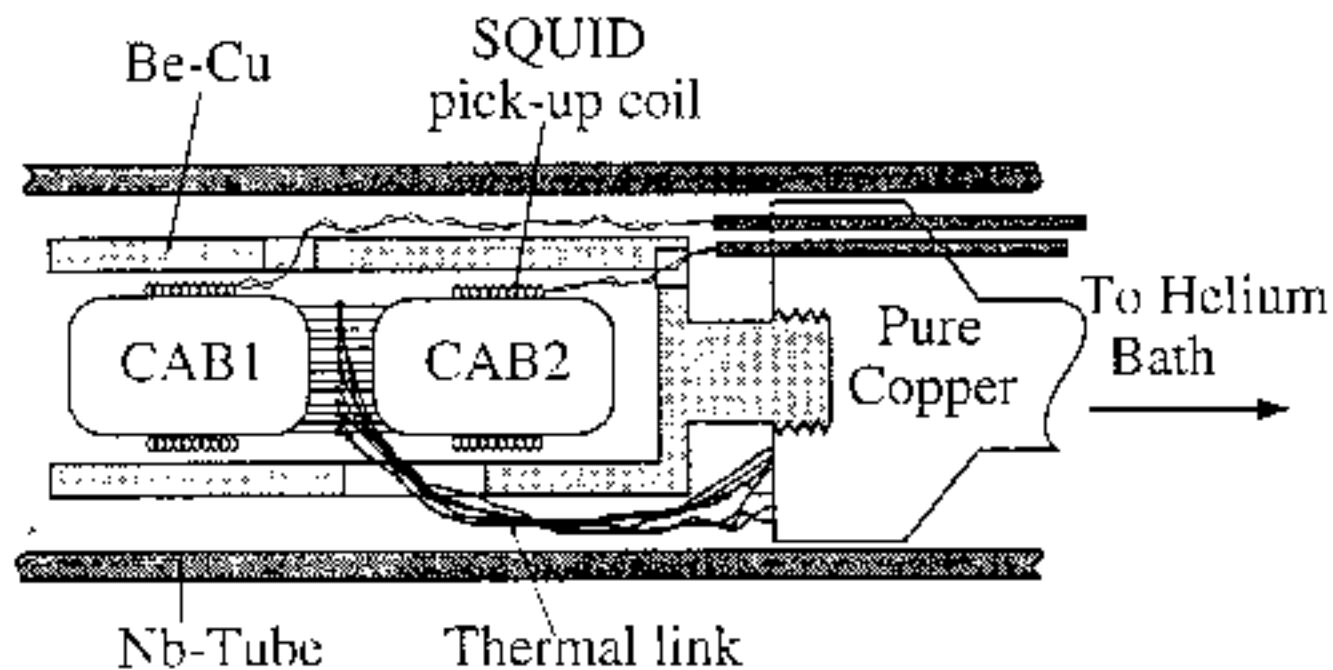


Fig. 2. Schematic view of the two-salt HRT.

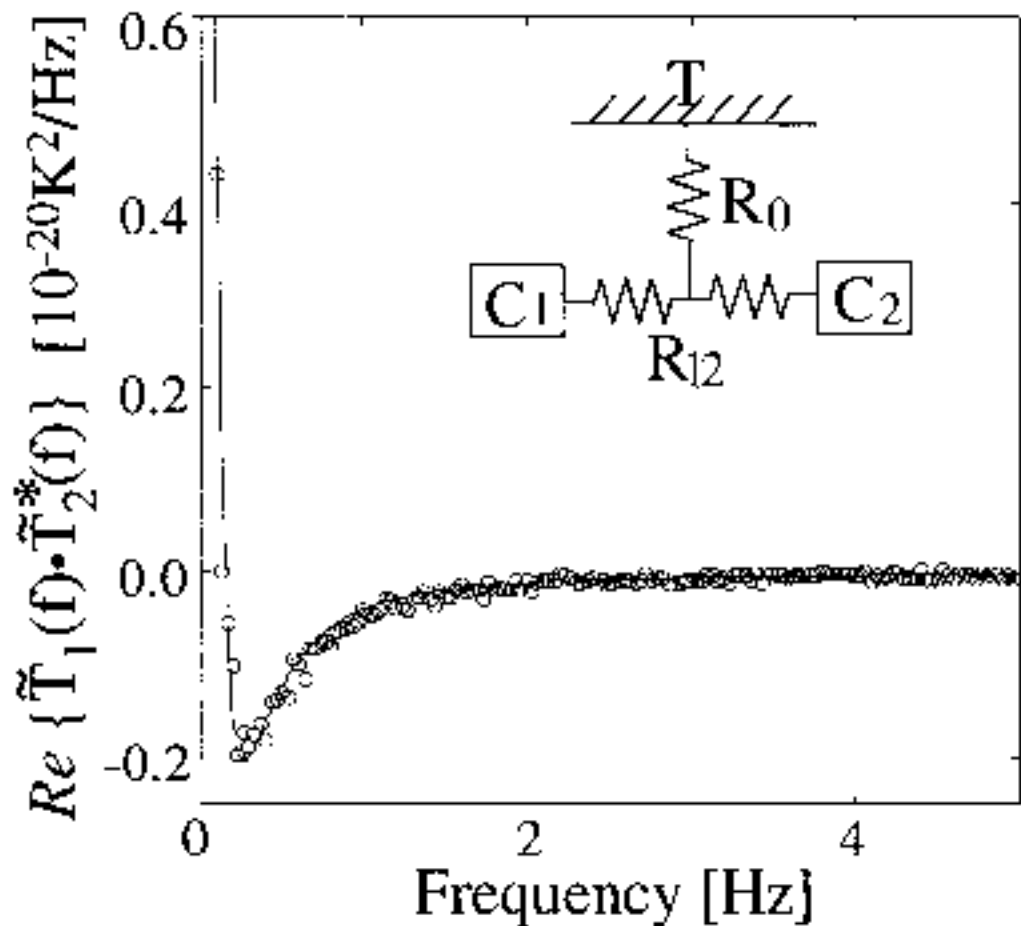
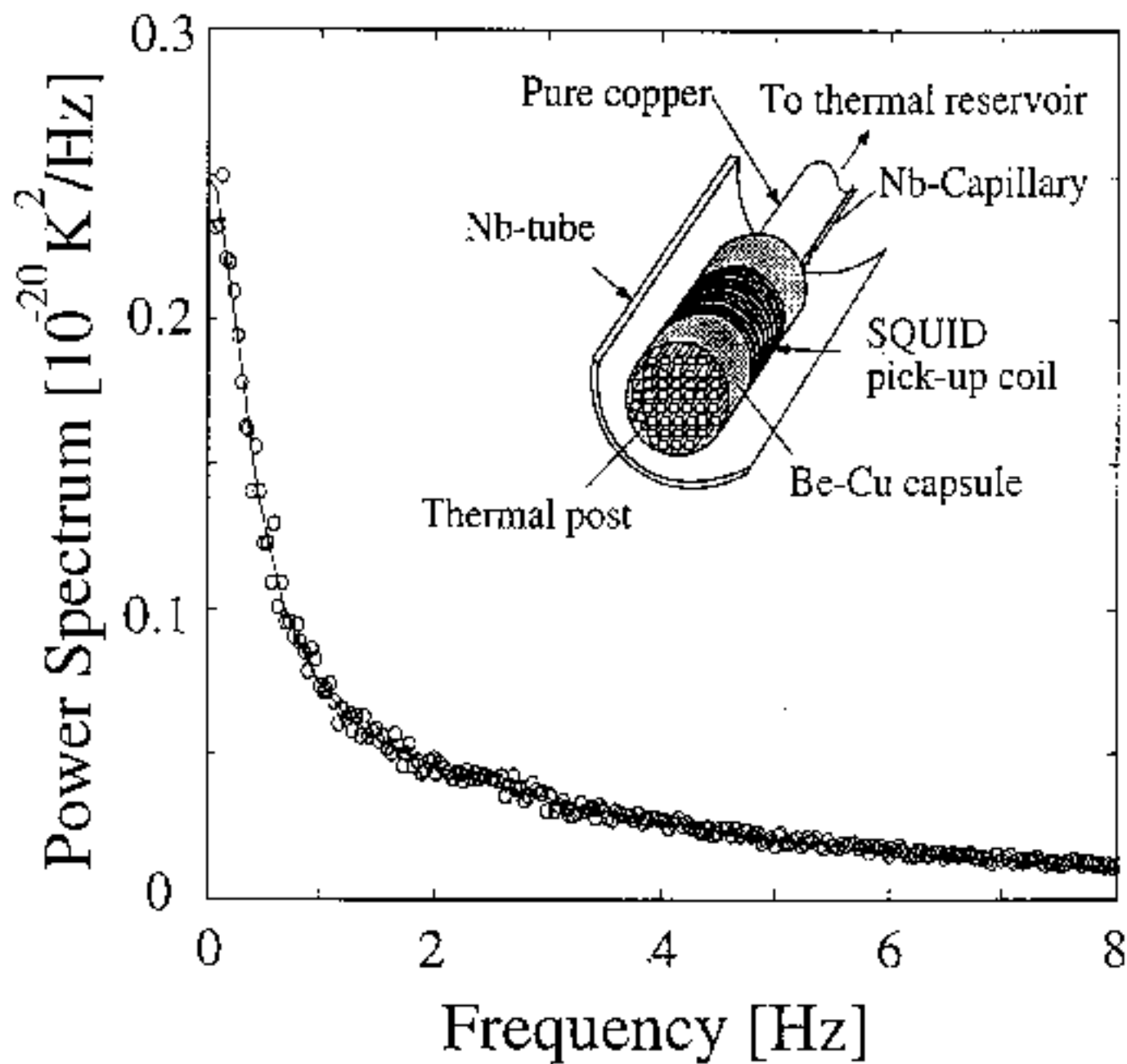


Fig. 3. Cross-power spectrum of the outputs from the two-salt HRT. Inset: the thermal model of the two-salt thermometer.



APPENDIX

In this appendix we present a motivating argument for the form of eq. 2.

Conjugate thermodynamic variables can be represented by a generalized displacement, \bar{x} , and an applied generalized force, F . The response function, $\bar{x}(F)$, is constant when F is fixed, but x itself can fluctuate around its mean. In such a representation, the free energy, defined by $df(\bar{x}) = Fd\bar{x}$, is the generalized work. The FDT predicts that if F is fixed by an experiment, then the PSD of x is:

$$\tilde{x}(f) \bullet \tilde{x}^*(f) = \frac{4\tau k_B T (dx/dF)_{f=0}}{(1 + 4\pi^2 \tau^2 f^2)}, \quad (3)$$

where τ is the response time of x to a sudden application of F . Here we should note that eq. (3) only applies when the motion of \bar{x} can be approximated by a lumped model. When the FDT is applied to the case of a subsystem of heat capacity C linked to a reservoir, F and x translate into $-S$ and T respectively, where S is the entropy and T is the temperature. Thus,

$$\tilde{T}(f) \bullet \tilde{T}^*(f) = \frac{4\tau k_B T^2}{C(1 + 4\pi^2 \tau^2 f^2)} \quad (4)$$

where τ is the thermal relaxation time of the subsystem given by $\tau = RC$, and R is the thermal resistance between the subsystem and the reservoir. Since the FDT requires that F be kept constant, and $S = \frac{1}{T} \int (Q_{in} - Q_{out}) dt$, the only way to keep S constant is for $Q_{in} = Q_{out}$. Therefore any applied heating power, Q_{in} , must be balanced by the heat flow, Q_{out} , out of the subsystem into the reservoir. In order to apply the FDT, R must be finite. Regardless of the magnitude of R , eq. (4)

integrates to give the result $\langle(\Delta T)^2\rangle = k_B T^2 / C$. Since R will be finite in any feasible experiment, and the lumped differential equation describing the thermal relaxation process is:

$$Q = -\frac{\Delta T}{R} = C \frac{d\Delta T}{dt} \quad (5)$$

where ΔT is the deviation of the temperature of the subsystem from the reservoir, eq. (4) is a realistic prediction for the fluctuations of T .

In perfect isolation, the temperatures of two masses with heat capacities C_1 and C_2 connected by a thermal link, R_{12} , are governed by the differential equation:

$$Q_{12} = -\frac{T_{12}}{R_{12}} = -C_{eff} \frac{dT_{12}}{dt} \quad (6)$$

where $T_{12} = T_1 - T_2$, $C_{eff} = C_1 C_2 / (C_1 + C_2)$. If T_1 and T_2 are initially different, they equilibrate to a uniform temperature in time $\tau_{eff} = R_{12} C_{eff}$. By comparing eq. (6) to eq. (5), we can conclude that the two equations of motion are the same if the variables ΔT and C are replaced by the variables T_{12} and C_{eff} . Thus, with this mapping, eq. (6) becomes:

$$\tilde{T}_{12}(f) \bullet \tilde{T}_{12}^* = \frac{4\tau_{eff} k_B T^2 (C_1 + C_2)}{C_1 C_2 (1 + 4\pi^2 \tau_{eff}^2 f^2)} \quad (7)$$

Alternatively, one can derive eq. (7) directly from the generalized FDT. In the isolated system, T_{12} is held at zero by removing all heat sources, and the free energy is defined by $df(T_{12}) = -SdT_{12}$. The entropy is changed by heat flow from one mass to the other, thus $TdS = Q_{12}dt$. But from equation (6), $Q_{12}dt = C_{eff}dT_{12}$. Thus dF/dx in the generalized FDT is, in this case:

$$\frac{dS}{dT_{12}} = \frac{C_{eff}}{T} \quad (8)$$

and eq. (7) is obtained.

The conservation of energy requires that the heat that leaves one mass enters the other, implying that $C_1\Delta T_1 = -C_2\Delta T_2$ and therefore $\tilde{T}_1(f) \bullet \tilde{T}_2^*(f) = -C_1 \bullet [\tilde{T}_1(f) \bullet \tilde{T}_1^*(f)] / C_2 = -C_2 \bullet [\tilde{T}_2(f) \bullet \tilde{T}_2^*(f)] / C_1$. These relations indicate that:

$$\tilde{T}_{12}(f) \bullet \tilde{T}_{12}^*(f) = \tilde{T}_1(f) \bullet \tilde{T}_1^*(f) + \tilde{T}_2(f) \bullet \tilde{T}_2^*(f) - 2[\tilde{T}_1(f) \bullet \tilde{T}_2^*(f)] \quad (9)$$

$$= \left(-\frac{C_2}{C_1} - \frac{C_1}{C_2} - 2 \right) \bullet [\tilde{T}_1(f) \bullet \tilde{T}_2^*(f)] \quad (10)$$

$$= - \left[\frac{(C_1 + C_2)^2}{C_1 C_2} \right] \bullet [\tilde{T}_1(f) \bullet \tilde{T}_2^*(f)] \quad (11)$$

From this expression we can solve for the anti-correlated component of the cross-correlation function:

$$\tilde{T}_{1-}(f) \bullet \tilde{T}_{2-}^*(f) = - \frac{4\tau_{eff} k_B T^2}{(C_1 + C_2)(1 + 4\pi^2 \tau_{eff}^2 f^2)}, \quad (12)$$

where we have added the subscript “-“ to denote that this component of the fluctuations arises from energy transfer between the two salt pills. If the two masses were completely isolated from their surroundings, there would be no correlated component. However, out of the necessity to stabilize the temperature of the salt pills for the measurement, the coupled masses are connected to the heat reservoir by another thermal link. For low frequencies, the system therefore behaves like a single mass with heat capacity equal to $C_1 + C_2$. This introduces an additional low frequency spectrum of:

$$\tilde{T}_{1+}(f) \bullet \tilde{T}_{2+}^*(f) = \frac{4\tau_0 k_B T^2}{(C_1 + C_2)(1 + 4\pi^2 \tau_0^2 f^2)} \quad (13)$$

where $\tau_0 = (R_0 + R_{12}/4) / (C_1 + C_2)$, and the subscript “+” denotes that this component of the fluctuations arises from energy exchange with the reservoir.

Since the two sources both contribute to the total thermal fluctuations in each salt pill, $\Delta T_1 = \Delta T_{1+} + \Delta T_{1-}$, and $\Delta T_2 = \Delta T_{2+} + \Delta T_{2-}$. Thus

$$\tilde{T}_1(f)\tilde{T}_2^*(f) = \tilde{T}_{1+} \bullet \tilde{T}_{2+}^* + \tilde{T}_{1-} \bullet \tilde{T}_{2-}^* + \tilde{T}_{1+} \bullet \tilde{T}_{2-}^* + \tilde{T}_{1-} \bullet \tilde{T}_{2+}^* . \quad (14)$$

The last two terms are zero because the “+” and the “-” fluctuations are independent and uncorrelated. We therefore obtain the total cross-correlation function:

$$\tilde{T}_1(f)\bullet\tilde{T}_2^*(f) = \frac{4k_B T^2}{(C_1 + C_2)} \left[\frac{\tau_0}{(1 + 4\pi^2 \tau_0^2 f^2)} - \frac{\tau_{eff}}{(1 + 4\pi^2 \tau_{eff}^2 f^2)} \right] \quad (15)$$