

The size-Ramsey number of cubic graphs

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Abstract

We show that the size-Ramsey number of any cubic graph with n vertices is $O(n^{8/5})$, improving a bound of $n^{5/3+o(1)}$ due to Kohayakawa, Rödl, Schacht, and Szemerédi. The heart of the argument is to show that there is a constant C such that a random graph with Cn vertices where every edge is chosen independently with probability $p \geq Cn^{-2/5}$ is with high probability Ramsey for any cubic graph with n vertices. This latter result is best possible up to the constant.

1 Introduction

We say that a graph G is *Ramsey* for another graph H , and write $G \rightarrow H$, if every colouring of the edges of G with two colours contains a monochromatic copy of H . In this paper, we study the quantity $\hat{r}(H)$, the *size-Ramsey number* of H , defined as the smallest $m \in \mathbb{N}$ such that there exists a graph G with m edges which is Ramsey for H . As any sufficiently large complete graph is Ramsey for H , this notion is well-defined.

The study of size-Ramsey numbers was initiated by Erdős, Faudree, Rousseau, and Schelp [14] in 1978. Their paper already contains the observation, which they attribute to Chvátal, that $\hat{r}(K_n) = \binom{r(K_n)}{2}$, where $r(K_n)$ is the usual *Ramsey number*, i.e., the smallest integer N such that $K_N \rightarrow K_n$. In other words, when $H = K_n$ one cannot do better than taking G to be the smallest complete graph that is Ramsey for H . A significantly more interesting example is when $H = P_n$, the path on n vertices, for which Beck [4] has shown that $\hat{r}(P_n) = O(n)$. Subsequent work has extended this result to many other families of graphs, including bounded-degree trees [15], cycles [21], and, more recently, powers of paths and bounded-degree trees [6] and long subdivisions [12]. For some further recent developments, see [8, 19, 20, 23, 26].

Moving away from trees and tree-like graphs, Beck [5] asked whether the size-Ramsey number of every bounded-degree graph is linear in its order. Rödl and Szemerédi [31] answered this in the negative by showing that there exists a constant $c > 0$ and, for every n , an n -vertex *cubic graph* H , that is, a graph with maximum degree three, such that $\hat{r}(H) \geq n(\log n)^c$. To date, this remains the best known general lower bound for the size-Ramsey number of bounded-degree graphs. Despite this, the conjecture of Rödl and Szemerédi [31] that there exists a constant $\varepsilon > 0$ and, for every n , an n -vertex cubic graph H such that $\hat{r}(H) \geq n^{1+\varepsilon}$ is widely believed.

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Regarding upper bounds, a result of Kohayakawa, Rödl, Schacht, and Szemerédi [25] shows that $\hat{r}(H) \leq n^{2-1/\Delta+o(1)}$ for any graph H on n vertices with maximum degree Δ . Here we improve this bound for cubic graphs.

Theorem 1.1. *There exists a constant K such that $\hat{r}(H) \leq Kn^{8/5}$ for every cubic graph H with n vertices.*

Seeing as there is still a very large gap between the upper and lower bounds for size-Ramsey numbers of cubic graphs, our result warrants some justification. The key point is that all previous work on upper bounds for size-Ramsey numbers has relied upon showing that suitable random graphs (or, in some cases, graphs derived from random graphs by taking appropriate powers and blow-ups) are, with high probability,¹ Ramsey for the required target graph H . Recall that $G_{n,p}$ stands for the probability distribution over all graphs with $n \in \mathbb{N}$ vertices in which each pair of vertices forms an edge independently with probability $p = p(n) \in (0, 1)$. We will use $G_{n,p}$ interchangeably to describe both this distribution and an actual graph sampled from it. With this notation, our real main result is then the following theorem about Ramsey properties of random graphs.

Theorem 1.2. *There exist $c, K > 0$ such that if $p \geq Kn^{-2/5}$, then, with high probability, $G_{n,p} \rightarrow H$ for every cubic graph H with at most cn vertices.*

It is easy to see that Theorem 1.2 implies Theorem 1.1, since $G_{n,p}$ typically has $\Theta(n^2p)$ edges, by standard concentration inequalities. Moreover, Theorem 1.2 is optimal: a classic result of Rödl and Ruciński [29, 30] shows that if $p = o(n^{-2/5})$, then, with high probability, $G_{n,p}$ is not Ramsey for K_4 . Hence, the statement of Theorem 1.1 is the most one can get out of using vanilla random graphs (that is, without modifying them further). We will come back to this point, and a further discussion on the limits of our method, in Section 6.

The rest of the paper is organised as follows. In the next section, we give a high-level overview of our argument. In Section 3, we collect several results about random graphs and sparse regular pairs that will be needed in the proof. Section 4 contains our two main building blocks, which will allow us to thread trees and cycles through prescribed sets of vertices. In Section 5, we then combine these building blocks with a decomposition result for cubic graphs to complete the proof of Theorem 1.2. Finally, as mentioned above, we will discuss the limits of our method and the potential next steps in Section 6.

2 Overview of the proof

Assume that the host graph $\Gamma \sim G_{n,p}$ satisfies a number of the properties that typically hold in random graphs, such as expansion, no dense spots, concentration of degrees and number of edges in subsets, etc. In their proof, Kohayakawa, Rödl, Schacht, and Szemerédi [25] start by showing that in any red/blue colouring of the edges of Γ , one can find disjoint sets $V_1, \dots, V_{20} \subseteq V(\Gamma)$, each of order αn for some $\alpha > 0$, such that each pair (V_i, V_j) is (ε, p) -regular with at least $|V_i||V_j|p/3$ edges between V_i and V_j in one of the colours, say red. That is, the density of edges between any two sufficiently large sets $U_i \subseteq V_i$ and $U_j \subseteq V_j$ is roughly the same as the density

¹We say that a property holds with high probability (or w.h.p. for brevity) if the probability it holds tends to 1 as $n \rightarrow \infty$.

between V_i and V_j , with a discrepancy of at most εp (see Definition 3.2 below). This is a standard step in the regularity method and we refer the reader to [7, 25] for more details.

Suppose, therefore, that we have a collection of large sets of vertices such that the distribution of red edges between any two of them is fairly uniform. We wish to use this structure to show that R , the subgraph of Γ consisting of all the red edges, contains any particular cubic graph H on at most cn vertices. The strategy used in [25] at this point follows an idea of Alon and Füredi [2]: split the vertex set of H into, say, 20 independent sets and then embed these sets one at a time using Hall's matching criteria, with the i -th set being embedded into V_i . When we come to embed the last such set, every remaining vertex of degree three has to be mapped into the common neighbourhood of three previously embedded vertices. However, if $p = o(n^{-1/3})$, three typical vertices in the random graph will have an empty common neighbourhood. It is for this reason that the methods of [25] break down at this point.

To circumvent this issue, we borrow an idea from the work of the first two authors together with Ferber and Škorić [10]. Assume that H is connected and remove from it an induced cycle of length at least four. This leaves us with a 2-degenerate graph H' . That is, we can order the vertices of H' in such a way that every vertex has at most two of its neighbours preceding it. One might then hope that $p \gg n^{-1/2}$ is sufficient to embed H' . Having embedded H' , we then need to replace the deleted cycle, which we suppose for illustration is a C_4 . As the graph is cubic, each vertex v of such a C_4 already has at most one embedded neighbour, so there will be a *candidate set* of order roughly np in which we can embed v . If all four candidate sets are disjoint and span four (ε, p) -regular pairs of density $\Theta(p)$ in the correct cyclic order, then a result of Gerke, Kohayakawa, Rödl, and Steger [16] implies that we can find the desired copy of C_4 between these sets, provided $p \gg (np)^{-2/3}$ (see Lemma 3.7 below). Rearranging, this gives $p \gg n^{-2/5}$ — precisely the bound promised by Theorem 1.2. Note that, crucially, the cycle we embed at the end is of length at least four. If it were a triangle instead, we would need $p \gg (np)^{-1/2}$ in the last step, which leads back to the bound $p \gg n^{-1/3}$.

In practice, our actual approach takes the idea of partitioning even further. Instead of taking out one cycle and relying on the fact that the remaining graph H' is 2-degenerate, we partition H (which we may assume contains no K_4 's, since they can be set aside and dealt with separately) into *blocks*: first removing a maximal collection of disjoint induced cycles of length at least four and then partitioning what remains into induced paths (see Lemma 5.1 below). Moreover, these blocks B_1, \dots, B_t can be placed in a '1-degenerate ordering', meaning that each vertex in B_i has at most one neighbour in $B_1 \cup \dots \cup B_{i-1}$ for every $i \in \{2, \dots, t\}$. We then find a copy of H by embedding one whole block at a time. Crucially, whenever we are about to embed a block B_i , every $v \in B_i$ has at most one previously embedded neighbour, so the candidate set for v has order $\Omega(np)$. This is large enough that the regularity property is inherited by any relevant pair of candidate sets, which then allows us to embed B_i , which is either a path or a cycle, in one sweep (see Lemmas 4.1 and 4.2 below).

3 Preliminaries

In this section, we collect several results about random graphs that we will need, with a particular focus on the properties of regular pairs in random graphs. For a thorough, although now somewhat outdated, treatment of the latter topic, we refer the reader to the survey of Gerke and Steger [18]. Though many of the results hold in greater generality, we have tailored the

statements towards their later use in the proof of Theorem 1.2. Where statements depart from their usual form in the literature, we also give, at the very least, a sketch of the proof.

We begin with a standard concentration result, which easily follows from combining Chernoff's inequality (see, e.g., [22, Corollary 2.3]) with a union bound. Here and throughout we write $\hat{N}_G(X, Y)$ for the set of *common neighbours* of the vertices from X in Y , that is, $\hat{N}_G(X, Y) := Y \cap \bigcap_{x \in X} N_G(x)$ and $\hat{N}_G(X) = \hat{N}_G(X, V(G) \setminus X)$.

Lemma 3.1. *For every $d \in \mathbb{N}$ and $\delta \in (0, 1)$, there exists a positive constant K such that, for $p \geq (K \log n/n)^{1/d}$, the random graph $\Gamma \sim G_{n,p}$ w.h.p. has the following property. For every family of disjoint d -sets $\mathcal{P} \subseteq \binom{V(\Gamma)}{d}$ of size $|\mathcal{P}| \leq \delta/p^d$,*

$$\left| \bigcup_{S \in \mathcal{P}} \hat{N}_\Gamma(S) \right| = (1 \pm \delta) |\mathcal{P}| np^d.$$

3.1 Properties of (ε, p) -regular pairs

We have already referred to (ε, p) -regular pairs several times. The formal definition is as follows.

Definition 3.2. Let G be a graph and let $V_1, V_2 \subseteq V(G)$ be disjoint subsets. We say that the pair (V_1, V_2) is (ε, p) -regular for some $0 < \varepsilon, p \leq 1$ if, for every $U_1 \subseteq V_1$, $U_2 \subseteq V_2$ with $|U_1| \geq \varepsilon|V_1|$, $|U_2| \geq \varepsilon|V_2|$, we have

$$|d_G(U_1, U_2) - d_G(V_1, V_2)| \leq \varepsilon p,$$

where $d_G(A, B) = e_G(A, B)/(|A||B|)$ denotes the edge density of a given pair.

It follows from the definition that if $d = d_G(V_1, V_2) = \Theta(p)$ and ε is sufficiently small, then there cannot be more than $\varepsilon|V_1|$ vertices in V_1 which have fewer than, say, $d|V_2|/2$ neighbours in V_2 . The next result shows that we can even find large subsets $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$ such that each vertex in V'_1 has at least $d|V_2|/2$ neighbours in V'_2 and vice versa and, moreover, that this property can be achieved for many pairs simultaneously.

Lemma 3.3. *For every $\Delta \in \mathbb{N}$ and $\gamma > 0$, there exists $\varepsilon_0 > 0$ such that the following holds for any $0 < \varepsilon \leq \varepsilon_0$ and $p \in (0, 1)$. Let H be a graph with maximum degree Δ and let $\{V_i\}_{i \in V(H)}$ be a family of subsets of some graph G such that (V_i, V_j) is (ε, p) -regular of density $d \geq \gamma p$ (with respect to G) for every $ij \in H$. Then, for every $i \in V(H)$, there exists $V'_i \subseteq V_i$ of order $|V'_i| \geq (1 - \Delta\varepsilon)|V_i|$ such that $\deg_G(v, V'_j) \geq d|V_j|/2$ for every $v \in V'_i$ and all $ij \in H$.*

Sketch of the proof. Set $B_i = \emptyset$ for every $i \in V(H)$ and repeat the following process: as long as there is an edge $ij \in H$ and a vertex $v \in V_i \setminus B_i$ which has fewer than $d|V_j|/2$ neighbours in $V_j \setminus B_j$, add v to B_i . Suppose that at some point one of the sets B_i becomes larger than $\Delta\varepsilon|V_i|$ and, once this happens, we terminate the process. By the pigeonhole principle, there must be a subset $B'_i \subseteq B_i$ of order $|B'_i| \geq \varepsilon|V_i|$ and an edge $ij \in H$ such that every $v \in B'_i$ satisfies $\deg_G(v, V_j \setminus B_j) < d|V_j|/2$. This implies that the density of the pair $(B'_i, V_j \setminus B_j)$ is less than $d/2$. On the other hand, as $|V_j \setminus B_j| \geq (1 - \Delta\varepsilon)|V_j|$, the (ε, p) -regularity property tells us that this density is close to d , a contradiction. In particular, the procedure terminates with each B_i being of order at most $\Delta\varepsilon|V_i|$ and the statement of the lemma follows. \square

3.1.1 Regularity inheritance

The following lemma is usually referred to as the *slicing lemma* and follows directly from the definition of (ε, p) -regularity.

Lemma 3.4. *Let $0 < \varepsilon_1 < \varepsilon_2 \leq 1/2$, $p \in (0, 1)$, and let (X, Y) be an (ε_1, p) -regular pair. Then any two subsets $X' \subseteq X$ and $Y' \subseteq Y$ of order $|X'| \geq \varepsilon_2|X|$ and $|Y'| \geq \varepsilon_2|Y|$ form an $(\varepsilon_1/\varepsilon_2, p)$ -regular pair of density $d(X, Y) \pm \varepsilon_1p$.*

In other words, sufficiently large subsets of a regular pair again induce a regular pair. This then allows us to take subsets of our regular pairs and yet still assume that they are (ε, p) -regular with ε sufficiently small, a fact that we will use implicitly in our main proof.

The next lemma captures a key feature of sparse regularity, that, for subgraphs of random graphs, the regularity property is typically inherited between neighbourhoods of vertices.

Lemma 3.5. *For all $\varepsilon', \alpha, \gamma, \delta > 0$, there exist $\varepsilon_0 = \varepsilon_0(\varepsilon', \gamma, \delta)$ and $K = K(\varepsilon', \alpha, \gamma)$ such that, for every $0 < \varepsilon \leq \varepsilon_0$ and $p \geq K(\log n/n)^{1/2}$, the random graph $\Gamma \sim G_{n,p}$ w.h.p. has the following property.*

Suppose $G \subseteq \Gamma$ and $V_1, V_2 \subseteq V(\Gamma)$ are disjoint subsets of order $\tilde{n} = \alpha n$ such that (V_1, V_2) is (ε, p) -regular of density $d \geq \gamma p$ with respect to G . Then there exists $B \subseteq V(\Gamma)$ of order $|B| \leq \delta \tilde{n}$ such that for each $v, w \in V(\Gamma) \setminus (V_1 \cup V_2 \cup B)$ (not necessarily distinct) the following holds: for any two subsets $N_v \subseteq N_\Gamma(v, V_1)$ and $N_w \subseteq N_\Gamma(w, V_2)$ of order $\tilde{n}d/4$, both (N_v, V_2) and (N_v, N_w) are (ε', p) -regular of density $(1 \pm \varepsilon')d$ with respect to G .

To prove Lemma 3.5, we need the following lemma of Škorić, Steger, and Trujić [33], itself based on an earlier result of Gerke, Kohayakawa, Rödl, and Steger [16]. It enables us to say something about regularity inheritance for subsets of order $o(n)$, a regime in which Lemma 3.4 tells us nothing.

Lemma 3.6 (Corollary 3.5 in [33]). *For all $0 < \beta, \varepsilon', \gamma < 1$, there exist positive constants $\varepsilon_0 = \varepsilon_0(\beta, \varepsilon', \gamma)$ and $D = D(\varepsilon')$ such that, for every $0 < \varepsilon \leq \varepsilon_0$ and $p = p(n) \in (0, 1)$, the random graph $\Gamma \sim G_{n,p}$ w.h.p. has the following property. Suppose $G \subseteq \Gamma$ and (V_1, V_2) is an (ε, p) -regular pair of density $d \geq \gamma p$ with respect to G . Then, for all $q_1, q_2 \geq Dp^{-1} \log n$, there are at least*

$$(1 - \beta^{\min\{q_1, q_2\}}) \binom{|V_1|}{q_1} \binom{|V_2|}{q_2}$$

sets $Q_i \subseteq V_i$ of order $|Q_i| = q_i$, $i \in \{1, 2\}$, which induce an (ε', p) -regular pair of density $(1 \pm \varepsilon')d$ with respect to G .

Intuitively, Lemma 3.6 tells us that if we have an (ε, p) -regular pair (V_1, V_2) with respect to a subgraph of $G_{n,p}$ and two vertices v and w which have neighbourhoods of order $\Theta(np)$ in V_1 and V_2 , respectively, then, for $p = \Omega((\log n/n)^{1/2})$, it is extremely unlikely that these two neighbourhoods do not inherit regularity. In fact, it is so unlikely that a union bound over all possible situations suffices to prove Lemma 3.5.

Proof of Lemma 3.5. We only prove that there exists a B which ensures that (N_v, N_w) is always (ε', p) -regular with the required density. The other case follows from similar calculations, so we omit it.

For two disjoint subsets $V_1, V_2 \subseteq V(\Gamma)$ of order \tilde{n} , let $\mathcal{E}(V_1, V_2)$ be the event that there exists $G \subseteq \Gamma$ for which the conclusion of the lemma fails for this particular choice of V_1 and V_2 . We will show that $\Pr[\mathcal{E}(V_1, V_2)] \leq \exp(-c\tilde{n}^2p)$ for some $c > 0$, which is more than enough to beat the union bound over all choices of V_1 and V_2 .

Suppose G is a witness for $\mathcal{E}(V_1, V_2)$. Consider the following process: start with $\mathcal{B} = \emptyset$ and $B = \emptyset$ and, as long as there exist (not necessarily distinct) vertices $v, w \in V(G) \setminus (B \cup V_1 \cup V_2)$ which violate the desired property, add the pair (v, w) to \mathcal{B} and add v and w to B . As soon as $|B|$ becomes at least $\delta\tilde{n}$, we stop the procedure. Note that the procedure must go on at least this long, by the choice of G . Then \mathcal{B} contains at least $\delta\tilde{n}/2$ and at most $\delta\tilde{n}$ pairs.

Therefore, if G is a witness for $\mathcal{E}(V_1, V_2)$, then there exists a set of pairs of vertices \mathcal{B} of size $\delta\tilde{n}/2 \leq |\mathcal{B}| \leq \delta\tilde{n}$ such that all elements of \mathcal{B} are pairwise disjoint and, for every $(v, w) \in \mathcal{B}$, there exist $N_v \subseteq N_\Gamma(v, V_1)$ and $N_w \subseteq N_\Gamma(w, V_2)$, both of order $\tilde{n}d/4$, such that (N_v, N_w) does not span an (ε', p) -regular pair of density $(1 \pm \varepsilon')d$ with respect to G . Let us bound the probability that such a configuration exists in $G_{n,p}$.

We first expose the edges between V_1 and V_2 in Γ . Choose a subset $E' \subseteq E_\Gamma(V_1, V_2)$ of these edges such that (V_1, V_2) is (ε, p) -regular of density $d \geq \gamma p$ with respect to E' . Conditioning on $|E_\Gamma(V_1, V_2)| \leq 2\tilde{n}^2p$ (which we may assume holds w.h.p. by a standard application of Chernoff's inequality and the union bound), there are at most $2^{2\tilde{n}^2p}$ choices for E' . Next, there are at most $\delta\tilde{n} \cdot (n!)^2 < 2^{3n \log n}$ choices for \mathcal{B} . Finally, for each $(v, w) \in \mathcal{B}$, choose the 'bad' subsets $N_v \subseteq V_1$ and $N_w \subseteq V_2$ of order $x = \tilde{n}d/4$. As (V_1, V_2) satisfies the conclusion of Lemma 3.6 and $\tilde{n}d/4 \geq D_{3.6}(\varepsilon')p^{-1} \log n$ by the assumption on p from the statement of the lemma and the fact that K is sufficiently large, there are at most

$$\left(\beta^x \binom{\tilde{n}}{x} \right)^{|\mathcal{B}|}$$

such choices. But we also need that each such set N_v lies in the neighbourhood of v in Γ and similarly for N_w and w . Using the fact that any two pairs in \mathcal{B} are disjoint and, within each pair, we consider neighbours into disjoint sets V_1 and V_2 , the probability of this happening is exactly $p^{2|\mathcal{B}|x}$. Putting all this together, we get that

$$\Pr[\mathcal{E}(V_1, V_2)] \leq 2^{2\tilde{n}^2p} \cdot 2^{3n \log n} \cdot \beta^{|\mathcal{B}|\tilde{n}d/4} \left(\frac{\tilde{n}}{\tilde{n}d/4} \right)^{2|\mathcal{B}|} \cdot p^{2|\mathcal{B}|\tilde{n}d/4}.$$

Using the standard bound $\binom{a}{b} \leq \left(\frac{ea}{b}\right)^b$, we conclude that by taking β sufficiently small we can make this be at most $\exp(-c\tilde{n}^2p)$, as desired. \square

3.1.2 The KLR conjecture in random graphs

The following result from [11] gives sufficient conditions for the existence of a small, fixed graph between an appropriate collection of (ε, p) -regular pairs in a random graph. This result also follows from a celebrated conjecture of Kohayakawa, Łuczak, and Rödl [24] (the so-called *KLR conjecture*), which was fully resolved by Balogh, Morris, and Samotij [3] and, independently, Saxton and Thomason [32] (though see the recent paper [28] for another proof). However, we will only invoke the lemma when H is either a cycle or K_4 , both of which were known before the full conjecture was proved (see [16] and [17], respectively).

Lemma 3.7 (The KLR conjecture). *For every graph H and every $\gamma > 0$, there exist $\varepsilon_0, K > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_0$ and $p \geq Kn^{-1/m_2(H)}$, where*

$$m_2(H) = \max \left\{ \frac{e(F) - 1}{v(F) - 2} : F \subseteq H, v(F) \geq 3 \right\},$$

the random graph $\Gamma \sim G_{n,p}$ w.h.p. has the following property.

Suppose $G \subseteq \Gamma$ and $\{V_i\}_{i \in V(H)}$ is a family of disjoint subsets of $V(G)$, each of order $\tilde{n} \geq \max\{(K/p)^{m_2(H)}, K \log n/p\}$. Suppose also that, for each $ij \in H$, the pair (V_i, V_j) is (ε, p) -regular of density at least γp with respect to G . Then there exists a copy of H in G which maps each vertex i to V_i .

Sketch of the proof. Theorem 1.6 in [11] gives $K > 1$ such that the conclusion of the lemma holds with probability at least $1 - \exp(-bn^2p)$ in the special case where $\tilde{n} = n/v(H)$, for some constant $b > 0$ depending on H and γ . As every subgraph of $G_{n,p}$ with s vertices is distributed as $G_{s,p}$, we can apply the above to such a subset as long as $p \geq Ks^{-1/m_2(H)}$ or, equivalently, $s \geq (K/p)^{m_2(H)}$. In particular, a subgraph of $G_{n,p}$ induced by a vertex subset S of order $s \geq (K/p)^{m_2(H)}$ fails to have the required property in the case $\tilde{n} = s/v(H)$ with probability at most $\exp(-bs^2p)$. By the assumption $\tilde{n} \geq K \log n/p$, and as we may assume K is sufficiently large in terms of b , this probability is at most n^{-2s} , which is sufficient to take a union bound over all possible choices of S . As every (ε, p) -regular constellation with each set of order \tilde{n} naturally lies in a subset S of order $v(H)\tilde{n}$, the claimed statement follows. \square

4 Trees and cycles through prescribed sets

In this section, we provide the two main building blocks used in the proof of Theorem 1.2. Both results are stated in greater generality than what is needed for Theorem 1.2, with the goal of making further claims in Section 6 more transparent.

Our first building block says that, under appropriate conditions, one can embed bounded-degree trees so that each vertex is mapped into a prescribed set. The embedding strategy used both here and in the main theorem, which we dub *first-free-bucket embedding*, originates from the second author's PhD thesis [27]. While the bound on p here could be lowered to, say, $n^{-1/2} \log^3 n$, we have decided not to complicate the proof any further.

Lemma 4.1. *For every $\alpha, \gamma, \xi > 0$ and $\Delta \in \mathbb{N}$, there exist $c = c(\alpha, \gamma, \xi, \Delta), \varepsilon_0 = \varepsilon_0(\gamma, \Delta) > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_0$ and $p \geq n^{-1/2+\xi}$, the random graph $\Gamma \sim G_{n,p}$ w.h.p. has the following property.*

Let T be a tree with $t \leq cn$ vertices and maximum degree Δ . Let $G \subseteq \Gamma$ and, for each $v \in V(T)$, let $s_v \in V(G)$ be a specific vertex such that every $s \in V(G)$ is chosen at most Δ times. Then, for any collection of subsets $N_v \subseteq N_G(s_v)$ of order αnp such that (N_v, N_w) is (ε, p) -regular of density $d \geq \gamma p$ (with respect to G) for each $vw \in T$, there exists a copy of T in G which maps each $v \in V(T)$ to N_v .

Proof. By applying Lemma 3.3 and renaming the sets if necessary, we may assume that each vertex in N_v has at least $d|N_w|/2$ neighbours in N_w for each $vw \in T$ and $|N_v| \geq \tilde{n} := 0.9\alpha np$.

Let $U = \bigcup_{v \in V(T)} N_v$ and take an equipartition $U = U_0 \cup U_1 \cup \dots \cup U_z$ uniformly at random, where $z = \lceil 1/\xi \rceil$. Let $N_v^j := U_j \cap N_v$ for all $v \in V(T)$ and $j \in \{0, \dots, z\}$. By Chernoff's inequality

and the union bound, there exists a choice for the U_j such that every vertex in N_v has at least $d|N_w|/(4z)$ neighbours in each N_w^j for all $vw \in T$. In other words, for every $u \in N_v$, we have

$$|\hat{N}_G(\{u, s_w\}, N_w^j)| = |N_G(u, N_w^j)| \geq d\tilde{n}/(4z). \quad (1)$$

Let $\{v_1, \dots, v_t\}$ be an ordering of the vertices of T such that each v_i , for $i \geq 2$, has exactly one preceding neighbour and denote this neighbour by a_i . For brevity, rename $N_{v_i}^j$ as V_i^j and s_{v_i} as s_i and, for every $i \in [t]$, let $T^{<i} := \{v_1, \dots, v_{i-1}\}$. We construct the desired copy of T by defining an embedding φ as follows:

- (E1) assign $\varphi(v_1) := x$ for an arbitrary vertex $x \in V_1^0$,
- (E2) for every $i \geq 2$, sequentially, take an arbitrary vertex x from the first (smallest $j \in \{0, \dots, z\}$) non-empty set $N_G(\varphi(a_i), V_i^j) \setminus \varphi(T^{<i})$ and assign $\varphi(v_i) := x$, or
- (E3) if all such sets are empty, terminate.

If the process never reaches (E3), the embedding φ gives the desired copy of T . Therefore, suppose towards a contradiction that there is some $i \in [t]$ for which the process enters (E3), that is, such that $N_G(\varphi(a_i), V_i^j) \setminus \varphi(T^{<i}) = \emptyset$ for all $j \in \{0, \dots, z\}$.

Let $X_j := U_j \cap \varphi(T^{<i})$, for all $j \in \{0, \dots, z\}$, be the set of vertices in U_j ‘taken’ by the images of $\{v_1, \dots, v_{i-1}\}$ under φ . We claim that

$$|X_j| \leq \frac{c_j n}{(np^2)^j} \quad (\star)$$

for constants c_j with $c_0 = c = \alpha^2 \gamma^2 / (10^6 \Delta z^2)$ and $c_j = 400 \Delta z c_{j-1} / (\alpha \gamma)$ for all $j \geq 1$.

Under this assumption, since $(np^2)^z \geq n^2$, we get that $X_z = \emptyset$ and thus $N_G(\varphi(a_i), V_i^z) \setminus \varphi(T^{<i}) \neq \emptyset$, contradicting our assumption that (E3) happens at step i . Therefore, in order to complete the proof, it only remains to show (\star) .

Clearly, $|X_0| \leq c_0 n$ as $t \leq cn$. Consider now the smallest $j \in [z]$ for which (\star) is violated. Without loss of generality, we may assume that $|X_j| = \lceil c_j n / (np^2)^j \rceil$ (if not, we can take a subset of X_j of precisely that order). Let $I \subseteq [i-1]$ be the set of indices of vertices $v_k \in T^{<i}$ that are embedded into X_j , i.e.,

$$I := \{k \in [i-1] : \varphi(v_k) \in X_j\}.$$

Since $\Delta(T) \leq \Delta$ and each $s_k \in V(G)$ is chosen at most Δ times, there exists (by greedily taking indices) $I' \subseteq I$ of size at least $|I|/(4\Delta)$ such that the sets $\{\varphi(a_k), s_k\}$ are pairwise disjoint for all $k \in I'$. For simplicity in our notation, we will assume that I already has this property, noting that $|I| \geq |X_j|/(4\Delta)$. For every $k \in I$, we set $\mathcal{P}_k = \{\varphi(a_k), s_k\}$.

Since, for every $k \in I$, the vertex v_k is embedded into X_j and not X_{j-1} , we have

$$|X_{j-1}| \geq \left| \bigcup_{k \in I} \hat{N}_G(\mathcal{P}_k, V_k^{j-1}) \right|.$$

Our goal is to show that this implies X_{j-1} is larger than what is claimed in (\star) , contradicting our assumption that j is the smallest index violating (\star) .

As $\hat{N}_G(\mathcal{P}_k, V_k^{j-1}) \subseteq \hat{N}_G(\mathcal{P}_k) \subseteq \hat{N}_\Gamma(\mathcal{P}_k)$, we have

$$\left| \bigcup_{k \in I} \hat{N}_G(\mathcal{P}_k, V_k^{j-1}) \right| \geq \left| \bigcup_{k \in I} \hat{N}_\Gamma(\mathcal{P}_k) \right| - \left| \bigcup_{k \in I} (\hat{N}_\Gamma(\mathcal{P}_k) \setminus \hat{N}_G(\mathcal{P}_k, V_k^{j-1})) \right|. \quad (2)$$

Note that $|I| \leq |X_j|$ and that $|X_j| \leq \lceil c_1/p^2 \rceil$ for $j = 1$ and $|X_j| = o(1/p^2)$ for $j \geq 2$. Hence, by Lemma 3.1 with $d = 2$, $\delta = 2c_1$, and $\mathcal{P} = \{\mathcal{P}_k\}_{k \in I}$, the first term on the right-hand side of (2) can be bounded by

$$\left| \bigcup_{k \in I} \hat{N}_\Gamma(\mathcal{P}_k) \right| \geq (1 - \delta)|I|np^2.$$

Lemma 3.1 also gives $|\hat{N}_\Gamma(\mathcal{P}_k)| \leq (1 + \delta)np^2$. Combining the previous two bounds with (1), we get that

$$|X_{j-1}| \geq (1 - \delta)|I|np^2 - |I|((1 + \delta)np^2 - d\tilde{n}/(4z)) \geq |I|d\tilde{n}/(8z) > |I|\alpha\gamma np^2/(100z),$$

where we used that $c_1 \leq \alpha\gamma/(300z)$. Finally, from $|I| \geq |X_j|/(4\Delta)$ and $|X_j| = \lceil c_j n/(np^2)^j \rceil$, we arrive at

$$|X_{j-1}| > \frac{c_j n}{4\Delta(np^2)^j} \cdot \frac{\alpha\gamma np^2}{100z} \geq \frac{c_{j-1}n}{(np^2)^{j-1}},$$

which contradicts our assumption that j was the smallest index violating (\star) . \square

The following lemma gives the same statement when T is a cycle C_t instead of a tree. Here, the bound on p comes from the fact that we apply Lemma 3.7 with $\tilde{n} = \Theta(np)$, for which we need $p = \Omega((np)^{-1/m_2(C_t)})$.

Lemma 4.2. *For every $\alpha, \gamma > 0$ and $\ell \geq 3$, there exist $\varepsilon_0 = \varepsilon_0(\gamma, \ell)$, $c = c(\alpha, \gamma, \ell)$, $K = K(\alpha, \gamma, \ell) > 0$ such that the statement of Lemma 4.1 holds if T is a cycle of length $t \in [\ell, cn]$ and $p \geq Kn^{-(\ell-2)/(2\ell-3)}$.*

Sketch of the proof. We first deal with the case where T is a cycle of length $t \in [\ell, 4\ell]$. Greedily take $N'_v \subseteq N_v$, each of order $|N'_v| = |N_v|/(2t)$, such that they are pairwise disjoint. By Lemma 3.4, these sets are (ε', p) -regular with density close to d and so Lemma 3.7 implies the existence of the desired copy of T .

The case $t > 4\ell$ requires a bit more work. Let us denote the vertices of T by v_1, \dots, v_t , in the natural order. Again, for each $i \in \{1, \dots, \ell + 1\} \cup \{t, \dots, t - \ell + 1\}$, choose a subset $N'_{v_i} \subseteq N_{v_i}$ of order $|N'_{v_i}| = |N_{v_i}|/(4\ell)$ such that they are pairwise disjoint. Let V' be the union of all these sets and, for every other v , set $N'_v = N_v \setminus V'$. As $(np^2)^\ell \gg np$ by the assumption on p , one can expect that the ℓ -th neighbourhood of each vertex $v \in N'_{v_1}$ contains almost all vertices in both $N'_{v_{\ell+1}}$ and $N'_{v_{t-\ell+1}}$. A minor modification of [13, Corollary 2.5] gives precisely this. Namely, it shows that there exists a vertex $v \in N'_{v_1}$ such that, for all but $\varepsilon|N'_{v_{\ell+1}}|$ vertices $w \in N'_{v_{\ell+1}}$, there exists a path from v to w (in G) with one vertex in each of $N'_{v_2}, \dots, N'_{v_\ell}$ and similarly for $N'_{v_{t-\ell+1}}$ with paths through $N'_{v_t}, \dots, N'_{v_{t-\ell+2}}$. Let us denote the set of such vertices reachable from v by $N''_{v_{\ell+1}}$ and $N''_{v_{t-\ell+1}}$. Now we can apply Lemma 4.1 to find a path with one vertex in each of $N''_{v_{\ell+1}}, N'_{v_{\ell+2}}, \dots, N'_{v_{t-\ell}}, N''_{v_{t-\ell+1}}$. Together with the paths to v from $N''_{v_{\ell+1}}$ and $N''_{v_{t-\ell+1}}$, this forms the desired cycle. \square

5 Proof of Theorem 1.2

Following the overview in Section 2, we first prove a decomposition result for cubic graphs.

Lemma 5.1. *Let H be a connected cubic graph which is not isomorphic to K_4 . Then there exists a partition $V(H) = B_1 \cup \dots \cup B_t$, for some t which depends on H , such that the following hold:*

- the subgraph of H induced by each B_i is either a path (of any length, even 0) or a cycle of length at least four;
- for every $i \in \{2, \dots, t\}$, each vertex in B_i has at most one neighbour in $B_1 \cup \dots \cup B_{i-1}$.

Proof. For simplicity in the proof, we specify the B_i 's in reverse order: for every $i \in \{1, \dots, t-1\}$, each vertex in B_i will have at most one neighbour in $B_{i+1} \cup \dots \cup B_t$.

Let $\{B_i\}_{i \in [m]}$, for some $m \in \mathbb{N}$, be a maximal family of disjoint sets such that each B_i induces a cycle of length at least four. Note that no matter how we specify the remaining sets, the desired degree property holds for B_1, \dots, B_m due to H having maximum degree three.

Let $H' = H \setminus \bigcup_{i \in [m]} B_i$ and set $B_{m+1} \subseteq V(H')$ to be a largest subset which induces a path in H' . We aim to show that each vertex in B_{m+1} has at most one neighbour in $R = V(H') \setminus B_{m+1}$. Suppose, towards a contradiction, that $v \in B_{m+1}$ has two neighbours $x, y \in R$, noting that v must be an endpoint of the path $H'[B_{m+1}]$. Then both x and y necessarily have at least one more neighbour in B_{m+1} , as otherwise we could extend $H'[B_{m+1}]$ to a longer path. Let $v_x \in B_{m+1}$ be a neighbour of x in $B_{m+1} \setminus \{v\}$ which is closest to v and define v_y similarly, where the distance is measured along the path $H'[B_{m+1}]$. If vv_x is not an edge, then the path from v to v_x in $H'[B_{m+1}]$ together with xv and xv_x forms an induced cycle of length at least four, contradicting the assumption that H' has no such cycle. Therefore, vv_x is an edge and, similarly, vv_y is an edge. But then we must have that $v_x = v_y$ is the other endpoint of the path $H'[B_{m+1}]$ and so $H'[B_{m+1}]$ consists of a single edge. Note also that xy is not an edge, as otherwise we would have that $H = K_4$. But then $\{x, v, y\}$ forms an induced path longer than $H'[B_{m+1}]$, contradicting the choice of B_{m+1} . Therefore, each vertex in B_{m+1} has at most one neighbour in R , as required.

We now repeat this process, first by considering a longest induced path in R , until the entire vertex set is partitioned. \square

An example of a partition of $V(H)$ as given by the procedure described above is depicted in Figure 1.

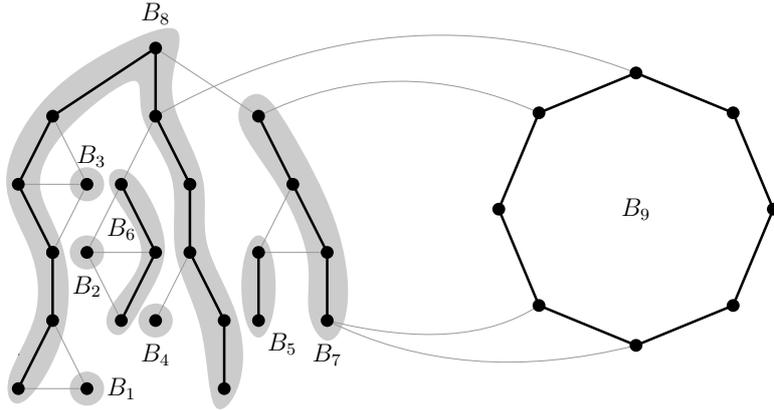


Figure 1: An example of a graph H with a partition produced by Lemma 5.1.

We are now in a position to prove our main result. Recall the statement, that there exist $c, K > 0$ such that if $p \geq Kn^{-2/5}$, then, with high probability, $G_{n,p} \rightarrow H$ for every cubic graph H with at most cn vertices.

Proof of Theorem 1.2. As mentioned in Section 2, by following a standard reduction process, it suffices to show that $\Gamma \sim G_{n,p}$ with high probability satisfies the following: if a subgraph

$G \subseteq \Gamma$ contains disjoint sets of vertices $V_1, \dots, V_{20} \subseteq V(G)$, each of order $|V_i| = \tilde{n} = \alpha n$ for some constant $\alpha > 0$, such that each pair (V_i, V_j) is (ε, p) -regular of density at least $p/3$ (with respect to G), then G contains H . Here we take $\varepsilon > 0$ to be as small as necessary for all our arguments to go through. This influences the choice of α and, consequently, decides how small c has to be (recall that H is a graph on cn vertices), but the exact dependencies are not important. We can also assume that the density of each pair is exactly $d = p/3$ (see [18, Lemma 4.3] or consider taking a random subset of the edges). We take this as our starting point.

Using Lemma 3.7, we can find cn vertex-disjoint copies of K_4 in G . Therefore, from now on we can assume that H is K_4 -free and connected. Let $B_1 \cup \dots \cup B_t = V(H)$ be a partition of $V(H)$ as given by Lemma 5.1 and let $\varphi: V(H) \rightarrow [10]$ be an arbitrary colouring of H such that if v and w are distinct vertices at distance at most two in H , then $\varphi(v) \neq \varphi(w)$.

Remove from G the ‘bad’ set given by Lemma 3.5 for every choice of (V_i, V_j) and, afterwards, take the subsets given by Lemma 3.3. After ‘cleaning’ the sets V_1, \dots, V_{20} and subsequently renaming what remains, we are left with sets $\{V_i^0, V_i^1\}_{i \in [10]}$ such that $0.9\tilde{n} \leq |V_i^*| \leq \tilde{n}$, for $* \in \{0, 1\}$, and the following properties hold for $(a, b) \in \{0, 1\}^2$:

- (P1) $\deg_G(v, V_j^b) \geq \tilde{n}d/2$ for every $v \in V_i^a$ and all $i \neq j$;
- (P2) for each $i \neq j$ and any two vertices $v, w \notin V_i^0 \cup V_i^1 \cup V_j^0 \cup V_j^1$, the pairs (N_v, V_j^b) and (N_v, N_w) are (ε', p) -regular with density at least $d/2$ for any subsets $N_v \subseteq N_G(v, V_i^a)$ and $N_w \subseteq N_G(w, V_j^b)$ of order $\tilde{n}d/4$.

We sequentially embed the blocks B_1, \dots, B_t , where for each block B_i we do the following:

- (Q1) For each $v \in B_i$, let $u_v \in V(G)$ be the image of its already embedded neighbour a_v from $B_1 \cup \dots \cup B_{i-1}$. Without loss of generality, we can assume that v always has such a neighbour.
- (Q2) Choose the smallest $z_v \in \{0, 1\}$ such that $N_G(u_v, V_{\varphi(v)}^{z_v})$ has at least $\tilde{n}d/4$ vertices which are not already taken by the embedding of any vertex in $B_1 \cup \dots \cup B_{i-1}$. If no $z_v \in \{0, 1\}$ has this property, terminate the procedure. Otherwise, let us denote the set of such ‘free’ neighbours of u_v by F_v .
- (Q3) As $|F_v| \geq \tilde{n}d/4$ for each $v \in B_i$, property (P2) ensures that the conditions of Lemma 4.1 (‘tree embedding’) and Lemma 4.2 (‘cycle embedding’) are satisfied, so there exists a copy of $H[B_i]$ in G which maps each vertex $v \in B_i$ into its corresponding set F_v . Fix such an embedding of $H[B_i]$ and proceed. (Minor technical detail: here we used the fact that if $w \in B_i$ is a neighbour of $v \in B_i$, then $\varphi(a_v) \neq \varphi(w)$, so that (P2) can be applied for $F_v \subseteq V_{\varphi(v)}^{z_v}$ and $F_w \subseteq V_{\varphi(w)}^{z_w}$.)

Assuming that the procedure did not terminate in step (Q2), we have successfully found a copy of H in G . It therefore remains to show that the process does not terminate early. As in the proof of Lemma 4.1, we will again use the first-free-bucket embedding strategy. However, the proof here is simpler as we only have two buckets to consider.

Recall that H has cn vertices, so the set X of all occupied vertices in $V_1^0 \cup \dots \cup V_{10}^0$ is of order at most cn . Let Z_1 denote the vertices $v \in V(H)$ for which $z_v = 1$ (as defined in (Q2)) and let $U = \{u_v : v \in Z_1\}$. Then $|Z_1|$ is an upper bound on the number of occupied vertices in $V_1^1 \cup \dots \cup V_{10}^1$. We claim that $|Z_1| = O(1/p)$, which, together with the fact that each u_v has at least $\tilde{n}d/2 \gg 1/p$ neighbours in $V_{\varphi(v)}^1$ (property (P1)), implies that the procedure does not terminate in step (Q2).

Suppose, towards a contradiction, that $|Z_1| = 3C/p$ for a sufficiently large constant C and so $|U| \geq C/p$. Then, by (P1) and (Q2), we have that each vertex in U has at least $\tilde{n}d/4$ neighbours in X . Hence, $e_G(U, X) \geq |U|\tilde{n}d/4$. On the other hand, for C sufficiently large, a simple application of Chernoff's inequality and the union bound implies that w.h.p. the number of edges in $\Gamma \sim G_{n,p}$ between any set U of order at least C/p and any set X of order cn is at most $2|U||X|p$. Therefore, since $G \subseteq \Gamma$ (and taking a superset of X of sufficient size if necessary), $e_G(U, X) \leq e_\Gamma(U, X) < 2|U|cnp$. Both the upper and lower bounds on $e_G(U, X)$ are of the same order of magnitude, but only the upper bound depends on c . Thus, for c sufficiently small with respect to α (which is hidden in \tilde{n}), we get a contradiction. \square

6 Concluding remarks

As in [25], the proof of Theorem 1.2 easily extends to more than two colours and, more importantly, it actually gives that in every q -colouring of $G_{n,p}$ one of the colours is *universal* for the family of cubic graphs with at most cn vertices, that is, it contains all such graphs simultaneously. It is known that any graph with this property has to have $\Omega(n^{4/3})$ edges (see, for instance, [1]). In other words, in order to go past that bound, one has to consider different host graphs for different H . However, we are nowhere near that bound yet.

Recall that the argument for why $n^{8/5}$ is the best one can get from random graphs is that, for $p = o(n^{-2/5})$, a typical $G_{n,p}$ is not Ramsey for K_4 . However, K_4 is the unique connected cubic graph with a fixed number of vertices which requires such a bound on p : for any other such graph, $p = \Theta(n^{-1/2})$ suffices (see [29, 30]). Therefore, it is not inconceivable that by adding some additional structure on top of the random graph to deal with the K_4 's, one could obtain a bound of type $n^{8/5-\varepsilon}$ or better.

Note that the only places where we used $p \geq Kn^{-2/5}$ in the proof of Theorem 1.2 were to invoke Lemma 4.2 to embed C_4 and to embed all components of H isomorphic to K_4 . In particular, if H does not contain K_4 and we can ensure in Lemma 5.1 that all cycles are of length at least five, then the same proof goes through with $p = \Theta(n^{-3/7})$. For longer cycles, this bound decreases even further, being governed by the bound in Lemma 4.2. Thus, we have the following improvements over Theorem 1.1.

Theorem 6.1. *There exists a constant K such that $\hat{r}(H) \leq Kn^{11/7}$ for every triangle-free cubic graph H . Moreover, if H is a cubic bipartite graph, then $\hat{r}(H) \leq Kn^{14/9}$.*

Sketch of the proof. We first show that if H is a connected triangle-free cubic graph with at least 7 vertices, then it has a decomposition as in Lemma 5.1, but where each cycle is now of length at least 5. Following the proof of Lemma 5.1, we start by removing from H a maximal family of disjoint induced cycles of length at least 5. Let H' denote the resulting graph and, for each vertex $v \in V(H')$ of degree two (in H'), add a temporary new vertex which is joined only to v . Therefore, each vertex in this temporarily expanded H' has degree either zero, one, or three. Consider a longest induced path in H' , say v_1, \dots, v_t . We will show that it has the desired property that every v_i has at most one neighbour in $V(H') \setminus \{v_1, \dots, v_t\}$, in which case we can remove it from H' , together with all the temporary vertices, and proceed. Note that at least one vertex in $\{v_1, \dots, v_t\}$ is an 'original' vertex, so we indeed make progress.

The only two vertices which can violate the desired property are v_1 and v_t . Without loss of generality, consider v_1 and suppose it has two neighbours $x, y \in V(H') \setminus \{v_2\}$. Then both of these

vertices need to have a neighbour in $\{v_2, \dots, v_t\}$, as otherwise we would get a longer induced path. Let us denote such a neighbour of x closest to v by v_x and of y by v_y . Then we necessarily have $v_x = v_y = v_3$, as otherwise we get an induced cycle of length at least 5. Therefore, the longest induced path is of length two (i.e., $t = 3$). As x has degree three, it must also have another neighbour z , which, as H is triangle-free, lies outside of $\{y, v_1, v_2, v_3\}$ (Figure 2).

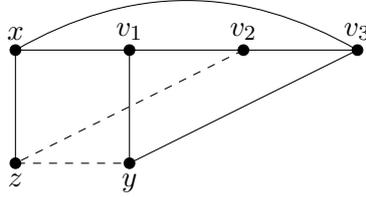


Figure 2: Finishing the decomposition.

Now we are almost done: if yz is not an edge, then we get a longer induced path y, v_1, x, z , while if zv_2 is not an edge, then we get the path z, x, v_1, v_2 . We must therefore have that yz and zv_2 are both edges, which implies that H' , without the temporary vertices, is a connected graph with 6 vertices where every vertex has degree three. But then $H' = H$, contradicting our assumption that H has at least 7 vertices.

Therefore, there exists a decomposition with each cycle being of length at least 5, as required. Moreover, if H is bipartite, each such cycle has to have length at least 6. To embed H , we then proceed by first embedding all connected components with at most 6 vertices using Lemma 3.7, which is possible already at $p = \Theta(n^{-1/2})$, as no such component is isomorphic to K_4 . To embed cycles through subsets of order $\Theta(np)$ using Lemma 4.2, we require $p = \Theta((np)^{-3/4})$ if they are of length at least 5 and $p = \Theta((np)^{-4/5})$ if they are of length at least 6, estimates which return the required bounds on p . The rest of the proof is then identical to the proof of Theorem 1.2. \square

One can also obtain better bounds than those in [25] if H has a special structure. For example, Clemens, Miralaei, Reding, Schacht, and Taraz [9] showed that if H is a $\sqrt{n} \times \sqrt{n}$ grid, then $\hat{r}(H) = O(n^{3/2+o(1)})$, which is essentially the best one can get from random graphs in this case. Our methods immediately imply this result, as the grid graph has a decomposition into induced paths (given by ‘vertical’ lines) which admit a ‘1-degenerate’ ordering (i.e., the second property of Lemma 5.1). One notable feature of this argument is that even though H has many copies of C_4 , we do not embed them directly using Lemma 4.2.

Finally, it is worth noting that the problem of improving the bound from [25] for general bounded-degree graphs remains open. For the class of triangle-free graphs H with maximum degree Δ , a rather convoluted argument from the second author’s PhD thesis [27] shows that $\hat{r}(H) = O(n^{2-1/\Delta-\varepsilon_\Delta})$ for some $\varepsilon_\Delta > 0$. Regarding the methods presented here, analogues of Lemmas 4.1 and 4.2 with each N_v now contained in the neighbourhood of at most $\Delta - 2$ vertices go through provided $p = \Theta(n^{-1/(\Delta-0.5)})$. Moreover, an easy modification of the proof of Lemma 5.1 shows that a connected graph H with maximum degree Δ , not isomorphic to $K_{\Delta+1}$, admits a decomposition into blocks B_1, \dots, B_t such that each B_i is either a path or a cycle of length at least four and each vertex in B_i has at most $\Delta - 2$ neighbours in $B_1 \cup \dots \cup B_{i-1}$. But, since these are the main ingredients of our proof in the $\Delta = 3$ case, why does the proof not go through for larger Δ ?

The answer is somewhat technical, but the heart of the matter is that, for $\Delta = 3$, no two

vertices in B_i can have a common neighbour in $B_{i+1} \cup \dots \cup B_t$, which means that the inheritance properties promised by Lemma 3.5 are sufficient to continue the embedding process. If $\Delta > 3$, it can happen that two (or more) vertices in B_i have such a common neighbour, which means that much more care is needed to ensure that the candidate set for each vertex has the desired size and that pairs inherit regularity in the correct way. It is entirely possible that this could be done, but we have chosen not to pursue the matter here. Another, more concrete, reason we did not pursue the matter further is that, if embedding $K_{\Delta+1}$ is again the main obstacle, what one would really like to show is that there are $c, K > 0$ such that if $p \geq Kn^{-2/(\Delta+2)}$, then, with high probability, $G_{n,p} \rightarrow H$ for any graph H with at most cn vertices and maximum degree Δ . This is Theorem 1.2 for $\Delta = 3$, but it is almost certain that the techniques described here are not sufficient to meet this bound for larger values of Δ .

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