Set-coloring Ramsey numbers and error-correcting codes near the zero-rate threshold

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Abstract

For positive integers $n, r, s$ with $r > s$, the set-coloring Ramsey number $R(n; r, s)$ is the minimum $N$ such that if every edge of the complete graph $K_N$ receives a set of $s$ colors from a palette of $r$ colors, then there is a subset of $n$ vertices where all of the edges between them receive a common color. If $n$ is fixed and $\frac{r}{s}$ is less than and bounded away from $1 - \frac{1}{n-1}$, then $R(n; r, s)$ is known to grow exponentially in $r$, while if $\frac{r}{s}$ is greater than and bounded away from $1 - \frac{1}{n-1}$, then $R(n; r, s)$ is bounded. Here we prove bounds for $R(n; r, s)$ in the intermediate range where $\frac{r}{s}$ is close to $1 - \frac{1}{n-1}$ by establishing a connection to the maximum size of error-correcting codes near the zero-rate threshold.

1 Introduction

Two of the central problems in discrete mathematics are that of estimating the maximum size of error-correcting codes with given parameters and that of estimating Ramsey numbers. Here, building on recent work by an overlapping set of authors [6], we find a close connection between these two problems. More precisely, we show that the problem of estimating set-coloring Ramsey numbers, a natural generalization of the usual Ramsey numbers, and that of estimating the size of error-correcting codes near the zero-rate threshold are essentially the same problem.

To say more, let $A_q(m, d)$ be the maximum size of a code $C \subseteq [q]^m$ of length $m$ in which any two codewords have Hamming distance at least $d$, i.e., they differ in at least $d$ coordinates. Such a code is called a $q$-ary code of length $m$ and distance $d$. The rate of the code is then defined as $(\log_q |C|)/m$. A result going back to work of Plotkin [14], who treated the binary case, says that there are codes of positive rate, that is, with exponentially many elements, if $d < (1 - \frac{1}{q} - \epsilon)m$ for any fixed $\epsilon > 0$ and no such codes if $d \geq (1 - 1/q)m$. That is, there is a threshold at distance $(1 - 1/q)m$ where the rate becomes zero.

On the other hand, for any positive integers $n, r, s$ with $r > s$, we define the set-coloring Ramsey number $R(n; r, s)$ to be the minimum $N$ such that if every edge of $K_N$ receives a set of $s$ colors from a palette of $r$ colors, then there is guaranteed to be a monochromatic clique on $n$ vertices, that is, a copy of $K_n$ whose edges all share a common color. As a shorthand, it will be convenient for us to refer to such a set-coloring as an $(r, s)$-coloring of $K_N$.

A priori, it is not clear that these quantities should have anything to do with one another. However, in [6], it was shown how to use the Gilbert–Varshamov bound, a standard lower bound for the size of codes, to show that for any $\epsilon > 0$ there exists $c > 0$ such that $R(n; r, s) \geq 2^{ctn}$ for any $r$ and $s$ with $cr < s < (1 - \epsilon)r$.

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and $n$ sufficiently large in terms of $r$, a result which is tight up to the constant $c$ (see also [3] for an alternative approach with an improved bound on the constant $c$ in terms of $c$). Moreover, the following result was noted.

**Theorem 1.1** ([6]). For all positive integers $q, r, s$ with $r > s$, $R(q + 1; r, s) \geq A_q(r, s) + 1$.

In particular, if $q$ is fixed, we see that, provided $s/r \leq 1 - 1/q - \varepsilon$ for some fixed $\varepsilon > 0$, $R(q + 1; r, s)$ grows at least exponentially in $r$. Moreover, it was also shown in [6] that if $s/r \geq 1 - 1/q + \varepsilon$ for some fixed $\varepsilon > 0$, then $R(q + 1; r, s)$ is at most a constant depending only on $q$ and $\varepsilon$. That is, for $q$ fixed, there is a threshold for $s/r$ at $1 - 1/q$ where the set-coloring Ramsey number $R(q + 1; r, s)$ goes from growing exponentially in $r$ to being bounded.

In [6], it was suggested that perhaps Theorem 1.1 is almost tight when $s/r$ is close to $1 - 1/q$. That this is indeed the case is our first new result.

**Theorem 1.2.** For any positive integer $q$ and any $\varepsilon > 0$, there is $c > 0$ such that if $r, s$ are positive integers with $s \leq (1 - 1/q)r$ and $j = (1 - 1/q)r - s + 1$, then

$$R(q + 1; r, s) \leq \max((1 + \varepsilon)A_q(r, s - cj), cs).$$

Furthermore, if $q = p^i$ and $r = p^j$ are powers of a prime $p$ with $r \geq q$, then $R(q + 1; r, s) \leq (1 + \varepsilon)A_q(r, s - cj)$.

We suspect that there may even be equality in Theorem 1.1 when $s$ is sufficiently close to $(1 - 1/q)r$, though our methods fall somewhat short of proving this.

Having established this connection, we can use it to prove bounds on $R(q + 1; r, s)$ when $s$ is close to $(1 - 1/q)r$ by studying the bounds for $A_q(r, s)$ in the same range. It turns out that the study of such bounds is a well-established topic in coding theory, particularly in the binary case. We have already mentioned the work of Plotkin above. More precisely, he showed that $A_2(r, r/2) \leq 2r$ and that $A_2(r, s) \leq 2s/2s - r)$ for $s > r/2$, both of which are sometimes tight by considering Hadamard codes (see, for instance, [11, Chapter 2]). More generally, Blake and Mullin [4] showed that $A_q(r, s) \leq \frac{qn}{2s - r(q - 1)}$ when $s > (1 - 1/q)r$ and it can also be shown that $A_q(r, (1 - 1/q)r) \leq 2qr$.

There has also been a great deal of work, particularly in the binary case, for $s$ of the form $(1 - 1/q)(r - j)$. For instance, using the linear programming bound, McEliece (see [11, Chapter 17]) showed that $A_2(r, (r - j)/2) \leq (1 + o(1))r(j + 2)$ for $j = o(r^{1/2})$. Sidel’nikov [15] constructed a code showing that McEliece’s bound is asymptotically tight when $j = \Theta(r^{1/3})$. In particular, he showed that $A_2(r, (r - j)/2) \geq r(j + 2) + 1$ for $r = (2^m - 1)/(2^m + 1)$ and $j = 2^m - 1$. Later, Tietäväinen [16] (see also [7]) showed that $A_2(r, (r - j)/2) = O(r \log(j + 1))$ when $j = o(r^{1/3})$ and conjectured that $A_2(r, (r - j)/2) = O(r)$ in this range.

In a recent paper, Pang, Mahdavifar and Pradhan [13] showed that $A_2(r, (r - 2\sqrt{r})/2) \geq r^c$ and that $A_2(r, (r - 2\sqrt{r})/2) = O(r^{7/2})$ and $A_2(r, (r - 4\sqrt{r})/2) = O(r^{15/2})$. We improve these bounds and, more generally, establish good bounds for $A_q(r, (1 - 1/q)(r - j))$ when $j$ is on the order of $\sqrt{r}$. In particular, because of Theorems 1.1 and 1.2, we get analogues, both of the bounds here and those mentioned above, for the corresponding set-coloring Ramsey numbers $R(q + 1; r, s)$.

**Theorem 1.3.** If $k$ is a positive integer and $j \leq \sqrt{(k - 1)r/(q - 1)}$, then

$$A_q(r, (1 - 1/q)(r - j)) = O_{q,k}(r^k).$$

On the other hand, for any prime power $q$, there are infinitely many $r$ such that, for $j \geq (k - 1)\sqrt{r}/q$,

$$A_q(r, (1 - 1/q)(r - j)) \geq (rq)^{k/2}.$$

As a warm-up to our main result, in the next section we will prove a tight result for $R(3; 2s, s)$ (see also [6, Proposition 4.3] for another tight result). This quantity was recently studied, independently of the work in [6], in the master’s thesis of Le [8]. She showed that if there is a Hadamard matrix of order $2s$, then $R(3; 2s, s) \geq 4s + 1$. In the other direction, she gave an upper bound on $R(3; 2s, s)$ which grows exponentially in $s$ and asked whether the gap can be closed. We answer this question by proving the following.
Theorem 1.4. For all \( s > 1 \), \( R(3; 2s, s) \leq 4s + 1 \).

Note that the assumption \( s > 1 \) is needed in Theorem 1.4 as \( R(3; 2, 1) = R(3; 2) = 6 \). Moreover, since there is a Hadamard matrix of order \( 2s \) whenever \( s = q + 1 \) with \( q \equiv 3 \pmod{4} \) a prime power, we see that \( R(3; 2s, s) = 4s + 1 \) for infinitely many \( s \) and also that \( R(3; 2s, s) = (4 + o(1))s \).

2 A tight result infinitely often

In this short section, we prove Theorem 1.4, that \( R(3; 2s, s) \leq 4s + 1 \) for all \( s > 1 \), which, by Le’s construction [8], is sharp for infinitely many \( s \). We begin with the following result, which is essentially a special case of [6, Proposition 4.1].

Lemma 2.1. If \( r < 2s \), then \( R(3; r, s) \leq \frac{2s}{2r-1} + 1 \). In particular, \( R(3; 2s - 1, s) \leq 2s + 1 \).

Proof. Consider an \((r, s)\)-coloring of the edges of the complete graph on \( N \) vertices without any monochromatic triangle. As each of the \( r \) color classes is triangle-free, each color class has at most \( N^2/4 \) edges, so the total number of colors used on all edges is at most \( rN^2/4 \). On the other hand, as \( s \) colors are used on each edge, the total number of colors used on all edges is \( s(N^2) \). Hence, \( s(N^2) \leq rN^2/4 \). Simplifying, we get that \( 1 - 1/N \leq r/2s \) and so \( N \leq \frac{2s}{2r-1} \). Thus, \( R(3; r, s) \leq \frac{2s}{2r-1} + 1 \). \( \square \)

In the proof above, we used Mantel’s theorem, the statement that any triangle-free graph on \( N \) vertices has at most \( \lceil N^2/4 \rceil \) edges. It is known that equality holds in Mantel’s theorem if and only if the graph is a balanced complete bipartite graph. If, instead, we restrict attention to non-bipartite graphs, Mantel’s theorem can be improved very slightly. Indeed, a result of Brouwer [6] implies that any non-bipartite triangle-free graph on \( N \) vertices has at most \( N^2/4 - \lceil N/2 \rceil + 1 \) edges. In particular, when \( N \) is odd, any such graph has at most \( N^2/4 - N/2 + 5/4 \) edges. With this, we can now prove Theorem 1.4.

Proof of Theorem 1.4. Consider a \((2s, s)\)-coloring of the edges of the complete graph on \( N = 4s + 1 \) vertices and suppose, for the sake of contradiction, that it has no monochromatic triangle. If one of the color classes has an independent set \( S \) of size \( 2s + 1 \), then the coloring restricted to \( S \) is a \((2s - 1, s)\)-coloring and so, by Lemma 2.1, the coloring restricted to \( S \) must contain a monochromatic triangle, a contradiction. Since any bipartite graph on \( 4s + 1 \) vertices contains an independent set with \( 2s + 1 \) vertices, to complete the proof it suffices to show that at least one of the color classes is bipartite. But, if each color class is non-bipartite, Brouwer’s result implies that each color class has at most \( N^2/4 - N/2 + 5/4 \) edges, so the total number of colors on edges is at most \( 2s(N^2/4 - N/2 + 5/4) \). As the total number of colors on edges equals \( s(N^2) \), we would then obtain \( s(N^2) \leq 2s(N^2/4 - N/2 + 5/4) \). This simplifies to \( N \leq 5 \) or, equivalently, \( s \leq 1 \), contradicting our assumption that \( s > 1 \) and completing the proof. \( \square \)

As a quick corollary of Theorem 1.4 applied in combination with Theorem 1.1, we see that \( A_2(2s, s) \leq R(3; 2s, s) - 1 \leq 4s \), which is exactly the Plotkin bound in the binary case. As the Plotkin bound is known to be tight whenever there is a Hadamard matrix of order \( 2s \), this also returns Le’s lower bound [8] for \( R(3; 2s, s) \).

3 Codes from set colorings

In this section, we prove Theorem 1.2, showing that the connection between codes and set-coloring Ramsey numbers discovered in [6] goes both ways near the zero-rate threshold. We first state and prove a certain stability version of Turán’s theorem.
3.1 Stability for Turán’s theorem

Turán’s theorem is the natural generalization of Mantel’s theorem to larger cliques. If we write $T_{N,q}$ for the Turán graph, the balanced complete $q$-partite graph on $N$ vertices, Turán’s theorem \cite{Tur} then states that the Turán graph $T_{N,q}$ is the unique $K_{q+1}$-free graph on $N$ vertices with the maximum number of edges. This maximum is therefore at most $(1 - \frac{1}{q})N^2/2$, with equality if and only if $N$ is a multiple of $q$.

We wish to prove a stability version of Turán’s theorem, saying that any graph on $N$ vertices with nearly as many edges as $T_{N,q}$ can be made $q$-partite by deleting a small number of vertices. In the proof, we will make use of the Andrásfai–Erdős–Sós theorem \cite{AES}, which says that every $K_{q+1}$-free graph on $N$ vertices with minimum degree larger than $\frac{3q-4}{3q-1}N = (1 - \frac{q}{q-1})N$ is $q$-partite.

**Lemma 3.1.** Every $K_{q+1}$-free graph $G$ on $N \geq 12q^2$ vertices has at most $(1 - \frac{1}{q})N^2/2 - \frac{Nf_q(G)}{8q^2}$ edges, where $f_q(G)$ is the minimum $f$ such that $f$ vertices can be deleted from $G$ so that the remaining induced subgraph is $q$-colorable.

**Proof.** Let $G(0) = G$. After defining $G(i)$, if $G(i)$ has a vertex $v_i$ of degree at most $\frac{3q-4}{3q-1}|G(i)|$, then let $G(i+1)$ be obtained from $G(i)$ by deleting $v_i$. Let $f = f_q(G)$. We must eventually define $G(f)$, as otherwise the process stops at some $G(i)$ with $i < f$ of minimum degree larger than $\frac{3q-4}{3q-1}|G(i)|$. But, by the Andrásfai–Erdős–Sós Theorem, this $G(i)$ is a $q$-partite graph obtained from $G$ by deleting $i < f = f_q(G)$ vertices, contradicting the definition of $f_q(G)$.

Since $G(f)$ is $K_{q+1}$-free, Turán’s theorem implies that $G(f)$ has at most $(1 - \frac{1}{q})|G(f)|^2/2$ edges. Hence, since the degree of $v_i$ in $G(i)$ is at most $\frac{3q-4}{3q-1}(N - i)$ and $\frac{3q-4}{3q-1} = (1 - \frac{1}{q}) - \frac{1}{q(3q-1)}$, the number of edges in $G$ is at most

$$e(G(f)) + \sum_{i=0}^{f-1} \frac{3q-4}{3q-1}(N - i) \leq \left(1 - \frac{1}{q}\right)N^2/2 + \frac{f}{2} - \frac{Nf}{2q(3q-1)} \leq \left(1 - \frac{1}{q}\right)N^2/2 - \frac{Nf}{8q^2},$$

as required. \hfill $\Box$

3.2 From set colorings to error-correcting codes

We will deduce Theorem 1.2 from the following result.

**Theorem 3.2.** Let $\lambda > 1$ and $N = R(q+1; r, s) - 1 \geq 12q^2$. If $b = \lfloor 4\lambda q^2 \left((1 - \frac{1}{q})r - s + \frac{s}{q}\right) \rfloor \geq 0$, then

$$A_q(r, s - 2b) \geq \left(1 - \frac{1}{\lambda}\right)N.$$

**Proof.** Consider an $(r,s)$-coloring of $K_N$ with $N = R(q+1; r, s) - 1$ without a monochromatic $K_{q+1}$. Such a coloring exists from the definition of the set-coloring Ramsey number. Consider the $r$ graphs $G_1, \ldots, G_r$ on $V(K_N)$ where $E(G_i)$ is the set of edges of $K_N$ whose set of colors contains color $i$, so that $G_i$ is $K_{q+1}$-free and each edge of $K_N$ is an edge of at least $s$ of these $r$ graphs. For each $G_i$, there is a set $U_i$ of $f_q(G_i)$ vertices such that the induced subgraph of $G_i$ upon deleting $U_i$ is $q$-partite. If we write $V_{i1}, \ldots, V_{iq}$ for the $q$ resulting independent sets in $G_i$, then $V(K_N) = U_i \cup V_{i1} \cup \cdots \cup V_{iq}$. For each vertex $v \in V(K_N)$, let $x_i(v) = j$ if $v \in V_{ij}$ and, otherwise, let $x_i(v)$ be an arbitrary element of $[q]$. Then, for each $v \in V(K_N)$, we let $x(v) = (x_1(v), \ldots, x_r(v)) \in [q]^r$.

By counting over each edge $e$, the number of pairs $(e, i)$ with $e \in E(G_i)$ is $\binom{N}{2}$. On the other hand, counting over the color classes, the number of such pairs is also $\sum_{i=1}^{r} e(G_i)$. Hence,

$$\binom{N}{2} = \sum_{i=1}^{r} e(G_i) \leq \sum_{i=1}^{r} \left((1 - \frac{1}{q})N^2/2 - \frac{Nf_q(G_i)}{8q^2}\right) = \left(1 - \frac{1}{q}\right)rN^2/2 - \frac{N}{8q^2} \sum_{i=1}^{r} f_q(G_i),$$

4.
where the inequality is by Lemma 3.1. Multiplying both sides by $\frac{qs}{r}$ and rearranging, we get

$$N^{-1} \sum_{i=1}^{r} f_q(G_i) \leq 4q^2 \left( \left(1 - \frac{1}{q}\right) r - s + \frac{s}{N}\right) := M.$$  

Since $\sum_{i=1}^{r} |U_i| = \sum_{i=1}^{r} f_q(G_i)$, Markov’s inequality now implies that the number of vertices $v$ for which $v \in U_i$ for at least $\lambda M$ values of $i$ is at least $N/\lambda$. Hence, the set $V'$ of vertices $v$ for which $v \in U_i$ for at most $\lambda M$ values of $i$ satisfies $|V'| \geq (1 - \lambda^{-1})N$. Consider the collection of codewords $C = \{x(v) : v \in V'\}$. For each pair of distinct vertices $u, v \in V'$, we have that $(u,v)$ is an edge of at least $s$ graphs $G_i$. For each $G_i$ for which $(u,v) \in E(G_i)$ and neither $u$ nor $v$ is in $U_i$, we have $x_i(u) \neq x_i(v).$ Since $u$ and $v$ are each in at most $b = [\lambda M]$ of the sets $U_i$, there are at least $s - 2b$ coordinates for which $u$ and $v$ must differ. Hence, since $C$ is a collection of codewords in $[q]^r$ in which each pair has distance at least $s - 2b$, $|C| \leq A_q(r,s - 2b)$. Since $|C| = |V'| \geq (1 - \lambda^{-1})N$, this completes the proof.

Proof of Theorem 1.3. If $N = R(q + 1;r,s) - 1 < \epsilon s$, then we are already done. Otherwise, assume that $N \geq \epsilon s$. We apply Theorem 3.2 with $\lambda = 2/\epsilon$ to obtain

$$A_q(r,s - 2b) \geq (1 - \epsilon/2)N,$$

where $b = \left\lfloor 4\lambda q^2 \left( \left(1 - \frac{1}{q}\right) r - s + \frac{s}{N}\right) \right\rfloor$. Note that $b \leq cj/2$ where $j = \left(1 - \frac{1}{q}\right) r - s + 1$ for an appropriate constant $c > 0$ depending only on $q$ and $\epsilon$. This implies that

$$R(q + 1;r,s) \leq (1 + \epsilon)A_q(r,s - cj),$$

as desired.

In the case where $q = p^i$ and $r = p^j$ are powers of the same prime $p$ with $r \geq q$, we show that $N > s$, which immediately gives the desired conclusion. Based on generalized Hadamard matrices, it is shown in [10] that for such $q$ and $r$ there are codes over $F_q^r$ with size $qr$ and distance $(1 - 1/q)r$. By Theorem 1.1, this implies that $N \geq A_q(r,s) \geq qr > s$, as required.

## 4 Codes with large distance

In this section, we prove our upper and lower bounds for $A_q(r,s)$, and hence $R(q + 1;r,s)$, when $s$ is close to $(1 - 1/q)r$. For the upper bound, we will make use of Delarte’s linear programming bound, following a technique of McEliece, Rodemich, Rumsey and Welch [12] (see also Theorem 35 in [11] Chapter 17) and its extension to $q$-ary codes in [11]. If we define the Krawtchouk polynomials by

$$K_{i}^{q,r}(x) = \sum_{j=0}^{i} (-1)^j(q-1)^{i-j} \binom{x}{j} \binom{r-x}{i-j}$$

for any $0 \leq i \leq r$, Delarte’s bound then says that if $P(x) = \sum_{i} \beta_i K_{i}^{q,r}(x)$ is a linear combination of the $K_{i}^{q,r}$ with $\beta_0 > 0$ and $\beta_i \geq 0$ for all $i \geq 1$ such that $P(d) \leq 0$ for all $D \leq d \leq r$, then

$$A_q(r,D) \leq P(0)/\beta_0.$$  

We are now ready to prove the upper bound in Theorem 1.3.

**Theorem 4.1.** If $k$ is a positive integer and $j \leq \sqrt{(k-1)r/(q-1)}$, then

$$A_q(r,(1 - 1/q)(r-j)) = O_q(r^k).$$
Proof. Our argument will largely follow the proof from [12]. We refer to this paper and to [9] for the standard properties of Krawtchouk polynomials. Note that throughout the proof, for clarity of presentation, we will systematically omit the superscripts in the notation for Krawtchouk polynomials.

For $a < (1 - 1/q)(r - j)$, consider the polynomial

$$P(x) = \frac{(K_i(a)K_{i+1}(x) - K_{i+1}(a)K_i(x))^2}{a - x},$$

noting that $P(d) \leq 0$ for all $d \geq (1 - 1/q)(r - j)$. By the Christoffel–Darboux formula (see [9] Corollary 3.5)

$$\frac{K_i(a)K_{i+1}(x) - K_{i+1}(a)K_i(x)}{x - a} = -\frac{q}{i + 1} \binom{r}{i} (q - 1)^i \sum_{j=0}^{i} \frac{K_j(x)K_j(a)}{\binom{q}{j}(q - 1)^j},$$

we have

$$P(x) = \frac{q}{i + 1} \binom{r}{i} (q - 1)^i(K_i(a)K_{i+1}(x) - K_{i+1}(a)K_i(x)) \sum_{j=0}^{i} \frac{K_j(x)K_j(a)}{\binom{q}{j}(q - 1)^j} + \frac{q}{i + 1} \binom{r}{i} (q - 1)^i \sum_{j=0}^{i} \frac{K_j(x)K_{i+1}(x)}{\binom{q}{j}(q - 1)^j} \cdot \frac{K_j(a)K_i(a)}{\binom{q}{j}(q - 1)^j}.$$

We will make use of the following properties of the Krawtchouk polynomials: $K_i(x)K_j(x)$ is a nonnegative combination of the $K_i(x)$ if the $K_i$ are orthogonal under the bilinear form $(f, g) = \sum_{j=0}^{r} \binom{q}{j}(q - 1)^j f(j)g(j)$ with $(K_i, K_i) = q^r(q - 1)^r\binom{q}{i}$; and if $\rho_i$ is the smallest positive root of $K_i$, then $\rho_i > \rho_{i+1}$ and there are no other roots of $K_{i+1}$ in $(\rho_{i+1}, \rho_i)$. We also have

$$\beta_0 = q^{-r} \sum_{x=0}^{r} \binom{r}{x} (q - 1)^x P(x), \quad K_i(0) = (q - 1)^i \binom{r}{i}.$$

If $a$ is such that $\rho_{i+1} < a < \rho_i$, then $K_j(a)K_{i+1}(a) \leq 0$ and $K_j(a)K_i(a) \geq 0$ for all $j \leq i$. Therefore, $P(x) = \sum_{i} \beta_i K_i^{q,r}(x)$ is a linear combination of the $K_i^{q,r}$ with $\beta_i \geq 0$ for all $i \geq 1$. Moreover, by orthogonality, we have that

$$\beta_0 = q^{-r} \sum_{x=0}^{r} \binom{r}{x} (q - 1)^x P(x) = -q^{-r} \frac{q}{i + 1} \binom{r}{i} (q - 1)^i \cdot \frac{K_i(a)K_{i+1}(a)}{\binom{q}{i}(q - 1)^i} \cdot q^r(q - 1)^i \binom{r}{i}$$

and

$$P(0) = \frac{(K_i(a)K_{i+1}(0) - K_{i+1}(a)K_i(0))^2}{a}.$$

Hence, if $\rho_{i+1} < a < \rho_i$, Delsarte’s bound implies that

$$A_q(r, (1 - 1/q)(r - j)) \leq P(0)/\beta_0 = -\frac{(K_i(a)K_{i+1}(0) - K_{i+1}(a)K_i(0))^2}{a \frac{q}{i+1} (q - 1)^i \binom{q}{i} K_i(a)K_{i+1}(a)}.$$

Noting that $K_{i+1}(a)/K_i(a)$ ranges from 0 to $-\infty$ as $a$ goes from $\rho_{i+1}$ to $\rho_i$, we can find $a_0$ in that range such that $K_{i+1}(a_0)/K_i(a_0) = -K_{i+1}(0)/K_i(0)$ and

$$\frac{(K_i(a_0)K_{i+1}(0) - K_{i+1}(a_0)K_i(0))^2}{a_0 \frac{q}{i+1} (q - 1)^i \binom{q}{i} K_i(a_0)K_{i+1}(a_0)} = \frac{4K_i(0)K_{i+1}(0)}{a_0 \frac{q}{i+1} (q - 1)^i \binom{q}{i}} = \frac{4(q - 1)^{i+1} \binom{r}{i+1}}{a_0 \frac{q}{i+1} \binom{q}{i+1} \rho_{i+1}} \leq \frac{4(i + 1)(q - 1)^{i+1} \binom{r}{i+1}}{q \rho_{i+1}}.$$

This should be taken as meaning that the values of the two polynomials are equal for all $x = 0, 1, \ldots, r$, but this is sufficient for our application of Delsarte’s bound.
Now we note that $K_1(x) = (q - 1)(r - x) - x$, so that $\rho_1 = (1 - 1/q)r$ and $K_2(x) = (q - 1)^2(r^2 - x) - (q - 1)x(r - x) + \binom{q}{2}$, so that $\rho_2 \leq (1 - 1/q)(r - \sqrt{r/(q - 1)})$. Writing $h_k$ for the largest root of the $k$-th Hermite polynomial, we also have the general bound (see [9] Corollary 6.1)

$$\rho_k \leq (1 - 1/q)r - \frac{q - 2}{2q}h_k^2 - \frac{\sqrt{2(q - 1)(r - k + 2)}h_k}{q} \leq (1 - 1/q)(r - \sqrt{(k - 1)r/(q - 1)}),$$

where we used that $h_k > \sqrt{(k - 1)/2}$ for $k > 2$ and that $r$ may be taken sufficiently large in $q$ and $k$. For $j \leq \sqrt{(k - 1)r/(q - 1)}$, we may therefore pick any $a$ with $\rho_{k+1} < a < \rho_k$ and it will automatically satisfy $a < (1 - 1/q)(r - j)$. Therefore, taking $a = a_0$ as in the calculation above, we find that

$$A_q(r, (1 - 1/q)(r - j)) \leq \frac{4(k + 1)(q - 1)^{k+1}(r/q)}{q\rho_{k+1}} = O_{q,k}(r^k),$$

where we used that $\rho_{k+1} = \Omega_{q,k}(r)$ (see [9] Equation 125)).

We now prove the lower bound in Theorem 1.3 which follows from concatenating appropriate codes.

**Theorem 4.2.** For any positive integer $k$ and any prime power $q$, there are infinitely many $r$ such that, for $j \geq (k - 1)\sqrt{r/q}$,

$$A_q(r, (1 - 1/q)(r - j)) \geq (rq)^{k/2}.$$

**Proof.** Based on generalized Hadamard matrices, it is shown in [11] that if $q = p^i$ and $u = p^i$ for some $i \geq 1$ there exist codes over $\mathbb{F}_q^u$ with size $qu$ and distance $(1 - 1/q)u$. We also recall that the Reed–Solomon code is a code over $\mathbb{F}_q^n$ with size $s^k$ and distance at least $n - k + 1$, where $s$ is a prime power at least $n$.

We consider a concatenation code with the generalized Hadamard code as the inner code and the Reed–Solomon code as the outer code. More explicitly, let $C_i$ be a code over $\mathbb{F}_q^n$ with size $qu$ and distance $(1 - 1/q)u$ and let $C \subseteq \mathbb{F}_q^n$ be the Reed–Solomon code where we choose $s = n = qu$ to be a prime power. The concatenation code is formed by considering $C_s$ as a subset of $[qu]^n$ through a bijection $\phi : [s] \rightarrow qu$ and then using the inner code $C_i$ to map each element of $[qu]^n$ term by term to a subset of $([qu]^u)^n = \mathbb{F}_q^{un}$. Since it is easy to see that the distance of a concatenated code is the product of the distances of the inner and outer codes, this gives a code in $\mathbb{F}_q^{un}$ with distance at least $(n - k + 1)(1 - 1/q)u$ and size at least $s^k$. Letting $r = qu^2$, we see that the distance of the code is at least $(1 - 1/q)(r - (k - 1)\sqrt{r/q})$ and the size of the code is at least $(rq)^{k/2}$.

5 Concluding remarks

Using Theorems 1.3 and 1.2 to turn back to $R(3; r, (r - j)/2)$, the picture that emerges is a rather complex one, with the function exhibiting a range of different behaviours depending on the value of $j$. When $r$ is even and $j = 0$, Theorem 1.4 gives the exact value $R(3; r, r/2) = 2r + 1$. Increasing $j$, by the result of Tietäväinen [10], discussed in the introduction, $R(3; r, (r - j)/2)$ remains nearly linear in $r$ until $j$ reaches roughly $r^{1/3}$, where the result of Sidel’nikov [15] shows that the value jumps to $r^{1+\epsilon}$ for some $\epsilon > 0$. The function is at most quadratic in $r$ until $j$ passes $\sqrt{r}$, where the results of Pang, Mahdavifar and Pradhan [18], which our Theorem 1.3 refines, show that the function starts to grow as an arbitrary power of $r$. By the time $j$ is linear in $r$, the bound becomes exponential in $r$ and it is possible (though not generally expected) that the bound jumps to superexponential as $j$ approaches $r$.

This summary raises many questions, not least of which is whether there are further jumps in behaviour as $j$ passes from $\sqrt{r}$ to $r$. It would also be interesting to decide whether any aspects of the picture drawn above change as the clique size goes from 3 to 4 and beyond. For instance, does the shift from the near-linear to the polynomially superlinear regime for $R(4; r, 2(r - j)/3)$ still happen when $j$ is roughly $r^{1/3}$?

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References


