# Everywhere unbalanced configurations 

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#### Abstract

An old problem in discrete geometry, originating with Kupitz, asks whether there is a fixed natural number $k$ such that every finite set of points in the plane has a line through at least two of its points where the number of points on either side of this line differ by at most $k$. We give a negative answer to a natural variant of this problem, showing that for every natural number $k$ there exists a finite set of points in the plane together with a pseudoline arrangement such that each pseudoline contains at least two points and there is a pseudoline through any pair of points where the number of points on either side of each pseudoline differ by at least $k$.


## 1 Introduction

Does there exist a positive integer $k$ such that for every finite set of points in the plane there is a line containing at least two of these points where the number of points on either side of this line differ by at most $k$ ? This basic problem was first raised by Kupitz [9] in the late 1970s and has since been reiterated by many authors, including Alon [1], Erdős [3], Green [7], Kalai [8], Pach [10] and Pinchasi [11]. It was also singled out for inclusion in the book of Brass, Moser and Pach [2] that surveys many of the most interesting open problems in discrete geometry.

If a point set has an odd number of points with no three of them on a line, then the number of points on either side of each line through two of the points must differ by at least one. Kupitz's original conjecture [9] was that this should be the extremal case, that is, that there should always be a line containing at least two points of any finite point set where the number of points on either side differ by at most one. This was disproved by Alon [1], who showed that there are finite point sets where the number of points on either side of each line determined by the set differ by at least two. No better example is known, though in Figure 1 we give a different example to Alon's with the same property, but, unlike his examples, containing an odd number of points. This example can easily be extended to any larger odd number of points by adding an equal number of points on both sides of some line.

In the opposite direction, improving earlier unpublished results of Alon and Perles, it was shown by Pinchasi [11] that there exists an absolute constant $C$ such that every $n$-point set determines a line where the number of points on either side of the line differ by at most $C \log \log n$. In fact, his result applies in the much broader context of generalised configurations. Recall that a pseudoline arrangement is a collection of two-way unbounded simple curves any two of which meet in at most

[^0]

Figure 1: A set of 23 points where the number of points on either side of each line determined by the set differ by at least two.
one point. A generalised configuration is then a finite set of points in the plane together with a pseudoline arrangement such that each pseudoline contains at least two points and there is a pseudoline through any pair of points. With this terminology, we may now state Pinchasi's result, of which the result stated above is clearly a special case.

Theorem 1.1 (Pinchasi). There exists an absolute constant $C$ such that every generalised configuration with $n$ points contains a pseudoline where the number of points on either side of the pseudoline differ by at most $C \log \log n$.

Our main result says that Pinchasi's result is tight up to the constant.
Theorem 1.2. There exists a positive constant c such that for every sufficiently large natural number $n$ there is a generalised configuration with $n$ points where the number of points on either side of each pseudoline differ by at least $c \log \log n$.

In particular, for any positive integer $k$, there is a generalised configuration where the number of points on either side of each pseudoline differ by at least $k$.

The reason why Pinchasi's result applies to pseudolines as well as lines is that he works throughout with allowable sequences of permutations. This is a sequence of permutations of $[n]:=\{1,2, \ldots, n\}$ starting with the identity permutation $1,2, \ldots, n$ and ending with its reverse $n, n-1, \ldots, 1$ where every permutation in the sequence arises from its predecessor by flipping the elements in one or more non-overlapping increasing subsequences of consecutive elements. For example, the sequence

$$
12345 \rightarrow 21354 \rightarrow 25314 \rightarrow 52341 \rightarrow 54321
$$

is an allowable sequence of permutations of [5].
Introduced by Goodman and Pollack [5], the importance of allowable sequences is that every generalised configuration of points gives rise to such a sequence and, conversely [6], that every allowable sequence can be realised as a generalised configuration. Some intuition can be gained by thinking about the generalised configuration consisting of a point set and the line arrangement determined by that point set. If we fix another line $\ell$ such that the orthogonal projection of the points of our set onto $\ell$ are all distinct, we can label the points as $1,2, \ldots, n$ in that order. If we now rotate $\ell$ counterclockwise about a fixed point, then the order of the orthogonal projections of our points onto $\ell$ shifts as we rotate, jumping to a different permutation every time $\ell$ moves through the perpendicular to any line in our collection, ultimately arriving at $n, n-1, \ldots, 1$ after rotating through $180^{\circ}$. The construction of an allowable sequence when we instead have a pseudoline arrangement is similar, though we refer the reader to [4] for more details.

In this language, Pinchasi's result may be stated as saying that there is an absolute constant $C$ such that every allowable sequence of permutations of $[n]$ uses a flip $[a, b]$ where $\left|\frac{a+b}{2}-\frac{n+1}{2}\right| \leq$ $C \log \log n$, where here the flip $[a, b]$ is understood as taking a permutation $\pi_{1}, \pi_{2}, \ldots, \pi_{n}$ of $[n]$ and reversing the elements in the block $\pi_{a}, \pi_{a+1}, \ldots, \pi_{b}$. Similarly, the result that we will actually prove can be stated as follows.

Theorem 1.3. There exists a positive constant $c$ such that for every sufficiently large natural number $n$ there is an allowable sequence of permutations of $[n]$ where each flip $[a, b] h a s\left|\frac{a+b}{2}-\frac{n+1}{2}\right| \geq$ $c \log \log n$.

As already mentioned, a result of Goodman and Pollack [6] says that any allowable sequence of permutations of $[n]$ can be realised as a generalised configuration with $n$ points. Moreover, when we flip $[a, b]$, the number of elements on either side of the corresponding pseudoline must differ by at least $|n-b-a+1|$, so Theorem 1.2 follows as an immediate corollary of Theorem 1.3. As such, the remainder of the paper will be concerned with proving this latter theorem. We will prove the theorem for sufficiently large odd $n$, but it trivially extends to all large $n$ by removing a single point when necessary.

## 2 Centred sequences, flips and blocks

We begin our proof of Theorem 1.3 by introducing some notation and justifying some standard procedures that will be used repeatedly in the construction. We first fix a positive integer $t$ that will appear throughout the argument.

Definition 2.1. A centred sequence $\mathbf{A}$ is an injective map $\mathbf{A}:[a, b] \rightarrow \mathbb{Z}$ for some integers $a \leq b$. For $a \leq n \leq b$, we denote $\mathbf{A}(n)$ by $\mathbf{A}_{n}$. The underlying sequence of $\mathbf{A}$ is then $\mathbf{A}_{a}, \mathbf{A}_{a+1}, \ldots, \mathbf{A}_{b}$.

Given such a centred sequence $\mathbf{A}$, a flip $F$ is an interval $[c, d]$, where $a \leq c \leq d \leq b$. The size of $F$ is $d-c+1$. Performing the flip $F$ on the sequence $\mathbf{A}$ gives a new centred sequence $\mathbf{A}^{\prime}:[a, b] \rightarrow \mathbb{Z}$ where the subsequence $[c, d]$ of $\mathbf{A}$ is reversed.

The flip $F$ is valid if the centred subsequence $\left.\mathbf{A}\right|_{[c, d]}$ is increasing and $\frac{c+d}{2}$ does not lie in the real interval $[-t, t]$.

Our aim will be to find some $n>t$ and, starting from the identity centred sequence $\mathbf{A}:[-n, n] \rightarrow$ $\mathbb{Z}$, to perform a sequence of valid flips, ending at the reverse of $\mathbf{A}$.

Definition 2.2. A block $B$ is a sequence $B_{1}, \ldots, B_{n}$ of distinct integers. The size of $B$, which we denote by $|B|$, is $n$.

We can again define a flip $[c, d]$ of $B$, where performing this flip will reverse the subsequence $B_{c}, B_{c+1}, \ldots, B_{d}$. We say that a flip is valid if the subsequence $B_{c}, \ldots, B_{d}$ is initially increasing.

Denote by $\bar{B}$ the block which is the reverse of $B$. If $|B|=m$, write $\mathbf{B}$ for the centred sequence $\mathbf{B}:[t-m+1, t] \rightarrow \mathbb{Z}$ with the same underlying sequence as $B$. Conversely, if $\mathbf{A}$ is a centred sequence, we write $A$ for the block corresponding to the underlying sequence of $\mathbf{A}$.

Definition 2.3. We say that a centred sequence $\mathbf{A}^{\prime}$ can be obtained from a centred sequence $\mathbf{A}$ if there is a sequence of valid flips sending $\mathbf{A}$ to $\mathbf{A}^{\prime}$. We say that a block $B^{\prime}$ can be obtained from a block $B$ if there is a sequence of valid flips sending $B$ to $B^{\prime}$. We also say that we can go from $\mathbf{A}$ to $\mathbf{A}^{\prime}$ or that we can go from $B$ to $B^{\prime}$.

Note that a valid flip of a block is equivalent to making several valid flips of size 2 each. Thus, to see if $B^{\prime}$ can be obtained from $B$, it suffices to consider only valid flips of size 2 .

Definition 2.4. Given two blocks $B, C$ of sizes $m, n$, their concatenation $B \wedge C$ is the block of size $m+n$ with sequence $B_{1}, \ldots, B_{m}, C_{1}, \ldots, C_{n}$.

Given a centred sequence $\mathbf{A}:[a, b] \rightarrow \mathbb{Z}$, the concatenation $\mathbf{A} \wedge B$ is the centred sequence $\mathbf{A}^{\prime}:[a, b+m] \rightarrow \mathbb{Z}$ with underlying sequence $\mathbf{A}_{a}, \ldots, \mathbf{A}_{b}, B_{1}, \ldots, B_{m}$. We can similarly concatenate a block to the left, denoted by $B \wedge \mathbf{A}$.

Definition 2.5. Given blocks or centred sequences $B, C$, we write $B \prec C$ if $\max _{i} B_{i}<\min _{j} C_{j}$. We also write $B \succ 0$ if $B_{i}>0$ for all $i$ and similarly for $B \prec 0$.

We say that a block or centred sequence $B$ is increasing (resp., decreasing) if its underlying sequence is increasing (resp., decreasing).

Observe that if $B, C$ are blocks such that $B \prec C$, then we can easily go from $B \wedge C$ to $C \wedge B$. We will constantly use this simple operation in what follows. We now describe another basic operation that we use repeatedly.

Lemma 2.6 (Shifting). For all $n \geq 3^{2 t}$, the following holds. Let $\mathbf{A}:[-t, t] \rightarrow \mathbb{Z}$ be a centred sequence and $B, C$ be blocks such that

- $|B|=n,|C|=2 t+1$,
- $B$ is increasing,
- $\mathbf{A} \prec B \prec C$.

Then one can go from $\mathbf{A} \wedge B \wedge C$ to $\mathbf{C} \wedge D$, where $D$ is decreasing and, in keeping with Definition 2.2, $\mathbf{C}:[-t, t] \rightarrow \mathbb{Z}$ is the centred sequence with the same underlying sequence as the block $C$.

A graphical depiction of this lemma can be seen in Figure 2. Throughout the paper, we will make extensive use of such figures to illustrate the steps in our construction. These figures will also accurately depict which blocks lie above which other blocks.

Proof. We will show by induction on $k=t, t-1, \ldots,-t$ that there is some $N_{k}$ such that, for all $n \geq N_{k}$, the following holds. Let $\mathbf{A}:[k, t] \rightarrow \mathbb{Z}$ be a centred sequence and $B, C$ be blocks such that

- $|B|=n,|C|=t-k+1$,


Figure 2: A graphical depiction of Lemma 2.6, Shifting.

- $B$ is increasing,
- $\mathbf{A} \prec B \prec C$.

Then one can go from $\mathbf{A} \wedge B \wedge C$ to $\mathbf{C} \wedge D$, where $D$ is decreasing and $\mathbf{C}:[k, t] \rightarrow \mathbb{Z}$ is the centred sequence with the same underlying sequence as the block $C$.

The base case $k=t$ is simple: we may take $N_{t}=0$ and perform a single flip on the whole centred sequence $\mathbf{A} \wedge B \wedge C$.

For $-t \leq k<t$, set $N_{k}=2\left(N_{k+1}+t-k\right)$. Write $\mathbf{A}=A_{1} \wedge \mathbf{A}^{\prime}$, where $A_{1}$ is a block of size 1 and $\mathbf{A}^{\prime}$ is the remaining centred sequence. Write $C=C_{1} \wedge C^{\prime}$, where $C_{1}$ is a block of size 1 and $C^{\prime}$ is the remaining block, and $B=B^{1} \wedge B^{2} \wedge B^{3} \wedge B^{4}$, where $\left|B^{1}\right|=N_{k+1}$ and $\left|B^{2}\right|=\left|B^{3}\right|=t-k$, so that $\left|B^{4}\right| \geq N_{k+1}$. Apply the induction hypothesis for $k+1$ on $\mathbf{A}^{\prime} \wedge B^{1} \wedge B^{2}$ to obtain $\mathbf{B}^{2} \wedge E^{1}$, where $E^{1}$ is decreasing (see Figure 3).

Since $E^{1} \prec B^{3}$, we may swap them and go from $E^{1} \wedge B^{3}$ to $B^{3} \wedge E^{1}$. By continuing to freely swap blocks this way, we can go from $A_{1} \wedge \mathbf{B}^{2} \wedge E^{1} \wedge B^{3} \wedge B^{4} \wedge C$ to $A_{1} \wedge \mathbf{B}^{\mathbf{2}} \wedge B^{3} \wedge C_{1} \wedge E^{1} \wedge B^{4} \wedge C^{\prime}$. We then perform the flip $[k, 2 t-k+1]$ to get

$$
C_{1} \wedge \overline{\mathbf{B}^{3}} \wedge \overline{B^{2}} \wedge A_{1} \wedge E^{1} \wedge B^{4} \wedge C^{\prime}
$$

(see Figure 4).
We can go from $\overline{B^{2}} \wedge A_{1} \wedge E^{1}$ to $E^{2}$, where $E^{2}$ is decreasing. Then we go from $E^{2} \wedge B^{4} \wedge C^{\prime}$ to $B^{4} \wedge C^{\prime} \wedge E^{2}$ to obtain

$$
C_{1} \wedge \overline{\mathbf{B}^{3}} \wedge B^{4} \wedge C^{\prime} \wedge E^{2}
$$

Finally, we again apply the induction hypothesis for $k+1$ to $\overline{\mathbf{B}^{3}} \wedge B^{4} \wedge C^{\prime}$ to obtain

$$
\mathbf{C} \wedge E^{3} \wedge E^{2}
$$

so we can set $D=E^{3} \wedge E^{2}$, which is decreasing (see Figure 5).
To finish the proof, note that a simple induction shows that $N_{k} \leq 3^{t-k}$. In particular, $N_{-t} \leq 3^{2 t}$, as required.


Figure 3: Proof of Lemma 2.6, part 1.


Figure 4: Proof of Lemma 2.6, part 2.


Figure 5: Proof of Lemma 2.6, part 3.


Figure 6: Reflection.

When we apply Lemma 2.6 , we will simply say that we shift $C$. With one additional step, we also obtain a procedure for bringing a block across 0 .

Lemma 2.7 (Reflection). For all $n \geq 3^{2 t}+4 t+2$, the following holds. Let $\mathbf{A}:[-t, t] \rightarrow \mathbb{Z}$ be $a$ centred sequence and $B, C, X$ be blocks such that

- $|B|=n$,
- $|C|=|X|$,
- $B, C, X$ are increasing,
- $X \prec \mathbf{A} \prec B \prec C$.

Then one can go from $X \wedge \mathbf{A} \wedge B \wedge C$ to $\bar{C} \wedge \mathbf{D} \wedge E$, where $\mathbf{D}:[-t, t] \rightarrow \mathbb{Z}$ is a decreasing centred sequence (with the same underlying sequence as the last $2 t+1$ elements of $B$ in reverse), $E$ is decreasing and $\mathbf{D} \succ E$ (see Figure 6).


Figure 7: Proof of Lemma 2.7, part 1.

Proof. Decompose $B=B^{1} \wedge B^{2} \wedge B^{3}$, where $\left|B^{2}\right|=\left|B^{3}\right|=2 t+1$. Then, $\left|B^{1}\right| \geq 3^{2 t}$, so, by shifting $B^{2}$ using Lemma 2.6, we can go from $\mathbf{A} \wedge B \wedge C$ to

$$
\mathbf{B}^{2} \wedge D^{\prime} \wedge B^{3} \wedge C
$$

where $\mathbf{B}^{\mathbf{2}}:[-t, t] \rightarrow \mathbb{Z}$ and $D^{\prime}$ is decreasing (see Figure 7).
Thus, after moving $D^{\prime}$ to the right, we can go from $X \wedge \mathbf{A} \wedge B \wedge C$ to $X \wedge \mathbf{B}^{2} \wedge B^{3} \wedge C \wedge D^{\prime}$. Now perform the flip $[-t-|C|, 3 t+1+|C|]$ to get

$$
\bar{C} \wedge \overline{\mathbf{B}^{3}} \wedge \overline{B^{2}} \wedge \bar{X} \wedge D^{\prime}
$$

from which we can go to $\bar{C} \wedge \mathbf{D} \wedge E$, where $\mathbf{D}=\overline{\mathbf{B}^{3}}$ and $E=\overline{B^{2}} \wedge D^{\prime} \wedge \bar{X}$ (see Figure 8).
When we apply Lemma 2.7 , we will simply say that we reflect $C$.
To close this section, we note that the operations above also hold if we mirror the blocks and centred sequences horizontally and vertically. For example, by mirroring Reflection, we obtain the following result: for $n \geq 3^{2 t}+4 t+2$, let $\mathbf{A}:[-t, t] \rightarrow \mathbb{Z}$ be a centred sequence and $B, C, X$ be blocks such that

- $|B|=n$,
- $|C|=|X|$,
- $B, C, X$ are increasing,
- $C \prec B \prec \mathbf{A} \prec X$.

Then one can go from $C \wedge B \wedge \mathbf{A} \wedge X$ to $E \wedge \mathbf{D} \wedge \bar{C}$, where $\mathbf{D}:[-t, t] \rightarrow \mathbb{Z}$ is a decreasing centred sequence, $E$ is decreasing and $\mathbf{D} \prec E$ (see Figure 9 ).


Figure 8: Proof of Lemma 2.7, part 2.


Figure 9: The mirrored version of Reflection.

## 3 The recursive step

In this section, we show how to perform a certain recursive step which lies at the heart of our construction and is arguably its most difficult part. To state the result, we need some further definitions.

Definition 3.1. Given a block $B$ of size $n$ with sequence $B_{1}, \ldots, B_{n}$, the width of $B$, denoted by width $(B)$, is the largest integer $k$ such that there exists $i_{1}<i_{2}<\cdots<i_{k}$ with $B_{i_{1}}>B_{i_{2}}>\cdots>$ $B_{i_{k}}$.

Note that, by Dilworth's theorem, this is the same as the smallest integer $k$ such that $B_{1}, \ldots, B_{n}$ can be partitioned into $k$ disjoint increasing subsequences.

Definition 3.2. Given a block $B$, denote by $B^{+}$the block corresponding to the subsequence of $B$ consisting of all the positive values. Denote by $B^{-}$the block corresponding to the subsequence of $B$ consisting of all the negative values.

Definition 3.3. Let $B$ be a block of size $n$ and $r \geq 0$ a real number. We say that $B$ is $r$-balanced if

1. the block $B^{-}$is increasing,
2. for each $1 \leq k \leq n$, the initial segment $B^{\prime}=\left.B\right|_{[k]}$ has $\left|B^{\prime-}\right| \geq r \cdot \operatorname{width}\left(B^{\prime+}\right)$.

The following key property of $r$-balanced blocks will be crucial to our main construction.
Lemma 3.4. Suppose $B$ is a block that is r-balanced for some real $r \geq 0$. Then there is some $k$ such that a block of the form

$$
C_{1} \wedge \cdots \wedge C_{k}
$$

can be obtained from $B$, where each block $C_{i}$ is increasing, $\left|C_{i}^{-}\right| \geq\lfloor r\rfloor$ and $C_{1}^{-} \prec C_{2}^{-} \prec \cdots \prec C_{k}^{-}$.
Proof. Note that if $B$ is $r$-balanced, then it is $r^{\prime}$-balanced for any $r^{\prime} \leq r$. Hence, without loss of generality, we can assume that $r$ is an integer.

Let $A$ be a block of size $n$. We shall decompose $A$ into a collection of increasing subsequences. First set $I_{0}=[n]$. We define an increasing subsequence $i_{1,1}, \ldots, i_{1, n_{1}}$ from $I_{0}$ as follows. Let $i_{1,1}$ be the smallest element of $I_{0}$. Given $i_{1, k}$, define $i_{1, k+1}$ to be the smallest element in $I_{0}$ larger than $i_{1, k}$ such that $A_{i_{1, k+1}}>A_{i_{1, k}}$. If no such $i_{1, k+1}$ exists, then we terminate the sequence and set $I_{1}=I_{0} \backslash\left\{i_{1,1}, \ldots, i_{1, n_{1}}\right\}$. We then define $i_{2,1}, \ldots, i_{2, n_{2}}$ similarly in terms of $I_{1}$ and set $I_{2}=I_{1} \backslash\left\{i_{2,1}, \ldots, i_{2, n_{2}}\right\}$. Repeating this procedure until we reach $d$ with $I_{d}=\emptyset$, we see that we have partitioned $A$ into $d$ increasing subsequences $A_{i_{k, 1}}, \ldots, A_{i_{k, n_{k}}}$ for $k=1, \ldots, d$.

We claim that $d=\operatorname{width}(A)$. In fact, we shall show that for any $1 \leq m \leq n$, if $k$ is the largest integer for which $i_{k, 1} \leq m$, then width $\left(\left.A\right|_{[m]}\right)=k$. Indeed, $\operatorname{width}\left(\left.A\right|_{[m]}\right) \leq k$, since we can cover $\left.A\right|_{[m]}$ with $k$ increasing subsequences $A_{i_{j, 1}}, \ldots, A_{i_{j, n_{j}}}$ for $j=1, \ldots, k$. To show that $\operatorname{width}\left(\left.A\right|_{[m]}\right) \geq$ $k$, we find a decreasing subsequence of length $k$. Let $j_{k}=i_{k, 1}$. Since $i_{k, 1} \notin\left\{i_{k-1,1}, \ldots, i_{k-1, n_{k-1}}\right\}$, there is some $l$ such that $i_{k-1, l}<j_{k}$ and $A_{i_{k-1, l}}>A_{j_{k}}$. Set $j_{k-1}=i_{k-1, l}$. By repeating this step, we eventually obtain a sequence $j_{k}>j_{k-1}>\cdots>j_{1}$ with $A_{j_{k}}<A_{j_{k-1}}<\cdots<A_{j_{1}}$, so we have the required decreasing subsequence of length $k$.

Now consider the above decomposition into increasing subsequences applied to $B^{+}$. If $k=$ width $\left(B^{+}\right)$, then we can partition $B^{+}$into $k$ increasing subsequences $S_{1}, \ldots, S_{k}$, the $j$ th of which
starts from $B_{i_{j}}$ for some $i_{j}$. Set $C_{1}$ to be the concatenation of the first $r$ elements of $B^{-}$and $S_{1}$. Set $C_{2}$ to be the concatenation of the next $r$ elements of $B^{-}$and $S_{2}$ and so on, until $C_{k}$, which we take to be the concatenation of the remaining elements of $B^{-}$and $S_{k}$. Note that, for each $j \leq k$, width $\left(\left.B\right|_{\left[i_{j}\right]}\right)=j$, so, since $B$ is $r$-balanced, there are at least $j r$ elements of $B^{-}$that come before $S_{j}$. Since $B^{-}$is also increasing by the definition of $r$-balancedness, we can go from $B$ to $C_{1} \wedge \cdots \wedge C_{k}$, where each $C_{i}$ is increasing and $\left|C_{i}^{-}\right| \geq r$. Furthermore, $C_{1}^{-} \prec C_{2}^{-} \prec \cdots \prec C_{k}^{-}$, since they are the elements of $B^{-}$in the right order.

The following lemma is the main recursive step in our construction (see Figure 10).
Lemma 3.5. Let $d, n \geq 1$, set $T=3^{2 t}$ and define sequences $\alpha_{0}, \alpha_{1}, \ldots$ and $\beta_{0}, \beta_{1}, \ldots$ inductively by

- $\alpha_{0}=T+4 t+2$,
- $\beta_{0}=0$,
- $\alpha_{i+1}=d \alpha_{i}+2 T d^{i}+d$,
- $\beta_{i+1}=d \beta_{i}+\frac{d^{i+1}}{3 T}$.

Suppose that $d \geq 9 T$. Then, for each $k \geq 0$ and any centred sequence $\mathbf{I}:[-t, t] \rightarrow \mathbb{Z}$, there are increasing blocks $X, Y$ of some sizes with $X \prec \mathbf{I} \prec Y$ and $X \prec 0 \prec Y$ such that one can go from $X \wedge \mathbf{I} \wedge Y$ to

$$
L_{k} \wedge W_{k} \wedge \mathbf{A}_{k} \wedge B_{k} \wedge R_{k}
$$

where:

1. $\mathbf{A}_{k}:[-t, t] \rightarrow \mathbb{Z}$,
2. $W_{k}, L_{k} \succ 0 \succ R_{k}$,
3. if $k>0$, then $\mathbf{A}_{k} \prec B_{k}$, while if $k=0$, then $\mathbf{A}_{k} \succ B_{k}$,
4. $W_{k}$ only contains elements from $Y$,
5. $W_{k}$ is of the form

$$
K_{1} \wedge \cdots \wedge K_{m} \wedge M_{m} \wedge \cdots \wedge M_{1}
$$

where $m=d^{k}$, each $K_{i}$ is decreasing, $\left|K_{i}\right|=n,\left|M_{i}\right|=1$ and

$$
K_{m} \succ M_{m} \succ K_{m-1} \succ M_{m-1} \succ \cdots \succ K_{1} \succ M_{1}
$$

6. $\operatorname{width}\left(B_{k}^{+}\right) \leq \alpha_{k}$,
7. $\left|B_{k}^{-}\right| \geq \beta_{k}$,
8. $B_{k}$ is $\left(\beta_{k} / \alpha_{k}\right)$-balanced.

Furthermore, one can take $|X|,|Y| \leq 10 d^{2 k+1} n$.
Though somewhat hidden in the detail of the induction hypothesis, the key point here is that one can make $B_{k} r$-balanced for any given $r$ by choosing $k$ and $d$ appropriately.


Figure 10: The main recursive step.

Proof. We induct on $k$. We first prove the base case $k=0$. Set $|X|=n+1$ and $Y=C_{1} \wedge C_{2} \wedge C_{3}$, where $\left|C_{1}\right|=T+2 t+1,\left|C_{2}\right|=2 t+1$ and $\left|C_{3}\right|=n+1$. By reflecting $C_{3}$, we can go from $X \wedge \mathbf{I} \wedge Y$ to

$$
\overline{C_{3}} \wedge \overline{\mathbf{C}_{2}} \wedge \overline{C_{1}} \wedge D \wedge \bar{X},
$$

where $D$ is the block we get by sorting $\mathbf{I}$ in decreasing order (see Figure 11). But this completes the base case by setting $L_{0}=\emptyset, W_{0}=\overline{C_{3}}, \mathbf{A}_{0}=\overline{\mathbf{C}_{\mathbf{2}}}, B_{0}=\overline{C_{1}} \wedge D^{+}$and $R_{0}=D^{-} \wedge \bar{X}$ and noting that $\operatorname{width}\left(B_{0}\right)=\left|B_{0}\right| \leq T+4 t+2=\alpha_{0}$.

We now proceed to the induction proper, assuming that the result holds for $k$ and then showing that it also holds for $k+1$. To make it more digestible, we shall describe the construction in several steps.

Step 1: Repeating the induction hypothesis $d$ times.
Write

$$
X \wedge \mathbf{I} \wedge Y=X^{\prime} \wedge P_{d} \wedge P_{d-1} \wedge \cdots \wedge P_{1} \wedge \mathbf{I} \wedge Q_{1} \wedge Q_{2} \wedge \cdots \wedge Q_{d} \wedge Y^{\prime}
$$

where $P_{i}, Q_{i}$ are blocks of suitable lengths for what follows. By our induction hypothesis (using $k$ and $n+1$ instead of $n$ ), we can go from $P_{1} \wedge \mathbf{I} \wedge Q_{1}$ to $L_{k}^{1} \wedge W_{k}^{1} \wedge \mathbf{A}_{k}^{1} \wedge B_{k}^{1} \wedge R_{k}^{1}$. Since $P_{i} \prec P_{1} \wedge \mathbf{I} \wedge Q_{1} \prec Q_{i}$ for $i>1$, we can therefore go from $X \wedge \mathbf{I} \wedge Y$ to

$$
L_{k}^{1} \wedge X^{\prime} \wedge W_{k}^{1} \wedge P_{d} \wedge \cdots \wedge P_{2} \wedge \mathbf{A}_{k}^{1} \wedge Q_{2} \wedge \cdots \wedge Q_{d} \wedge B_{k}^{1} \wedge Y^{\prime} \wedge R_{k}^{1}
$$

(see Figure 12).
Repeating our induction hypothesis on $P_{2} \wedge \mathbf{A}_{k}^{1} \wedge Q_{2}$ to get $L_{k}^{2} \wedge W_{k}^{2} \wedge \mathbf{A}_{k}^{2} \wedge B_{k}^{2} \wedge R_{k}^{2}$ and so on, we can eventually go to

$$
L^{\prime} \wedge X^{\prime} \wedge W_{k}^{1} \wedge \cdots \wedge W_{k}^{d} \wedge \mathbf{A}_{k}^{d} \wedge B^{\prime} \wedge Y^{\prime} \wedge R^{\prime}
$$



Figure 11: The case $k=0$.


Figure 12: Step 1, part 1.


Figure 13: Step 1, part 2.
where $L^{\prime}=L_{k}^{1} \wedge \cdots \wedge L_{k}^{d}, R^{\prime}=R_{k}^{d} \wedge \cdots \wedge R_{k}^{1}$ and $B^{\prime}=B_{k}^{d} \wedge \cdots \wedge B_{k}^{1}$ (see Figure 13). Since $W_{k}^{i}$ only contains elements from $Q_{i}$, we have $W_{k}^{1} \prec W_{k}^{2} \prec \cdots \prec W_{k}^{d}$, as depicted in Figure 13.

If $k=0$, then $\mathbf{A}_{k}^{i-1} \succ B_{k}^{i-1},\left(B_{k}^{i}\right)^{+}$only contains elements from $\mathbf{A}_{k}^{i-1} \wedge Q_{i}$ and $\left(B_{k}^{i}\right)^{-}$only contains elements from $P_{i}$. Otherwise, if $k>0$, then $\mathbf{A}_{k}^{i-1} \prec B_{k}^{i-1},\left(B_{k}^{i}\right)^{+}$only contains elements from $Q_{i}$ and $\left(B_{k}^{i}\right)^{-}$only contains elements from $P_{i} \wedge \mathbf{A}_{k}^{i-1}$. In either case, we have $\left(B_{k}^{d}\right)^{+} \succ\left(B_{k}^{d-1}\right)^{+} \succ \cdots \succ\left(B_{k}^{1}\right)^{+}$and $\left(B_{k}^{d}\right)^{-} \prec \cdots \prec\left(B_{k}^{1}\right)^{-}$, so $\left(B^{\prime}\right)^{-}$is increasing. Furthermore, $\left|B^{\prime-}\right|=\left|\left(B_{k}^{1}\right)^{-}\right|+\cdots+\left|\left(B_{k}^{d}\right)^{-}\right| \geq d \beta_{k}, \operatorname{width}\left(B^{\prime+}\right) \leq \operatorname{width}\left(\left(B_{k}^{1}\right)^{+}\right)+\cdots+\operatorname{width}\left(\left(B_{k}^{1}\right)^{+}\right) \leq d \alpha_{k}$ and, since each $B_{k}^{i}$ is $\left(\beta_{k} / \alpha_{k}\right)$-balanced, $B^{\prime}$ is $\left(\beta_{k} / \alpha_{k}\right)$-balanced.

Step 2: Moving the tails of each $W_{k}$ to the right.
Write $W_{k}^{i}=K_{1}^{i} \wedge K_{2}^{i} \wedge \cdots \wedge K_{m}^{i} \wedge M_{m}^{i} \wedge \cdots \wedge M_{1}^{i}$, where $m=d^{k}$ and $\left|K_{j}^{i}\right|=n+1$. Decompose $Y^{\prime}$ into $S_{1} \wedge \cdots \wedge S_{m} \wedge Y^{\prime \prime}$, where $\left|S_{i}\right|=T+d+4 t+2$. Further decompose each $S_{i}$ into $T_{i} \wedge U_{i}$ where $\left|T_{i}\right|=T+2 t+1$ and $\left|U_{i}\right|=d+2 t+1$. Go to

$$
L^{\prime} \wedge X^{\prime} \wedge W_{k}^{1} \wedge \cdots \wedge W_{k}^{d} \wedge \mathbf{A}_{k}^{d} \wedge T_{1} \wedge U_{1} \wedge B^{\prime} \wedge S_{2} \wedge \cdots \wedge S_{m} \wedge Y^{\prime \prime} \wedge R^{\prime}
$$

By performing suitable transpositions to move the points $M_{j}^{i}$, we can go to
$L^{\prime} \wedge X^{\prime} \wedge W^{\prime} \wedge M_{m}^{1} \wedge M_{m}^{2} \wedge \cdots \wedge M_{m}^{d} \wedge M_{m-1}^{1} \wedge \cdots \wedge M_{1}^{d} \wedge \mathbf{A}_{k}^{d} \wedge T_{1} \wedge U_{1} \wedge B^{\prime} \wedge S_{2} \wedge \cdots \wedge S_{m} \wedge Y^{\prime \prime} \wedge R^{\prime}$, where $W^{\prime}=K_{1}^{1} \wedge K_{2}^{1} \wedge \cdots \wedge K_{m}^{1} \wedge K_{1}^{2} \wedge \cdots \wedge K_{m}^{d}$ (see Figure 14).

We would now like to move all the $M_{j}^{i}$ to the right. Write $T_{1}=T_{1}^{\prime} \wedge J_{1}$, where $\left|J_{1}\right|=2 t+1$ and $\left|T_{1}^{\prime}\right|=T$. By shifting $J_{1}$, we can go to

$$
L^{\prime} \wedge X^{\prime} \wedge W^{\prime} \wedge M_{m}^{1} \wedge \cdots \wedge M_{1}^{d} \wedge \mathbf{J}_{1} \wedge C_{1} \wedge U_{1} \wedge B^{\prime} \wedge S_{2} \wedge \cdots \wedge S_{m} \wedge Y^{\prime \prime} \wedge R^{\prime}
$$

where $\mathbf{J}_{1}:[-t, t] \rightarrow \mathbb{Z}$ is increasing and $C_{1}$ is decreasing, noting that $\left|C_{1}\right|=\left|T_{1}^{\prime}\right|+\left|A_{k}^{d}\right|=T+2 t+1$. If $k=0$, then $\mathbf{A}_{0}^{d} \succ 0$ implies that $C_{1}^{-}=\emptyset$. Otherwise, $\mathbf{A}_{k}^{d} \prec B^{\prime}$, so $C_{1}^{-} \prec B^{\prime}$ and we can move $C_{1}^{-}$ all the way to the right to be part of $R^{\prime}$. Thus, we can assume that $C_{1} \succ 0$ and $\left|C_{1}\right| \leq T+2 t+1$ (see Figure 15).

We can go from $C_{1} \wedge U_{1}$ to $U_{1} \wedge C_{1}$. Decompose $U_{1}=U_{1}^{\prime} \wedge U_{1}^{\prime \prime}$, where $\left|U_{1}^{\prime}\right|=2 t+1$ and $\left|U_{1}^{\prime \prime}\right|=d$. Then, perform the flip $[-d-t, d+3 t+1]$ to go to

$$
L^{\prime} \wedge X^{\prime} \wedge W^{\prime} \wedge M_{m}^{1} \wedge \cdots \wedge M_{2}^{d} \wedge \overline{U_{1}^{\prime \prime}} \wedge \overline{\mathbf{U}_{1}^{\prime}} \wedge \overline{J_{1}} \wedge N_{1} \wedge C_{1} \wedge B^{\prime} \wedge S_{2} \wedge \cdots \wedge S_{m} \wedge Y^{\prime \prime} \wedge R^{\prime}
$$

where $N_{1}=M_{1}^{d} \wedge M_{1}^{d-1} \wedge \cdots \wedge M_{1}^{1}$. We can then move $\overline{U_{1}^{\prime \prime}}$ to the left to be absorbed by $L^{\prime}$ (see Figure 16).

Repeating this process, we can go to

$$
L^{\prime} \wedge X^{\prime} \wedge W^{\prime} \wedge \overline{\mathbf{U}_{\mathbf{m}}^{\prime}} \wedge C \wedge B^{\prime} \wedge Y^{\prime \prime} \wedge R^{\prime}
$$

where $C=\overline{J_{m}} \wedge N_{m} \wedge C_{m} \wedge \cdots \wedge \overline{J_{1}} \wedge N_{1} \wedge C_{1} \succ 0$ and $N_{i}=M_{i}^{d} \wedge M_{i}^{d-1} \wedge \cdots \wedge M_{i}^{1}$ (see Figure 17).
Before continuing with the construction, let us analyse the width of the block $C$. Observe that the width of interlaced blocks is at most the sum of the widths of the individual blocks, so

$$
\operatorname{width}(C) \leq \sum_{i=1}^{m} \operatorname{width}\left(\overline{J_{i}} \wedge C_{i}\right)+\operatorname{width}\left(N_{m} \wedge \cdots \wedge N_{1}\right)
$$



Figure 14: Step 2, part 1.


Figure 15: Step 2, part 2.


Figure 16: Step 2, part 3.


Figure 17: Step 2, part 4.

We have $\operatorname{width}\left(\overline{J_{i}} \wedge C_{i}\right) \leq\left|J_{i}\right|+\left|C_{i}\right| \leq T+4 t+2$. Moreover, $N_{m} \wedge \cdots \wedge N_{1}$ can be covered by $m+d-1$ increasing subsequences, so that $\operatorname{width}\left(N_{m} \wedge \cdots \wedge N_{1}\right) \leq m+d-1$. Overall, we have

$$
\operatorname{width}(C) \leq m(T+4 t+2)+m+d-1 \leq 2 m T+d
$$

Step 3: Forming the tail of $W_{k+1}$.
Move $X^{\prime}$ and $Y^{\prime \prime}$ towards the centre. In this step, we will largely focus on the centred subsequence $W^{\prime} \wedge X^{\prime} \wedge \overline{\mathbf{U}_{\mathbf{m}}^{\prime}} \wedge Y^{\prime \prime}$, only coming back to the full sequence towards the end.

Expand $W^{\prime}=K_{1} \wedge \cdots \wedge K_{m d}$, where each $K_{i}$ is decreasing and $\left|K_{i}\right|=n+1$. Write $K_{i}=K_{i}^{\prime} \wedge O_{i}$, where $\left|K_{i}^{\prime}\right|=n$ and $\left|O_{i}\right|=1$. Then we can go from $W^{\prime}$ to

$$
K_{1}^{\prime} \wedge \cdots \wedge K_{m d}^{\prime} \wedge O_{1} \wedge \cdots \wedge O_{m d}
$$

Write $X^{\prime}=X^{\prime \prime} \wedge P_{p} \wedge P_{p-1} \wedge \cdots \wedge P_{1}$ and $Y^{\prime \prime}=Q_{1} \wedge \cdots \wedge Q_{p}$, where $\left|P_{i}\right|=\left|Q_{i}\right|=1,\left|X^{\prime \prime}\right|=2 t+1$ and $p=\left\lfloor\frac{m d-T}{T+4 t+2}\right\rfloor \geq 1$, by using $d \geq 9 T$ (note that we have now fixed the original sizes of $X$ and $Y)$. Set $W^{\prime \prime}=K_{1}^{\prime} \wedge \cdots \wedge K_{m d}^{\prime}$. We shall group the $O_{i}$ as

$$
O_{1} \wedge \cdots \wedge O_{m d}=F \wedge G_{p} \wedge H_{p} \wedge G_{p-1} \wedge H_{p-1} \wedge \cdots \wedge G_{1} \wedge H_{1}
$$

where $\left|G_{i}\right|=2 t+1,\left|H_{i}\right|=T+2 t+1$ and $F$ is what remains, so that $|F| \geq T$ from the definition of $p$.

By rearranging the blocks, we can go to

$$
W^{\prime \prime} \wedge X^{\prime \prime} \wedge F \wedge P_{p} \wedge G_{p} \wedge H_{p} \wedge \cdots \wedge P_{1} \wedge G_{1} \wedge H_{1} \wedge \overline{\mathbf{U}_{\mathbf{m}}^{\prime}} \wedge Q_{1} \wedge \cdots \wedge Q_{p}
$$

By reflecting $P_{1}$ in the mirrored sense, we can go from $P_{1} \wedge G_{1} \wedge H_{1} \wedge \overline{\mathbf{U}_{\mathbf{m}}^{\prime}} \wedge Q_{1}$ to $Q_{1} \wedge \overline{U_{m}^{\prime}} \wedge$ $\overline{H_{1}} \wedge \overline{\mathbf{G}_{\mathbf{1}}} \wedge P_{1}$. Then, moving $Q_{1} \wedge \overline{U_{m}^{\prime}}$ to the left, we can go to

$$
Q_{1} \wedge \overline{U_{m}^{\prime}} \wedge W^{\prime \prime} \wedge X^{\prime \prime} \wedge F \wedge P_{p} \wedge G_{p} \wedge H_{p} \wedge \cdots \wedge P_{2} \wedge G_{2} \wedge H_{2} \wedge \overline{H_{1}} \wedge \overline{\mathbf{G}_{\mathbf{1}}} \wedge P_{1} \wedge Q_{2} \wedge \cdots \wedge Q_{p}
$$

(see Figure 18).
Move $\overline{H_{1}}$ to be between $W^{\prime \prime}$ and $X^{\prime \prime}$ and move $P_{1}$ to the right. Reflecting $P_{2}$ similarly, we can go from $P_{2} \wedge G_{2} \wedge H_{2} \wedge \overline{\mathbf{G}_{\mathbf{1}}} \wedge Q_{2}$ to $Q_{2} \wedge \overline{G_{1}} \wedge \overline{H_{2}} \wedge \overline{\mathbf{G}_{\mathbf{2}}} \wedge P_{2}$ (see Figure 19). Rearranging the blocks, we obtain

$$
\begin{aligned}
& Q_{1} \wedge Q_{2} \wedge \overline{U_{m}^{\prime}} \wedge W^{\prime \prime} \wedge \overline{H_{1}} \wedge \overline{G_{1}} \wedge \overline{H_{2}} \wedge X^{\prime \prime} \wedge F \wedge P_{p} \wedge G_{p} \wedge H_{p} \wedge \cdots \wedge P_{3} \wedge G_{3} \wedge H_{3} \wedge \\
& \quad \overline{\mathbf{G}_{\mathbf{2}}} \wedge Q_{3} \wedge \cdots \wedge Q_{p} \wedge P_{2} \wedge P_{1}
\end{aligned}
$$

Repeating all the way to $P_{p} \wedge G_{p} \wedge H_{p}$, we obtain

$$
Q \wedge \overline{U_{m}^{\prime}} \wedge W^{\prime \prime} \wedge \overline{H_{1}} \wedge \overline{G_{1}} \wedge \cdots \wedge \overline{H_{p}} \wedge X^{\prime \prime} \wedge F \wedge \overline{\mathbf{G}_{\mathbf{p}}} \wedge P
$$

where $P=P_{p} \wedge \cdots \wedge P_{1}$ and $Q=Q_{1} \wedge \cdots \wedge Q_{p}$ are increasing. By shifting $X^{\prime \prime}$ in the mirrored sense, we obtain

$$
Q \wedge \overline{U_{m}^{\prime}} \wedge W^{\prime \prime} \wedge \overline{H_{1}} \wedge \overline{G_{1}} \wedge \cdots \wedge \overline{H_{p}} \wedge \overline{G_{p}} \wedge \bar{F} \wedge \mathbf{X}^{\prime \prime} \wedge P
$$

(see Figure 20).


Figure 18: Step 3, part 1.


Figure 19: Step 3, part 2.


Figure 20: Step 3, part 3.

Overall, we managed to go from the original sequence $X \wedge \mathbf{I} \wedge Y$ to

$$
L^{\prime} \wedge Q \wedge \overline{U_{m}^{\prime}} \wedge W^{\prime \prime} \wedge O_{m d} \wedge \cdots \wedge O_{1} \wedge \mathbf{X}^{\prime \prime} \wedge P \wedge C \wedge B^{\prime} \wedge R^{\prime}
$$

Now set $L_{k+1}=L^{\prime} \wedge Q \wedge \overline{U_{m}^{\prime}}, W_{k+1}=W^{\prime \prime} \wedge O_{m d} \wedge \cdots \wedge O_{1}, \mathbf{A}_{k+1}=\mathbf{X}^{\prime \prime}, B_{k+1}=P \wedge C \wedge B^{\prime}$ and $R_{k+1}=R^{\prime}$. This concludes the construction, but we still need to check that the required conditions hold. Conditions (1), (2), (3), (4) and (5) are easy to check, so it only remains to show that $B_{k+1}$ satisfies conditions (6), (7) and (8).

For $l \geq 0$, let $\beta_{l}^{\prime}=\frac{d^{l+1}}{3 T}$ and $\alpha_{l}^{\prime}=2 d^{l} T+d$. In particular, $\beta_{k}^{\prime}=\frac{m d}{3 T} \leq\left\lfloor\frac{m d-T}{T+4 t+2}\right\rfloor=p$ and $\alpha_{k}^{\prime}=2 m T+d$, where the inequality holds since $d>9 T$. We have $\operatorname{width}\left(B_{k+1}^{+}\right) \leq \operatorname{width}(C)+$ $\operatorname{width}\left(B^{\prime+}\right) \leq \alpha_{k}^{\prime}+d \alpha_{k}=\alpha_{k+1}$, proving (6). We also have that $\left|B_{k+1}^{-}\right|=|P|+\left|B^{\prime-}\right| \geq \beta_{k}^{\prime}+d \beta_{k}=$ $\beta_{k+1}$, proving (7). To prove (8), we need the following claim.

Claim. For all $l \geq 0, \beta_{l}^{\prime} / \alpha_{l}^{\prime}>\beta_{l+1} / \alpha_{l+1}>\beta_{l} / \alpha_{l}$.
Proof. We will show by induction on $l$ that $\beta_{l}^{\prime} / \alpha_{l}^{\prime}>\beta_{l} / \alpha_{l}$ and the full result will fall out as a consequence. The base case $l=0$ is trivial, since $\beta_{0}=0$. To show the result for $l+1$, note that, since $\frac{\beta_{l+1}}{\alpha_{l+1}}=\frac{\beta_{l}^{\prime}+d \beta_{l}}{\alpha_{l}^{\prime}+d \alpha_{l}}$ and $\beta_{l}^{\prime} / \alpha_{l}^{\prime}>\beta_{l} / \alpha_{l}$, we have $\beta_{l}^{\prime} / \alpha_{l}^{\prime}>\beta_{l+1} / \alpha_{l+1}>\beta_{l} / \alpha_{l}$. But $\frac{\beta_{l+1}^{\prime}}{\alpha_{l+1}^{\prime}}=$ $\frac{d^{l+2}}{3 T\left(2 d^{l+1} T+d\right)} \geq \frac{d^{l+1}}{3 T\left(2 d^{l} T+d\right)}=\frac{\beta_{l}^{\prime}}{\alpha_{l}^{\prime}}$, so that $\beta_{l+1}^{\prime} / \alpha_{l+1}^{\prime}>\beta_{l+1} / \alpha_{l+1}$, as required.

Suppose now that $J$ is an initial segment of $B_{k+1}$. If $J$ is a subblock of $P$, then $J^{+}=\emptyset$. If $J$ contains $P$ and is a subblock of $P \wedge C$, then $\left|J^{-}\right|=p \geq \beta_{k}^{\prime}$ and width $\left(J^{+}\right) \leq \operatorname{width}(C) \leq \alpha_{k}^{\prime}$, so, by the claim, $\left|J^{-}\right| \geq\left(\beta_{k+1} / \alpha_{k+1}\right)$ width $\left(J^{+}\right)$. Suppose then that $J=P \wedge C \wedge J^{\prime}$, where $J^{\prime}$ is a nonempty initial segment of $B^{\prime}$. Since $B^{\prime}$ is $\left(\beta_{k} / \alpha_{k}\right)$-balanced, we have $\left|J^{\prime-}\right| \geq\left(\beta_{k} / \alpha_{k}\right)$ width $\left(J^{\prime+}\right)$. But $\left|J^{-}\right| \geq \beta_{k}^{\prime}+\left|J^{\prime-}\right|$ and width $\left(J^{+}\right) \leq \alpha_{k}^{\prime}+\operatorname{width}\left(J^{\prime+}\right)$, so, $\operatorname{since}$ width $\left(J^{\prime+}\right) \leq \operatorname{width}\left(B^{\prime+}\right) \leq d \alpha_{k}$, we have

$$
\frac{\left|J^{-}\right|}{\operatorname{width}\left(J^{+}\right)} \geq \frac{\beta_{k}^{\prime}+\left(\beta_{k} / \alpha_{k}\right) \operatorname{width}\left(J^{\prime+}\right)}{\alpha_{k}^{\prime}+\operatorname{width}\left(J^{\prime+}\right)} \geq \frac{\beta_{k}^{\prime}+d \beta_{k}}{\alpha_{k}^{\prime}+d \alpha_{k}}=\frac{\beta_{k+1}}{\alpha_{k+1}}
$$



Figure 21: Starting the construction.
Note that here, in the second inequality, we used that, since $\beta_{k}^{\prime} / \alpha_{k}^{\prime}>\beta_{k} / \alpha_{k}$, the function $\left(\beta_{k}^{\prime}+\right.$ $\left.\left(\beta_{k} / \alpha_{k}\right) x\right) /\left(\alpha_{k}^{\prime}+x\right)$ is decreasing in $x$. Since $B_{k+1}^{-}$is increasing, this implies that $B_{k+1}$ is $\left(\beta_{k+1} / \alpha_{k+1}\right)-$ balanced, proving (8).

Finally, we give bounds on $|X|$ and $|Y|$. Suppose, for given $d, n, k$, that the construction gives $|X|=x_{d, n, k}$ and $|Y|=y_{d, n, k}$. Then $x_{d, n, 0}=n+1$ and $y_{d, n, 0}=T+4 t+3+n$. Moreover, following the construction, we have that, for $k \geq 1$,

$$
\begin{aligned}
& x_{d, n, k}=d x_{d, n+1, k-1}+p+2 t+1 \leq d x_{d, n+1, k-1}+\frac{d^{k+1}}{2 T}+2 t+1 \\
& y_{d, n, k}=d y_{d, n+1, k-1}+m(T+d+4 t+2)+p \leq d y_{d, n+1, k-1}+2 d^{k+1},
\end{aligned}
$$

so we get the (loose) bounds $x_{d, n, k}, y_{d, n, k} \leq 10 d^{2 k+1} n$.

## 4 The full construction

We now proceed to the full construction. Set $T=3^{2 t}$. Start with the identity centred sequence $\mathbf{I}:[a, b] \rightarrow \mathbb{Z}$ for suitable $a<0<b$ to be decided later. Perform the flip $[-t, 3 t+1]$ to obtain

$$
X^{\prime} \wedge X \wedge \mathbf{I}^{\prime} \wedge J \wedge Y
$$

where $\mathbf{I}^{\prime}:[-t, t] \rightarrow \mathbb{Z},|J|=2 t+1$ and $X^{\prime}, X$ are of suitable lengths to be decided later. $J$ is an important piece that we will have to bring back to the centre at the end. For now, we set it aside by moving it to the right to get

$$
X^{\prime} \wedge X \wedge \mathbf{I}^{\prime} \wedge Y \wedge J
$$

(see Figure 21).
Applying Lemma 3.5 to $X \wedge \mathbf{I}^{\prime} \wedge Y$ with $n=1$ and some $d$ (with suitably chosen $a, b$ ), we can go to

$$
X^{\prime} \wedge L_{k} \wedge W_{k} \wedge \mathbf{A}_{k} \wedge B_{k} \wedge R_{k} \wedge J
$$

where $B_{k}$ is $r=\left(\beta_{k} / \alpha_{k}\right)$-balanced.


Figure 22: Transforming $B_{k}$.

We now show that for a suitable choice of $d$ and $k$, we can make $r$ arbitrarily large. Indeed, we have $\alpha_{1} \leq 3 d T$,

$$
\alpha_{i+1} \leq d \alpha_{i}+3 T d^{i}
$$

for $i \geq 1$ and

$$
\beta_{i+1} \geq d \beta_{i}+\frac{d^{i+1}}{3 T}
$$

so that $\alpha_{k} \leq 3 T d^{k}+3 k T d^{k-1}$ and $\beta_{k} \geq \frac{k}{3 T} d^{k}$. Hence, $r \geq \frac{k}{3 T(3 T+3 k T / d)}$. Setting $k=d$, we see that $r \geq \frac{d}{18 T^{2}}$ can be taken arbitrarily large. In practice, we will set $d=100 T^{3}$, so that $r \geq 3 T+1$.

By Lemma 3.4, we can go from $B_{k}$ to $C_{1} \wedge \cdots \wedge C_{m-1} \wedge C_{m}^{\prime}$ where each block $C_{i}$ and $C_{m}^{\prime}$ is increasing and $\left|C_{i}^{-}\right|,\left|C_{m}^{\prime-}\right| \geq\lfloor r\rfloor \geq 3 T$. We can decompose $C_{m}^{\prime}$ into $C_{m}^{\prime \prime} \wedge Z \wedge C_{m}^{\prime \prime \prime}$, where $C_{m}^{\prime \prime} \prec Z \prec 0 \prec C_{m}^{\prime \prime \prime},|Z| \geq T$ and $\left|C_{m}^{\prime \prime}\right| \geq 2 T$. We can go from $C_{m}^{\prime \prime} \wedge Z \wedge C_{m}^{\prime \prime \prime}$ to $C_{m}^{\prime \prime} \wedge C_{m}^{\prime \prime \prime} \wedge Z$. Now set $C_{m}=C_{m}^{\prime} \wedge C_{m}^{\prime \prime}$. Thus, we have gone from $B_{k}$ to $C_{1} \wedge \cdots \wedge C_{m} \wedge Z$, where $C_{1}^{-} \prec C_{2}^{-} \prec \cdots \prec$ $C_{m}^{-} \prec Z \prec 0$ and $\left|C_{i}^{-}\right| \geq 2 T$ for each $i$ (see Figure 22).

For each $i$, decompose $C_{i}$ into $D_{i} \wedge E_{i} \wedge F_{i}$, where $D_{i} \prec E_{i} \prec 0 \prec F_{i}$ and $\left|E_{i}\right|=2 t+1$, so that $\left|D_{i}\right| \geq T+2 t+1$. Decompose $X^{\prime}$ as $X^{\prime}=X_{m} \wedge \cdots \wedge X_{1}$ with $\left|X_{i}\right|=\left|F_{i}\right|$, thus fixing the value of $a$.

Note that $\mathbf{A}_{k} \prec C_{1}$. By bringing $X_{1}$ towards the centre and reflecting $F_{1}$, we can go from $X_{1} \wedge \mathbf{A}_{k} \wedge D_{1} \wedge E_{1} \wedge F_{1}$ to

$$
\overline{F_{1}} \wedge \overline{\mathbf{E}_{\mathbf{1}}} \wedge P_{1}
$$

for some decreasing $P_{1} \prec E_{1}$ (see Figure 23).
Note that $P_{1}$ consists of elements from $X_{1}, A_{k}$ and $D_{1}$, so $P_{1} \prec Z \wedge J$. Set $P_{1}$ aside by moving it all the way to the right, just before $Z$. Repeating this process for all the $C_{i}$, we can go from

$$
X^{\prime} \wedge L_{k} \wedge W_{k} \wedge \mathbf{A}_{k} \wedge B_{k} \wedge R_{k} \wedge J
$$

to

$$
L_{k} \wedge W_{k} \wedge \overline{F_{1}} \wedge \overline{F_{2}} \wedge \cdots \wedge \overline{F_{m}} \wedge \overline{\mathbf{E}_{\mathbf{m}}} \wedge P_{m} \wedge \cdots \wedge P_{1} \wedge Z \wedge R_{k} \wedge J
$$



Figure 23: Moving $C_{1}^{+}$across.

Bringing $Z \wedge J$ to the left, we can go to

$$
L_{k} \wedge W_{k} \wedge \overline{F_{1}} \wedge \cdots \wedge \overline{F_{m}} \wedge \overline{\mathbf{E}_{\mathbf{m}}} \wedge Z \wedge J \wedge P_{m} \wedge \cdots \wedge P_{1} \wedge R_{k}
$$

(see Figure 24).
Finally, shift $J$ to go from $\overline{\mathbf{E}_{\mathbf{m}}} \wedge Z \wedge J$ to $\mathbf{J} \wedge Z^{\prime}$, where $Z^{\prime} \prec 0$. Now sort $L_{k} \wedge W_{k} \wedge \overline{F_{1}} \wedge \cdots \wedge \overline{F_{m}}$ and $Z^{\prime} \wedge P_{m} \wedge \cdots \wedge P_{1} \wedge R_{k}$ in decreasing order to get the reverse of the identity. Since this must be a centred sequence $[a, b] \rightarrow \mathbb{Z}$, we must have $a=-b$. Moreover, we have $b=3 t+1+|Y|$ and, by Lemma 3.5, $|Y| \leq 10 d^{2 k+1} n$ with $k=d=100 T^{3}$ and $n=1$, giving $b=e^{e^{O(t)}}$, as required.


Figure 24: Almost done!

## 5 Concluding remarks

The main problem left open by our work is whether there are configurations like ours which can be realised by lines. In the literature, such configurations are called stretchable. While we suspect that our configurations, as given, are not stretchable, we firmly believe, and conjecture below, that there should be stretchable configurations, perhaps even suitable variants of our construction, where the number of points on either side of each line differs by at least $k$ for any given natural number $k$.

Conjecture 5.1. There exists an unbounded function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every natural number $n$ there is a set of $n$ points for which the number of points on either side of each line determined by the set differs by at least $f(n)$.

It may also be that we can take $f(n)=c \log \log n$ for some $c>0$, matching our Theorem 1.2. However, at present, we do not even know how to find point sets where the number of points on either side of each line determined by the set differs by at least 3 .

Our Theorem 1.2 also implies that another result of Pinchasi is essentially best possible, in that any improvement must somehow take into account the fact that one is dealing with lines rather than pseudolines. If we define $f(k)$ to be the maximum size of a finite point set in the plane, not contained in a line, with the property that there is no line through at least two of these points with at least $k$ points on either side of this line, then Pinchasi [11] showed that $f(k)<2 k+C \log \log k$ for some absolute constant $C$. Once again, this follows from a statement about allowable sequences and therefore holds in the broader context of generalised configurations. Moreover, in this form, the result easily implies Theorem 1.1, since if $n=2 k+C \log \log k$, then either every point is on some pseudoline, in which case we are done, or there exists a pseudoline with at least $k$ points on either side and the number of points on each side differs by at most $C \log \log k$. By the same argument, any improved bound for this result would give an analogous improvement to Theorem 1.1, but our Theorem 1.2 shows that this is already best possible.

## References

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