Difference sets in $\mathbb{R}^d$

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Abstract

Let $d \geq 2$ be a natural number. We show that

$$|A - A| \geq \left(2d - 2 + \frac{1}{d-1}\right)|A| - (2d^2 - 4d + 3)$$

for any sufficiently large finite subset A of $\mathbb{R}^d$ that is not contained in a translate of a hyperplane. By a construction of Stanchescu, this is best possible and thus resolves an old question first raised by Uhrin.

1 Introduction

Given two subsets A, B of an abelian group, the sumset $A + B$ is defined by

$$A + B = \{a + b : a \in A, b \in B\}$$

and the difference set $A - B$ is defined similarly. One of the fundamental results in additive combinatorics is Freiman's structure theorem, the statement that any finite set of integers A with small doubling, that is, with $|A+A| \leq K|A|$ for some fixed constant K, is contained in a generalised arithmetic progression of small size and dimension. The first step in Freiman's original proof [2] of this theorem is a simple lemma showing that if A is a finite d-dimensional subset of $\mathbb{R}^d$, then

$$|A + A| \geq (d + 1)|A| - d(d + 1)/2,$$

where we say that a subset A of $\mathbb{R}^d$ is *k-dimensional* and write $\dim(A) = k$ if the dimension of the affine subspace spanned by A is k. Freiman's result is tight, as may be seen by considering the union of d parallel arithmetic progressions with the same common difference.

Surprisingly, the analogous problem of estimating $|A - A|$ for d-dimensional subsets A of $\mathbb{R}^d$ has remained open, despite first being raised by Uhrin [13] in 1980 because of connections to the geometry of numbers and then reiterated many times (see, for example, [1, 3, 8, 9, 10]). However, the first few cases are well understood. Indeed, for $d = 1$, it is an elementary observation that $|A - A| \geq 2|A| - 1$, which is tight for arithmetic progressions, while, for $d = 2$, the bound $|A - A| \geq 3|A| - 3$, tight for the union of two parallel arithmetic progressions with the same length and

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common difference, was proven by Freiman, Heppes and Uhrin [3]. More generally, they showed that if A is a finite d-dimensional subset of $\mathbb{R}^d$, then

$$|A - A| \geq (d+1)|A| - d(d+1)/2,$$

in analogy with Freiman's result on $|A + A|$. This estimate was later generalised by Ruzsa [8], who showed that if $A, B \subset \mathbb{R}^d$ are finite sets such that $|A| \geq |B|$ and $\dim(A + B) = d$, then

$$|A + B| \geq |A| + d|B| - d(d+1)/2. \tag{1}$$

Finally, for $d = 3$, Stanchescu [9], making use of this inequality of Ruzsa, proved that $|A - A| \geq 4.5|A| - 9$ for any finite 3-dimensional subset A of $\mathbb{R}^3$. This is again tight, with the example now being a parallelogram of four parallel arithmetic progressions with the same length and common difference.

For higher dimensions, the best known construction is due to Stanchescu [10] and comes from a collection of $2d - 2$ carefully placed parallel arithmetic progressions with the same length and common difference. More precisely, set $T = \{e_0, e_1, \ldots, e_{d-2}\}$, where e_0 is the origin and $\{e_1, \ldots, e_d\}$ is the standard basis for $\mathbb{R}^d$, and, for any natural number k, let $A_k = (T \cup (a_k - T)) + P_k$, where $a_k = e_d - ke_{d-1}$ and $P_k = \{e_0, e_{d-1}, 2e_{d-1}, \ldots, (k-1)e_{d-1}\}$. Worked out carefully, this construction satisfies

$$|A_k - A_k| = \left(2d - 2 + \frac{1}{d-1}\right)|A_k| - (2d^2 - 4d + 3).$$

Supplanting an earlier conjecture of Ruzsa [8], Stanchescu proposed that this is best possible.

Conjecture 1.1 (Stanchescu [10])**.** *Suppose $d \geq 2$ and $A \subset \mathbb{R}^d$ is a finite set such that $\dim(A) = d$. Then*

$$|A - A| \geq \left(2d - 2 + \frac{1}{d-1}\right)|A| - (2d^2 - 4d + 3).$$

Until very recently, little was known about this conjecture for $d \geq 4$ besides the result of Freiman, Heppes and Uhrin [3]. However, the situation was considerably improved by Mudgal [6], who showed that

$$|A - A| \geq (2d - 2)|A| - o(|A|)$$

for any finite d-dimensional subset A of $\mathbb{R}^d$. Our main result, which builds on both Mudgal's work and earlier work of Stanchescu [9, 12], is a proof of Conjecture 1.1 in full provided only that $|A|$ is sufficiently large in terms of d, essentially resolving the problem of minimising the value of $|A - A|$ over all d-dimensional sets A of a given size.

Theorem 1.2. *Suppose $d \geq 2$ and $A \subset \mathbb{R}^d$ is a finite set such that $\dim(A) = d$. Then, provided $|A|$ is sufficiently large in terms of d,*

$$|A - A| \geq \left(2d - 2 + \frac{1}{d-1}\right)|A| - (2d^2 - 4d + 3).$$

We begin our proof of Theorem 1.2 in the next section with a result that we believe to be of independent interest, an extension of a result of Stanchescu [12] about the structure of d-dimensional subsets A of $\mathbb{R}^d$ with doubling constant smaller than $d + 4/3$ to asymmetric sums $A + B$.

2 An asymmetric version of a theorem of Stanchescu

Our starting point is with the following theorem of Stanchescu [12] (see also [11] for the $d = 3$ case).

Theorem 2.1 (Stanchescu [12]). *Suppose $d \geq 2$ and $A \subset \mathbb{R}^d$ is a finite set with $\dim(A) = d$. If $|A| > 3 \cdot 4^d$ and $|A + A| < (d + 4/3)|A| - \frac{1}{6}(3d^2 + 5d + 8)$, then A can be covered by d parallel lines.*

By considering the set $A = A_0 \cup \{e_3, \ldots, e_d\}$ with $A_0 = \{ie_1 + je_2 : 0 \leq i < n, 0 \leq j \leq 2\}$ for some natural number n, which satisfies $|A + A| = (d + 4/3)|A| - \frac{1}{6}(3d^2 + 5d + 8)$ and yet cannot be covered by d parallel lines, we see that Theorem 2.1 is tight. The main result of this section is an extension of Theorem 2.1 to asymmetric sums $A + B$. We begin with the two-dimensional case, whose proof relies in a critical way on the following result of Grynkiewicz and Serra [4, Theorem 1.3].

Lemma 2.2 (Grynkiewicz–Serra [4]). *Let $A, B \subset \mathbb{R}^2$ be finite sets, let l be a line, let r_1 be the number of lines parallel to l which intersect A and let r_2 be the number of lines parallel to l that intersect B. Then*

$$|A + B| \geq \left(\frac{|A|}{r_1} + \frac{|B|}{r_2} - 1\right)(r_1 + r_2 - 1).$$

In particular, we note that, since $|B| \geq r_2$ and $r_1 \geq 1$,

$$|A + B| \geq \frac{r_2}{r_1}|A|.$$

Lemma 2.3. *Let $A, B \subset \mathbb{R}^2$ be finite sets and l be a fixed line. Let r_1 be the number of lines parallel to l which intersect A. If $|A| \geq |B|$ and $|A + B| < |A| + 7|B|/3 - 5\sqrt{|A|}$, then either $r_1 \leq 2$ or $r_1 > |A|/4$.*

Proof. Notice that if A is at most 1 dimensional, then either $r_1 = 1$ or $r_1 = |A|$, so we may assume that $\dim(A) = 2$. Let r_2 be the number of lines parallel to l which intersect B. We consider 2 cases, depending on whether r_1 is at most $\sqrt{|A|}$ or not.

Case 1: $r_1 \leq \sqrt{|A|}$

We have $10|A|/3 \geq |A + B| \geq |A|r_2/r_1$, so $r_2 \leq 10r_1/3 \leq 4\sqrt{|A|}$. Thus, by Lemma 2.2 and the fact that $|A| \geq |B|$,

$$\begin{aligned} |A + B| &\geq \left(\frac{|A|}{r_1} + \frac{|B|}{r_2} - 1\right)(r_1 + r_2 - 1) \\ &= |A| + \frac{r_2 - 1}{r_1}|A| + \left(1 + \frac{r_1 - 1}{r_2}\right)|B| - r_1 - r_2 + 1 \\ &\geq |A| + \left(1 + \frac{r_2 - 1}{r_1} + \frac{r_1 - 1}{r_2}\right)|B| - 5\sqrt{|A|}. \end{aligned}$$

If $r_2 = 1$ and $r_1 \geq 3$, then this last expression is $|A| + r_1|B| - 5\sqrt{|A|} \geq |A| + 3|B| - 5\sqrt{|A|}$. If $r_2 = 2$ and $r_1 \geq 3$, then it is

$$|A| + \left(\frac{1}{2} + \frac{1}{r_1} + \frac{r_1}{2}\right)|B| - 5\sqrt{|A|} \geq |A| + \frac{7}{3}|B| - 5\sqrt{|A|}.$$

If $r_2 \geq 3$ and $r_1 \geq 3$, then it is at least

$$|A| + \left(3 - \frac{1}{r_1} - \frac{1}{r_2}\right)|B| - 5\sqrt{|A|} \geq |A| + \frac{7}{3}|B| - 5\sqrt{|A|}.$$

In each case, we contradict our assumption that $|A+B| < |A| + 7|B|/3 - 5\sqrt{|A|}$, so we must have $r_1 \leq 2$.

Case 2: $r_1 \geq \sqrt{|A|}$

Let $r'_1 = |A|/r_1$ and $r'_2 = |B|/r_2$, so that $r'_1 \leq \sqrt{|A|}$ and

$$|A+B| \geq \left(\frac{|A|}{r'_1} + \frac{|B|}{r'_2} - 1\right)(r'_1 + r'_2 - 1),$$

which is the same expression as in the previous case, but now r'_1, r'_2 may not be integers. Nevertheless, we still have $1 \leq r'_1 \leq |A|$ and $1 \leq r'_2 \leq |B|$, so that $|A+B| \geq \frac{r'_2}{r'_1}|A|$ and, therefore, $r'_2 \leq 4\sqrt{|A|}$ holds similarly. Expanding the equation above and using $|A| \geq |B|$, we have

$$\begin{aligned}|A+B| &\geq |A| + \left(1 + \frac{r'_2}{r'_1} + \frac{r'_1 - 1}{r'_2} - \frac{1}{r'_1}\right)|B| - 5\sqrt{|A|}\\ &\geq |A| + \left(1 + 2\sqrt{\frac{r'_1 - 1}{r'_1}} - \frac{1}{r'_1}\right)|B| - 5\sqrt{|A|}.\end{aligned}$$

Setting $c = \sqrt{\frac{r'_1 - 1}{r'_1}}$, we see that if $r_1 \leq |A|/4$ or, equivalently, $r'_1 \geq 4$, then $c \geq \frac{\sqrt{3}}{2}$ and the expression above is $|A| + (2c + c^2)|B| - 5\sqrt{|A|} \geq |A| + 7|B|/3 - 5\sqrt{|A|}$. But this again contradicts our assumption, so we must have $r_1 > |A|/4$. $\square$

For higher dimensions, we will use an induction scheme based on taking a series of compressions. Let us first say what a compression is in this context.

Definition 2.4. Let H be a hyperplane in $\mathbb{R}^d$ and $v \in \mathbb{R}^d$ a vector not parallel to H. For a finite set $A \subset \mathbb{R}^d$, the *compression of A onto H with respect to v*, denoted by $P(A) = P_{H,v}(A)$, is formed by replacing the points on any line l parallel to v which intersects A at $s \geq 1$ points with the points $u + jv$, $j = 0, 1, \ldots, s-1$, where u is the intersection of l with H.

By preserving the ordering of the points on each line, we may view the compression P as a pointwise map $A \to P(A)$, so we may talk about points of A being fixed by P. Note that it is clearly the case that $|P(A)| = |A|$. Moreover, sumsets cannot increase in size after applying this compression operation. That this is the case is our next result.

Lemma 2.5. *For finite sets $A, B \subset \mathbb{R}^d$ and a compression P,*

$$|P(A) + P(B)| \leq |A + B|.$$

Proof. Without loss of generality, we may assume that H passes through the origin. Let $p : \mathbb{R}^d \to H$ be the projection onto H along v. For $u \in p(A)$, let l_u be the line through u parallel to v and define $X_u = X \cap l_u$ for any set $X \subset \mathbb{R}^d$. Note that $p(P(A)) = p(A)$ and so $p(P(A) + P(B)) = p(A+B)$. It therefore suffices to show that $|(P(A)+P(B))_u| \le |(A+B)_u|$ for each $u \in p(A+B) = p(A)+p(B)$. Since $P(A)_x$ is a set of the form $\{x + jv \mid j = 0, \dots, s-1\}$, we have

$$\begin{aligned}
|(P(A)+P(B))_u| &= \max\{|P(A)_x + P(B)_y| \mid x \in p(A), y \in p(B), x+y=u\}\\
&= \max\{|P(A)_x| + |P(B)_y| - 1 \mid x \in p(A), y \in p(B), x+y=u\}\\
&= \max\{|A_x| + |B_y| - 1 \mid x \in p(A), y \in p(B), x+y=u\}\\
&\le |(A+B)_u|.
\end{aligned}$$

□

Our main compression lemma, which draws on ideas in the work of Stanchescu [11, 12], is now as follows.

Lemma 2.6. *Let $A, B \subset \mathbb{R}^d$ be finite sets such that $\dim(A) = d \ge 3$ and l be a fixed line. Suppose that there are exactly $s < |A|$ lines parallel to l which intersect A. Then there are sets $A', B' \subset \mathbb{R}^d$ satisfying the following properties:*

1. *$|A'| = |A|$, $|B'| = |B|$;*
2. *$|A' + B'| \le |A + B|$;*
3. *there are exactly s lines $l'_1, \dots, l'_s$ parallel to l intersecting A';*
4. *$\dim(A') = d$;*
5. *$l'_1, \dots, l'_{s-1}$ lie on a hyperplane;*
6. *l'_s intersects A' at a single point.*

Proof. The sets A', B' will be obtained by taking a series of compressions, so 1 and 2 will automatically be satisfied by Lemma 2.5. Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^d$. By applying an affine transformation if necessary, we may assume that l is the line $\mathbb{R}e_d$ and that A contains the set $S = \{0, e_1, \dots, e_d\}$ (this is possible since at least one line parallel to l intersects A in at least 2 points). For each i, let H_i be the hyperplane through 0 perpendicular to e_i. Let $P_i = P_{H_i, e_i}$ be the compression onto H_i with respect to e_i. Let $A_1 = P_d(A)$, noting that this set satisfies 3 and $s = |A_1 \cap H_d|$. Furthermore, for any compression P_i, $i < d$, $|P_i(A_1) \cap H_d| = s$, so $P_i(A_1)$ also satisfies 3. Now set $A_2 = P_1(P_2(\cdots P_{d-1}(A_1)\cdots))$. Then $A_2 \subset \mathbb{N}_0^d$ again satisfies 3 and, since $S \subseteq A_2$, $\dim(A_2) = d$ and it also satisfies 4. Moreover, A_2 has the property that if $(x_1, \dots, x_d) \in A_2$, then, for any $y_1, \dots, y_d \in \mathbb{N}_0$ with $y_i \le x_i$ for all i, $(y_1, \dots, y_d) \in A_2$.

We now show that a finite number of further compressions will give us a set additionally satisfying 5 and 6. Suppose A_2 can be covered by n hyperplanes parallel to H_{d-1}, i.e., the $(d-1)$th coordinate of all the points of A_2 is the set $\{0, 1, \dots, n-1\}$. Let $w = (w_1, \dots, w_{d-2}, 0, 0) \in A_2$ be such that $w_1 + \cdots + w_{d-2}$ is maximal. Then, whenever $tw + u \in A_2 \cap H_{d-1} \cap H_d$ for some $u \in \mathbb{N}_0^d$ and $t \ge 1$, we must have $u = 0$ and $t = 1$. Let P be the compression onto H_{d-1} with respect to $f = e_{d-1} - w$. Set $A_3 = P(A_2)$. Since f is parallel to H_d, $|A_3 \cap H_d| = |A_2 \cap H_d| = s$. The number of lines through A_3 parallel to l is $|A_3 \cap H_d| = s$, so 3 is still satisfied. Moreover, since $w \in A_2$, e_{d-1} is fixed by P, so $S \subseteq A_3$ and 4 is still satisfied. We now consider two cases:

<u>Case 1: $n = 2$</u>

We claim that A_3 is covered by H_{d-1} and the single line $e_{d-1} + \mathbb{R}e_d$, so that 5 is satisfied with $l'_s = e_{d-1} + \mathbb{R}e_d$. Indeed, by the maximality of $\|w\|_1$, the points of A_2 on any vertical line $u + \mathbb{R}e_d$

with $u \in H_d \setminus \{e_{d-1}\}$ are mapped by P into a vertical line contained in H_{d-1}. To see this, suppose $e_{d-1} + re_d + v \in A_2$ with $v \in H_{d-1} \cap H_d$ and $r \in \mathbb{N}_0$. Then $e_{d-1} + re_d + v$ is fixed by P iff $v + re_d + w \in A_2$. If $v \neq 0$, then $v + w \notin A_2$ by the maximality of w, so $v + re_d + w \notin A_2$ and $e_{d-1} + re_d + v$ is not fixed by the compression, being moved instead to $v + re_d + w$.

Case 2: $n > 2$

Suppose $(n-1)e_{d-1} + v \in A_2$ with $v \in H_{d-1}$. Then, since $(n-1)w + v \notin A_2$ as in Case 1, $(n-1)e_{d-1} + v$ is not fixed by the compression. Thus, A_3 is contained in fewer than n hyperplanes parallel to H_{d-1}. By repeatedly applying compressions of this type, we will eventually reach the previous case. Abusing notation very slightly, we shall still call the set obtained after these repeated compressions A_3.

Thus, A_3 is covered by H_{d-1} and the line $e_{d-1} + \mathbb{R}e_d$. Suppose now that $r > 0$ is the largest integer such that $re_d \in A_3$. Let P' be the compression with respect to $g = e_{d-1} - re_d$ and set $A_4 = P'(A_3)$. Then all points of A_3 in H_{d-1} and e_{d-1} are fixed by P', but $e_{d-1} + te_d$ is mapped to $(r+t)e_d$ for each $t > 0$. Thus, $A_4 \cap (e_{d-1} + H_{d-1}) = \{e_{d-1}\}$, so that A_4 satisfies 3-6. We may therefore set $A' = A_4$. Finally, to obtain B', we simply apply the same series of compressions to B that we applied to A. $\square$

We are now in a position to prove the main result of this section, the promised asymmetric version of Theorem 2.1.

Theorem 2.7. *Let $d \geq 2$, $A, B \subset \mathbb{R}^d$ be finite sets and l be a line. Let r be the number of lines parallel to l which intersect A. Suppose that A is d-dimensional, $|A| \geq |B|$ and $|A+B| < |A| + (d+1/3)|B| - 2^{d+1}\sqrt{|A|} - E_d$, where $E_d = (d+2)^{2^d-2}$. Then $r = d$ or $r > |A|/4$.*

Proof. Notice that since $\dim(A) = d$, we must have $r \geq d$. We shall induct on d. The case $d = 2$ was dealt with in Lemma 2.3. We may therefore assume that $d \geq 3$. E_d is chosen to satisfy the following inequalities:

1. $E_d \geq 2(E_{d-1} + 1)$,
2. $E_d \geq (d+2)(2^d + E_{d-1} + 1)^2$.

If $|A| \leq (2^d + E_{d-1} + 1)^2$, then $|A| + (d+1/3)|B| \leq (d+2)|A| \leq E_d$, so it is not possible that $|A+B| < |A| + (d+1/3)|B| - 2^{d+1}\sqrt{|A|} - E_d$. We may therefore assume that $|A| > (2^d + E_{d-1} + 1)^2$ and, thus, that $|A| - 2^d\sqrt{|A|} - E_{d-1} - 1 \geq 0$.

Suppose that $d < r \leq |A|/4$. By Lemma 2.6, replacing A with A', we can assume that $A = A_1 \cup \{e_d\}$, where A_1 lies on the hyperplane H defined by $x_d = 0$. Let $H_1, \ldots, H_s$ be the hyperplanes parallel to H that intersect B and let $B_i = B \cap H_i$.

If $s = 1$, then $|A+B| = |A_1 + B| + |B|$. Moreover, A_1 is $(d-1)$-dimensional and is covered by $r - 1 \leq |A_1|/4$ lines parallel to l. Thus, if $|B| \leq |A_1|$, our induction hypothesis implies that $|A_1 + B| \geq |A_1| + (d-1+1/3)|B| - 2^d\sqrt{|A_1|} - E_{d-1}$. If instead $|B| > |A_1|$, then $|B| = |A_1| + 1$, so, letting B' be B with an element removed, our induction hypothesis implies that $|A_1 + B| \geq |A_1 + B'| \geq |A_1| + (d-1+1/3)(|B|-1) - 2^d\sqrt{|A_1|} - E_{d-1}$. In either case, we have

$$\begin{aligned}|A+B| &\geq |A_1| + (d+1/3)(|B|-1) - 2^d\sqrt{|A_1|} - E_{d-1} \\ &\geq |A| + (d+1/3)|B| - 2^{d+1}\sqrt{|A|} - E_d.\end{aligned}$$

If $s \geq 2$, then $|A+B| \geq |A_1+B| = |A_1+B_1| + \cdots + |A_1+B_s|$. By our induction hypothesis, $|A_1+B_i| \geq |A_1| + (d-1+1/3)|B_i| - 2^d\sqrt{|A_1|} - E_{d-1}$ for each i and so

$$\begin{aligned}
|A+B| &\geq s|A_1| + (d-1+1/3)|B| - 2^d s\sqrt{|A_1|} - sE_{d-1} \\
&\geq 2|A| + (s-2)|A| - s + (d-1+1/3)|B| - 2^{d+1}\sqrt{|A|} - 2^d(s-2)\sqrt{|A|} - sE_{d-1} \\
&\geq |A| + (d+1/3)|B| - 2^{d+1}\sqrt{|A|} - 2(E_{d-1}+1) + (s-2)(|A| - 2^d\sqrt{|A|} - E_{d-1} - 1) \\
&\geq |A| + (d+1/3)|B| - 2^{d+1}\sqrt{|A|} - E_d. \qquad \square
\end{aligned}$$

3 Special cases of Theorem 1.2

In this section, we show that the conclusion of Theorem 1.2 holds if we make some additional assumptions about the structure of A. We begin with a simple example of such a result.

Lemma 3.1. *Let $A \subset \mathbb{R}^d$ be a finite set with* $\dim(A) = d$ *that can be covered by d parallel lines. Then*

$$|A-A| \geq \left(2d - 2 + \frac{2}{d}\right)|A| - (d^2 - d + 1).$$

Proof. Suppose $A = A_1 \cup \cdots \cup A_d$ where each A_i lies on a line parallel to some fixed line l. Let $a_i = |A_i|$ and assume, without loss of generality, that $a_1 \geq a_2 \geq \cdots \geq a_d$. Since A is d-dimensional, the d lines covering A are in general position, i.e., no k of them lie on a $(k-1)$-dimensional affine subspace for each $1 \leq k \leq d$. Thus, for $i \neq j$, the sets $A_i - A_j$ are pairwise disjoint and also disjoint from $A_1 - A_1$. Hence, we have

$$\begin{aligned}
|A-A| &\geq |A_1 - A_1| + \sum_{i\neq j} |A_i - A_j| \\
&\geq 2a_1 - 1 + \sum_{i \neq j}(a_i + a_j - 1) \\
&\geq 2a_1 - 1 + 2(d-1)\sum_i a_i - d(d-1) \\
&\geq \left(2d - 2 + \frac{2}{d}\right)|A| - (d^2 - d + 1). \qquad \square
\end{aligned}$$

We will use a common framework for the next two lemmas, with the following definition playing a key role.

Definition 3.2. Let $A \subset \mathbb{R}^d$ be a finite set with $\dim(A) = d$ and l be a fixed line. A hyperplane H is said to be a *supporting hyperplane* of A if all points of A either lie on H or on one side of H. A supporting hyperplane H of A is said to be a *major hyperplane of A (with respect to l)* if H is parallel to l and $|H \cap A|$ is maximal.

Suppose now that $A \subset \mathbb{R}^d$ is d-dimensional and l is a fixed line. Let H be a major hyperplane with respect to l and $H_1 = H, H_2, \ldots, H_r$ be the hyperplanes parallel to H that intersect A, arranged in the natural order. Let $A_i = A \cap H_i$ for $i = 1, \ldots, r$. Since $|A_1|$ is maximal, $|A_1| \geq |A_r|$. Let π be the projection along l onto a hyperplane perpendicular to l. Then $\dim(\pi(A)) = d-1$ and $\pi(H)$ is a maximal face of the convex hull of $\pi(A)$ (since $|H \cap A|$ is maximal), so $\dim(\pi(A_1)) = d-2$, which implies that there are at least $d-1$ lines parallel to l intersecting A_1. If any such line intersects A_1 in at least 2 points, then $\dim(A_1) = d-1$. Assuming this setup, the next lemma explores the situation where A is covered by two parallel hyperplanes.

Lemma 3.3. *Suppose that $r = 2$, $\dim(A_1) = d-1$ and there are s lines parallel to l intersecting A_1.*

1. *If $s = d-1$, then*

$$\begin{aligned} |A-A| &\geq (2d-2)\,|A| + \frac{2}{d-1}|A_1| - (2d^2-4d+3) \\ &\geq \left(2d-2+\frac{1}{d-1}\right)|A| - (2d^2-4d+3). \end{aligned}$$

2. *If $d \leq s \leq |A_1|/4$ and*

$$|A_1 - A_1| \geq \left(2d-4+\frac{1}{d-2}\right)|A_1| - (2d^2-8d+9),$$

then, given $0 < \epsilon < \min(\frac{2}{3}, \frac{1}{d-2}) - \frac{1}{d-1}$, there is some n_0 such that for $|A| \geq n_0$,

$$|A-A| \geq \left(2d-2+\frac{1}{d-1}+\epsilon\right)|A|.$$

Proof. For 1, note, by Lemma 3.1, that

$$|A_1 - A_1| \geq \left(2d-4+\frac{2}{d-1}\right)|A_1| - (d^2-3d+3).$$

By Ruzsa's inequality (1), $|A_1 - A_2| \geq |A_1| + (d-1)|A_2| - d(d-1)/2$ and so

$$\begin{aligned} |A-A| &\geq |A_1 - A_1| + 2|A_1 - A_2| \\ &\geq \left(2d-2+\frac{2}{d-1}\right)|A_1| + (2d-2)|A_2| - d(d-1) - (d^2-3d+3) \\ &\geq (2d-2)|A| + \frac{2}{d-1}|A_1| - (2d^2-4d+3) \\ &\geq \left(2d-2+\frac{1}{d-1}\right)|A| - (2d^2-4d+3). \end{aligned}$$

For 2, A_1 is $(d-1)$-dimensional and cannot be covered by $d-1$ lines, so this case only exists for $d \geq 3$. Since $|A_1| \geq |A_2|$, Theorem 2.7 implies that

$$|A_1 - A_2| \geq |A_1| + (d-2/3)|A_2| - 2^d\sqrt{|A_1|} - E_{d-1}.$$

But then, since $|A_1| \geq |A|/2$ can be taken sufficiently large,

$$\begin{aligned}
|A-A| &\geq |A_1 - A_1| + 2|A_1 - A_2| \\
&\geq \left(2d - 4 + \frac{1}{d-2}\right)|A_1| - (2d^2 - 8d + 9) + 2|A_1| + 2(d - 2/3)|A_2| - 2^{d+1}\sqrt{|A_1|} - 2E_{d-1} \\
&\geq \left(2d - 2 + \frac{1}{d-1} + \epsilon\right)|A| + \left(\frac{1}{(d-1)(d-2)} - \epsilon\right)|A_1| - (2d^2 - 8d + 9) - 2^{d+1}\sqrt{|A_1|} - 2E_{d-1} \\
&\geq \left(2d - 2 + \frac{1}{d-1} + \epsilon\right)|A|,
\end{aligned}$$

as required. □

We now consider the situation where every line parallel to l meets A in a reasonable number of points.

Lemma 3.4. *Let $0 < \epsilon < 1/(4d+1)(d-1)$. Suppose that every line parallel to l intersecting A intersects A in at least $4d$ points. Then there is a constant C_d such that either*

1.
$$|A-A| \geq \left(2d - 2 + \frac{1}{d-1} + \epsilon\right)|A| - C_d$$

   or

2. $r = 2$ *and*
$$|A-A| \geq (2d-2)\,|A| + \frac{2}{d-1}|H \cap A| - (2d^2 - 4d + 3).$$

In particular,

$$|A-A| \geq \left(2d - 2 + \frac{1}{d-1}\right)|A| - (2d^2 - 4d + 3)$$

for $|A|$ sufficiently large.

Proof. We shall induct on d and $|A|$. Let n_0 be chosen sufficiently large that the following conditions hold:

1. Lemma 3.3 holds with this n_0.
2. Whenever $B \subset \mathbb{R}^d$ has $\dim(B) = d - 1 > 1$, each line parallel to l intersecting B intersects it in at least $4(d-1)$ points and $|B| \geq n_0/2$, then
$$|B-B| \geq \left(2d - 4 + \frac{1}{d-2}\right)|B| - (2d^2 - 8d + 9).$$
   This is possible by induction since C_{d-1} is already determined.
3. $\epsilon n_0 \geq d(d-1)$.

Then $C_d \geq 2d^2 - 4d + 3$ is chosen sufficiently large that the first option in the lemma trivially holds for $|A| \leq n_0$.

The base case $d = 2$ and the inductive step will be handled together. If $|A| \leq n_0$, the lemma holds, so we may assume that $|A| > n_0$. Since $\dim(A_1) = d-1$, there are at least $d-1$ lines parallel to l intersecting A_1. Each such line intersects A_1 in at least $4d$ points, so we have $|A_1| \geq 4d(d-1)$.

First suppose $r = 2$. If A_1 is covered by s lines parallel to l, then, as above, $s \geq d-1$. If $s = d-1$, then, by Lemma 3.3,

$$|A - A| \geq (2d-2)|A| + \frac{2}{d-1}|A_1| - (2d^2 - 4d + 3).$$

If $s > d-1$, then we must have $d > 2$, since, for $d = 2$, $\dim(A_1) = 1$ and A_1 is covered by a single line. Since $\dim(A_1) = d - 1 > 1$ and $|A_1| \geq |A|/2 \geq n_0/2$, condition 2 implies that

$$|A_1 - A_1| \geq \left(2d - 4 + \frac{1}{d-2}\right)|A_1| - (2d^2 - 8d + 9).$$

Each line parallel to l passes through at least 4 points of A_1, so $s \leq |A_1|/4$. Thus, by Lemma 3.3 and condition 1,

$$|A - A| \geq \left(2d - 2 + \frac{1}{d-1} + \epsilon\right)|A|.$$

Now suppose $r > 2$. Let $B = A \setminus H_r$ and note that $\dim(B) = d$ and $|B| \geq |A|/2$. By our induction hypothesis,

$$|B - B| \geq \left(2d - 2 + \frac{1}{d-1}\right)|B| - C_d.$$

Let H' be a major hyperplane of B with respect to l (which is not necessarily a major hyperplane of A!), so that $|B \cap H'| \geq |A_1|$. If $|A_1| \geq 2\epsilon|A|$, then, using Ruzsa's inequality (1) and condition 3,

$$\begin{aligned}
|A - A| &\geq |B - B| + 2|A_1 - A_r| \\
&\geq \left(2d - 2 + \frac{1}{d-1}\right)|B| - C_d + 2|A_1| + (2d-2)|A_r| - d(d-1) \\
&\geq \left(2d - 2 + \frac{1}{d-1}\right)|A| + \left(2 - \frac{1}{d-1}\right)|A_1| - C_d - d(d-1) \\
&\geq \left(2d - 2 + \frac{1}{d-1} + 2\epsilon\right)|A| - C_d - d(d-1) \\
&\geq \left(2d - 2 + \frac{1}{d-1} + \epsilon\right)|A| - C_d.
\end{aligned}$$

We may therefore assume that $|A_1| < 2\epsilon|A|$.

If B cannot be covered by two translates of H', then, by our induction hypothesis,

$$|B - B| \geq \left(2d - 2 + \frac{1}{d-1} + \epsilon\right)|B| - C_d.$$

Thus, again using Ruzsa's inequality (1),

$$\begin{aligned}
|A-A| &\geq |B-B| + 2|A_1 - A_r| \\
&\geq \left(2d-2+\frac{1}{d-1}+\epsilon\right)|B| + 2|A_1| + (2d-2)|A_r| - d(d-1) - C_d \\
&\geq \left(2d-2+\frac{1}{d-1}+\epsilon\right)|A| + \left(2-\frac{1}{d-1}-\epsilon\right)|A_1| - d(d-1) - C_d \\
&\geq \left(2d-2+\frac{1}{d-1}+\epsilon\right)|A| - C_d,
\end{aligned}$$

since $|A_1| \geq 4d(d-1)$.

We may therefore assume that B is covered by two translates of H', say H' and H''. If $A_r \subseteq H' \cup H''$, then $A \subseteq H' \cup H''$, so one of $|A \cap H'|, |A \cap H''|$ is at least $|A|/2$, say $|A \cap H'| \geq |A|/2$. But H is a major hyperplane of A, so $|A_1| = |A \cap H| \geq |A \cap H'| \geq |A|/2$, contradicting our assumption that $|A_1| < 2\epsilon|A|$. Hence, $A_r \not\subseteq H' \cup H''$.

If

$$|B-B| \geq \left(2d-2+\frac{1}{d-1}+\epsilon\right)|B| - C_d,$$

then the above argument holds similarly. Thus, by our induction hypothesis, we must have that

$$|B-B| \geq (2d-2)|B| + \frac{2}{d-1}|H' \cap B| - (2d^2-4d+3).$$

Let $B_1 = B \cap H', B_2 = B \cap H''$, noting that $|B_1| \geq |B_2|$. Fix also a point $x \in A_r$ that does not lie on $H' \cup H''$. If x lies between H' and H'', then $x - B_1, B_1 - x, B - B$ are pairwise disjoint. If H' lies between x and H'', then $x - B_2, B_2 - x, B - B$ are pairwise disjoint. If H'' lies between x and H', then $x - B_1, B_1 - x, B - B$ are pairwise disjoint. In any case, there is some $i \in \{1, 2\}$ such that $x - B_i, B_i - x, B - B$ are pairwise disjoint. Since $|B_1| \geq |B_2|$,

$$\begin{aligned}
|A-A| &\geq |B-B| + 2|B_2| \\
&\geq (2d-2)|B| + \frac{2}{d-1}|B_1| - (2d^2-4d+3) + 2|B_2| \\
&\geq \left(2d-2+\frac{2}{d-1}\right)|B| - (2d^2-4d+3) \\
&= \left(2d-2+\frac{2}{d-1}\right)(|A| - |A_r|) - (2d^2-4d+3) \\
&\geq \left(2d-2+\frac{1}{d-1}+\epsilon\right)|A| - C_d,
\end{aligned}$$

where the last inequality follows from $|A_r| \leq |A_1| \leq 2\epsilon|A|$ and $\epsilon < 1/(4d+1)(d-1)$. □

4 Proof of Theorem 1.2

The final ingredient in our proof is the following structure theorem due to Mudgal [5, Lemma 3.2], saying that sets with small doubling in $\mathbb{R}^d$ can be almost completely covered by a reasonably small collection of parallel lines.

Lemma 4.1 (Mudgal [5])**.** *For any $c > 0$, there exist constants $0 < \sigma \leq 1/2$ and $C > 0$ such that if $A \subset \mathbb{R}^d$ is a finite set with $|A| = n$ and $|A + A| \leq cn$, then there exist parallel lines $l_1, l_2, \ldots, l_r$ with*

$$|A \cap l_1| \geq \cdots \geq |A \cap l_r| \geq |A \cap l_1|^{1/2} \geq C^{-1} n^{\sigma}$$

and

$$|A \setminus (l_1 \cup l_2 \cup \cdots \cup l_r)| < Ccn^{1-\sigma}.$$

We are now ready to prove Theorem 1.2, which, we recall, states that if $d \geq 2$ and $A \subset \mathbb{R}^d$ is a finite set such that $\dim(A) = d$, then, provided $|A|$ is sufficiently large,

$$|A - A| \geq \left(2d - 2 + \frac{1}{d-1}\right)|A| - (2d^2 - 4d + 3).$$

Proof of Theorem 1.2. We shall proceed by induction on d, starting from the known case $d = 2$ [3]. We will suppose throughout that n_0 is large enough for our arguments to hold. Our aim is to show that, for all $A \subset \mathbb{R}^d$ with $\dim(A) = d$,

$$|A - A| \geq \left(2d - 2 + \frac{1}{d-1}\right)|A| - \max(2d^2 - 4d + 3, D - |A|/3),$$

where $D \geq 2d^2 - 4d + 3$ is chosen so that the above inequality trivially holds for $|A| \leq n_0$. The result then clearly follows for $|A|$ sufficiently large. We will proceed by induction on $|A|$, where the base case $|A| \leq n_0$ trivially holds.

We may clearly assume that $|A - A| \leq (2d-1)|A|$, since otherwise we already have the required conclusion. By the Plünnecke–Ruzsa inequality, we then have $|A + A| \leq (2d-1)^2|A|$. Applying Lemma 4.1 with $c = (2d-1)^2$, we get parallel lines $l_1, \ldots, l_r$ and constants $0 < \sigma \leq 1/2$ and $C > 0$ such that

$$|A \cap l_1| \geq \cdots \geq |A \cap l_r| \geq |A \cap l_1|^{1/2} \geq C^{-1} n^{\sigma}$$

and

$$|A \setminus (l_1 \cup l_2 \cup \cdots \cup l_r)| < Ccn^{1-\sigma},$$

where $n = |A|$. Since $|A \cap l_i| \geq C^{-1} n^{\sigma}$ for each i, we have $n = |A| \geq rC^{-1}n^{\sigma}$ or $r \leq Cn^{1-\sigma}$. Let $A' = A \cap (l_1 \cup \cdots \cup l_r)$ and $S = A \setminus A'$, so that $|S| < Ccn^{1-\sigma}$. If $\dim(A') = d_1 < d$, then, by our induction hypothesis, for $|A|$ sufficiently large,

$$|A' - A'| \geq \left(2d_1 - 2 + \frac{1}{d_1 - 1}\right)|A'| - (2d_1^2 - 4d_1 + 3).$$

There are $a_1, \ldots, a_{d-d_1} \in S$ such that $\dim(A' \cup \{a_1, \ldots, a_{d-d_1}\}) = d$. This implies that $a_1, \ldots, a_{d-d_1}$ lie outside the affine span of A', so the sets

$$A' - A', A' - a_1, \ldots, A' - a_{d-d_1}, a_1 - A', \ldots, a_{d-d_1} - A'$$

are pairwise disjoint. Thus,

$$\begin{aligned}|A-A| &\geq |A'-A'| + \sum_{i=1}^{d-d_1}(|A'-a_i| + |a_i - A'|)\\ &\geq \left(2d_1 - 2 + \frac{1}{d_1-1}\right)|A'| - (2d_1^2 - 4d_1 + 3) + 2(d-d_1)|A'|\\ &\geq \left(2d - 2 + \frac{1}{d_1-1}\right)(|A| - |S|) - (2d_1^2 - 4d_1 + 3)\\ &\geq \left(2d - 2 + \frac{1}{d-1}\right)|A|\end{aligned}$$

for $|A| \geq n_0$ sufficiently large. Thus, we may assume that $\dim(A') = d$.

For n_0 sufficiently large, we may assume that each line l_i intersects A' in at least $4d$ points. Let H be a major hyperplane of A' with respect to l_1 and let $H_1 = H, H_2, \ldots, H_r$ be the translates of H covering A' in the natural order. Fix $0 < \epsilon < 1/(4d+1)(d-1)$. If we are in the case of Lemma 3.4 where

$$|A'-A'| \geq \left(2d - 2 + \frac{1}{d-1} + \epsilon\right)|A'| - C_d,$$

then, since $|S| = O(|A|^{1-\sigma})$ is sublinear, for $|A|$ sufficiently large,

$$\begin{aligned}|A-A| &\geq |A'-A'|\\ &\geq \left(2d - 2 + \frac{1}{d-1} + \epsilon\right)|A'| - C_d\\ &\geq \left(2d - 2 + \frac{1}{d-1}\right)|A|.\end{aligned}$$

Thus, we may assume that $r = 2$ and

$$|A'-A'| \geq (2d-2)|A'| + \frac{2}{d-1}|A_1'| - (2d^2 - 4d + 3).$$

Let $A_1' = A' \cap H_1$ and $A_2' = A' \cap H_2$. If $S \not\subseteq H_1 \cup H_2$, then there is a point $x \in S$ not lying on the hyperplanes H_1, H_2. But then $x - A_i', A_i' - x, A' - A'$ are pairwise disjoint for some $i \in \{1, 2\}$ and so, since $|A_1'| \geq |A_2'|$,

$$\begin{aligned}|A-A| &\geq |A'-A'| + 2|A_2'|\\ &\geq (2d-2)|A'| + \frac{2}{d-1}|A_1'| - (2d^2 - 4d + 3) + 2|A_2'|\\ &\geq \left(2d - 2 + \frac{2}{d-1}\right)|A'| - (2d^2 - 4d + 3)\\ &\geq \left(2d - 2 + \frac{1}{d-1}\right)|A|.\end{aligned}$$

We may therefore assume that $S \subseteq H_1 \cup H_2$.

Let $A_1 = A \cap H_1$ and $A_2 = A \cap H_2$. Let H' be a major hyperplane of A with respect to l_1 (possibly equal to H) and $H_1' = H', H_2', \ldots, H_s'$ be the translates of H' covering A, ordered naturally. Let $B_i = A \cap H_i'$ for $i = 1, \dots, s$. Since H_1, H_2 are both supporting hyperplanes of A, we must have $|B_1| \geq \max(|A_1|, |A_2|) \geq |A|/2 > |S|$, so B_1 must contain at least one point of A'. Hence, B_1 contains one of the lines $l_i \cap A$, each of which has at least 2 points, and so $\dim(B_1) = d - 1$.

Suppose $s = 2$. The number of lines parallel to l_1 intersecting B_1 is at most $r + |S| = O(|A|^{1-\sigma})$, which is smaller than $|B_1|/4$. Thus, for n_0 sufficiently large, by both cases of Lemma 3.3,

$$|A - A| \geq \left(2d - 2 + \frac{1}{d-1}\right)|A| - (2d^2 - 4d + 3).$$

We may therefore assume that $s > 2$. Let $B = A \setminus B_s$, noting that $|B| \geq |A|/2$ and $\dim(B) = d$. By our induction hypothesis,

$$|B - B| \geq \left(2d - 2 + \frac{1}{d-1}\right)|B| - D.$$

Thus, again using Ruzsa's inequality (1),

$$\begin{aligned}
|A - A| &\geq |B - B| + 2|B_1 - B_s| \\
&\geq \left(2d - 2 + \frac{1}{d-1}\right)|B| - D + 2|B_1| + (2d-2)|B_s| - d(d-1) \\
&\geq \left(2d - 2 + \frac{1}{d-1}\right)|A| + \left(2 - \frac{1}{d-1}\right)|B_1| - d(d-1) - D \\
&\geq \left(2d - 2 + \frac{1}{d-1}\right)|A| + \left(1 - \frac{1}{2(d-1)}\right)|A| - d(d-1) - D \\
&\geq \left(2d - 2 + \frac{1}{d-1}\right)|A| - D + |A|/3,
\end{aligned}$$

where the last inequality holds if $|A|/6 \geq n_0/6 \geq d(d-1)$. □

5 Concluding remarks

By carefully analysing our proof of Theorem 1.2, it is possible to deduce some structural properties of large sets $A \subset \mathbb{R}^d$ with $\dim(A) = d$ and

$$|A - A| \leq \left(2d - 2 + \frac{1}{d-1}\right)|A| + o(|A|).$$

In particular, such sets can be covered by two parallel hyperplanes H_1 and H_2, where, writing $A_1 = A \cap H_1$ and $A_2 = A \cap H_2$, we can assume that A_1 and A_2 have roughly the same size, differing by $o(|A|)$. We can also assume that $\dim(A_1) = d - 1$ and that A_1 can be covered by $d - 1$ parallel lines $l_1, \dots, l_{d-1}$, where the sets $A_1 \cap l_i$ all have approximately equal size, again up to $o(|A|)$.

In practice, H_1 will be a major hyperplane of A with respect to l_1, which, we recall, means that it is parallel to l_1, it is supporting, in the sense that all points of A lie either on or on one side of it, and $|H_1 \cap A|$ is as large as possible. Knowing this allows us to also deduce that $\dim(A_2) = d - 1$. Indeed, it must be the case that the affine span of A_2 is parallel to l_1, since otherwise $|A_1 - A_2|$ would be too large. But then, if $\dim(A_2) < d - 1$, there is a supporting hyperplane through A_2 and one of the $A_1 \cap l_i$ which contains more points than H_1, contradicting the fact that H_1 is a major hyperplane. Since $|A_1|$ and $|A_2|$ differ by $o(|A|)$, this then allows us to argue that A_2 is also covered by $d-1$ lines parallel to l_1 of approximately equal size.

In fact, we can deduce the very same structural properties for large sets $A \subset \mathbb{R}^d$ with $\dim(A) = d$ and

$$|A - A| \leq \left(2d - 2 + \frac{1}{d-1} + \epsilon\right)|A| + o(|A|)$$

for some $\epsilon > 0$, giving a difference version of Stanchescu's result about the structure of d-dimensional subsets of $\mathbb{R}^d$ with doubling constant smaller than $d + 4/3$, which we stated as Theorem 2.1. It would be interesting to determine the maximum value of ϵ for which this continues to hold.

Unfortunately, our methods tell us very little about how A_1 and A_2 are related, though we suspect that A_2 should be close to a translate of $-A_1$. Proving this, which will likely require a better understanding of when Ruzsa's inequality (1) is tight, may then lead to a determination of the exact structure of d-dimensional subsets A of $\mathbb{R}^d$ with $|A - A|$ as small as possible in terms of $|A|$, a problem that was already solved for $d = 2$ and 3 by Stanchescu [9].

Note added. Shortly after completing this paper, we learned from Akshat Mudgal that he had independently proved an asymptotic version of Conjecture 1.1. We refer the reader to his paper [7] for further details.

References

[1] E. S. Croot and V. F. Lev, Open problems in additive combinatorics, in Additive combinatorics, 207–233, CRM Proc. Lecture Notes, 43, American Mathematical Society, Providence, RI, 2007.

[2] G. A. Freiman, Foundations of a structural theory of set addition, Translations of Mathematical Monographs, Vol. 37, American Mathematical Society, Providence, RI, 1973.

[3] G. Freiman, A. Heppes and B. Uhrin, A lower estimation for the cardinality of finite difference sets in $\mathbb{R}^n$, in Number theory, Vol. I (Budapest, 1987), 125–139, Colloq. Math. Soc. János Bolyai, 51, North-Holland, Amsterdam, 1990.

[4] D. Grynkiewicz and O. Serra, Properties of two-dimensional sets with small sumset, *J. Combin. Theory Ser. A* **117** (2010), 164–188.

[5] A. Mudgal, Sums of linear transformations in higher dimensions, *Q. J. Math.* **70** (2019), 965–984.

[6] A. Mudgal, Difference sets in higher dimensions, *Math. Proc. Cambridge Philos. Soc.* **171** (2021), 467–480.

[7] A. Mudgal, New lower bounds for cardinalities of higher dimensional difference sets and sumsets, *Discrete Anal.* (2022), Paper No. 15, 19 pp.

[8] I. Z. Ruzsa, Sum of sets in several dimensions, *Combinatorica* **14** (1994), 485–490.

[9] Y. Stanchescu, On finite difference sets, *Acta Math. Hungar.* **79** (1998), 123–138.

[10] Y. V. Stanchescu, An upper bound for d-dimensional difference sets, *Combinatorica* **21** (2001), 591–595.

[11] Y. V. Stanchescu, Three-dimensional sets with small sumset, *Combinatorica* **28** (2008), 343–355.

[12] Y. V. Stanchescu, The structure of d-dimensional sets with small sumset, *J. Number Theory* **130** (2010), 289–303.

[13] B. Uhrin, Some useful estimations in the geometry of numbers, *Period. Math. Hungar.* **11** (1980), 95–103.