HYPERGRAPHS ACCUMULATE INFINITELY OFTEN

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ABSTRACT. We show that the set $\Pi^{(k)}$ of Turán densities of k-uniform hypergraphs has infinitely many accumulation points in $[0,1)$ for every $k \geqslant 3$. This extends an earlier result of ours showing that $\Pi^{(k)}$ has at least one such accumulation point.

§1. INTRODUCTION

For $k \in \mathbb{N}$, a *k-uniform hypergraph* (or *k-graph*) $H = (V, E)$ consists of a vertex set V and an edge set $E \subseteq V^{(k)} = \{e \subseteq V : |e| = k\}$. Given $n \in \mathbb{N}$ and a family of k-graphs $\mathcal{F}$, the *extremal number* $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in a k-graph H with n vertices that does not contain a copy of any graph in $\mathcal{F}$. The *Turán density* of $\mathcal{F}$ is then given by

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{k}},$$

where the limit is known, by a simple monotonicity argument [5], to be well-defined. If $\mathcal{F} = \{F\}$ for some k-graph F, we omit the parentheses, writing $\pi(\mathcal{F}) = \pi(F)$. The problem of determining these Turán densities is one of the oldest and most fundamental questions in extremal combinatorics.

When $k = 2$, that is, when $\mathcal{F}$ is a family of graphs, $\pi(\mathcal{F})$ is essentially completely understood, with the final result, the culmination of work by Turán [10], Erdős and Stone [3], and Erdős and Simonovits [2], saying that $\pi(\mathcal{F}) = \frac{\chi(\mathcal{F})-2}{\chi(\mathcal{F})-1}$, where $\chi(\mathcal{F})$ is the minimum chromatic number of an element of $\mathcal{F}$. If we set

$$\begin{aligned}
\Pi^{(k)} &= \{\pi(F) : F \text{ is a } k\text{-graph}\}\,, \\
\Pi^{(k)}_{\mathrm{fin}} &= \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a finite family of } k\text{-graphs}\}\,, \\
\Pi^{(k)}_{\infty} &= \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a family of } k\text{-graphs}\}\,,
\end{aligned}$$

then the Erdős–Stone–Simonovits theorem implies that

$$\Pi^{(2)} = \Pi^{(2)}_{\mathrm{fin}} = \Pi^{(2)}_{\infty} = \{0, 1/2, 2/3, 3/4, \dots\}.$$

In particular, each of these sets is well-ordered.

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For $k \geqslant 3$, there is no clear analogue of the Erdős–Stone–Simonovits theorem for any of the sets $\Pi^{(k)}$, $\Pi^{(k)}_{\text{fin}}$, or $\Pi^{(k)}_{\infty}$. For $\Pi^{(k)}_{\infty}$, this was shown by Frankl and Rödl [4], who proved that $\Pi^{(k)}_{\infty}$ is not well-ordered. That is, the set has downward accumulation points. In particular, this disproved the jumping conjecture of Erdős, which suggested that, like for $\Pi^{(2)}_{\infty}$, there should be a non-trivial gap or jump between any element of $[0,1)$ and the next element of $\Pi^{(k)}_{\infty}$. By applying the important result of Pikhurko [9] that $\Pi^{(k)}_{\infty}$ is the closure of $\Pi^{(k)}_{\text{fin}}$, it is easily seen that $\Pi^{(k)}_{\text{fin}}$ is also not well-ordered.

While it remains an intriguing open problem to show that $\Pi^{(k)}$ is again not well-ordered, a first step showing that $\Pi^{(k)}$ is indeed more complex than $\Pi^{(2)}$ was taken in the recent paper [1], where we proved the following result.

Theorem 1.1. *For every integer $k \geqslant 3$, the set $\Pi^{(k)}$ has an accumulation point in $[0,1)$.*

Here we extend this result, showing that $\Pi^{(k)}$ has infinitely many accumulation points. This goes another step further in showing how much more complex $\Pi^{(k)}$ gets for $k \geqslant 3$.

Theorem 1.2. *For every integer $k \geqslant 3$, the set $\Pi^{(k)}$ has infinitely many accumulation points in $[0,1)$.*

This is a consequence of the following result which states that, in addition, for each of these accumulation points α there is a family of k-graphs whose Turán density is α.

Theorem 1.3. *For every integer $k \geqslant 3$, there are infinitely many $\alpha \in [0,1)$ such that there are two sequences of k-graphs, $\{F_i\}_{i\in\mathbb{N}}$ and $\{G_i\}_{i\in\mathbb{N}}$, with the following properties:*

I. $\pi(F_i) \to \alpha$ and $\pi(F_i) < \alpha$ for all $i \in \mathbb{N}$.
II. For all $\varepsilon > 0$, there is some $i \in \mathbb{N}$ such that $\alpha \leqslant \pi(G_i) \leqslant \alpha + \varepsilon$.

The proof of this result will occupy the remainder of this short paper.

§2. Preliminaries

Given an integer t and a k-graph F, let $B(F,t)$ be the *t-blow-up* of F, the k-graph obtained from F by replacing every vertex by t copies of itself. The following phenomenon, which we make extensive use of, is well-known (see, for instance, Lemma 2.1 and Theorem 2.2 in [6], as well as the subsequent discussion).

Theorem 2.1 (Supersaturation). *(1) For every k-graph F and $\delta > 0$, there are $\varepsilon > 0$ and n_0 such that every k-graph on $n \geqslant n_0$ vertices with at least $(\pi(F)+\delta)\binom{n}{k}$ edges contains at least $\varepsilon n^{|V(F)|}$ copies of F.*

(2) For every integer t and k-graph F, $\pi(B(F,t)) = \pi(F)$.

(3) Let F be a k-graph and $\mathcal{F}$ be the (finite) family of k-graphs F' whose vertex set is a subset of $V(F)$ and for which there exists a homomorphism $\varphi : F \to F'$. Then $\pi(\mathcal{F}) = \pi(F)$.

(4) For every k-graph F and $\delta > 0$, there are $\varepsilon > 0$ and n_0 such that, for all $v \in V(F)$, every k-graph on $n \geqslant n_0$ vertices with at least $(\pi(F) + \delta)\binom{n}{k}$ edges contains the k-graph obtained from F by replacing v by εn copies of v.

We will also make use of expansions of hypergraphs. Setting $X = \{x_1, \dots, x_s\}$, the k-uniform *expansion* of $K_s^{(2)}$, the complete 2-graph on s vertices, is the k-graph $G_s^{(k)}$ with vertex set

$$X \,\dot\cup\, \{v_i^e :\ i \in [k-2],\ e \in X^{(2)}\}$$

and edge set

$$\{e \cup \{v_1^e, \dots, v_{k-2}^e\} :\ e \in X^{(2)}\}.$$

In other words, the k-uniform expansion of $K_s^{(2)}$ is obtained from $K_s^{(2)}$ by adding $k-2$ new vertices to each edge. We will need the following result of Mubayi [7] determining the Turán density of these expansions.

Theorem 2.2. *For all integers $s > k \geqslant 2$, $\pi(G_s^{(k)}) = \frac{(s-1)\cdot(s-2)\cdots(s-k)}{(s-1)^k}$.*

§3. Proof of Theorem 1.3

The proof makes use of some k-graphs that are obtained by gluing ladders and zycles, both of which we now define, in appropriate ways.

For $k, \ell \in \mathbb{N}$, we define the k-uniform *ladder of length ℓ* to be the k-graph $L_\ell^{(k)}$ with vertex set

$$V(L_\ell^{(k)}) = \{v_{ij} :\ i \in [\ell],\ j \in [k-1]\} \,\dot\cup\, \{t\}$$

and edge set

$$E(L_\ell^{(k)}) = \{v_{i1} \dots v_{ik-1} v_{i+1j} :\ i \in [\ell-1],\ j \in [k-1]\} \cup \{v_{\ell 1} \dots v_{\ell k-1} t\}\,.$$

For $k, \ell \in \mathbb{N}$ with $\ell \geqslant 2$, we define the k-uniform *zycle of length ℓ* to be the k-graph $Z_\ell^{(k)}$ with vertex set

$$V(Z_\ell^{(k)}) = \{v_{ij} : i \in \mathbb{Z}/\ell\mathbb{Z}, j \in [k-1]\}$$

and edge set

$$E(Z_\ell^{(k)}) = \{v_{i1} \dots v_{ik-1} v_{i+1j} :\ i \in \mathbb{Z}/\ell\mathbb{Z},\ j \in [k-1]\}\,.$$

Next we define a k-graph that, roughly speaking, is obtained by gluing a zycle of length ℓ to the last $(k-1)$-set of a ladder of length m. More formally, for $k, m, \ell \in \mathbb{N}$ with $\ell \geqslant 2$, the k-graph $LZ^{(k)}(m,\ell)$ is the k-graph with vertex set

$$\begin{aligned} V(LZ^{(k)}(m,\ell)) = &\{v_{ij} : \ i \in [m],\ j \in [k-1]\} \\ &\dot\cup \{w_{ij} : \ i \in [\ell] \setminus \{1\},\ j \in [k-1]\} \end{aligned}$$

and edge set

$$\begin{aligned} E(LZ^{(k)}(m,\ell)) = &\{v_{i1}\dots v_{ik-1}v_{i+1j} : \ i \in [m-1],\ j \in [k-1]\} \\ &\cup \{v_{m1}\dots v_{mk-1}w_{2j} : \ j \in [k-1]\} \\ &\cup \{w_{i1}\dots w_{ik-1}w_{i+1j} : \ i \in [\ell-1] \setminus \{1\},\ j \in [k-1]\} \\ &\cup \{w_{\ell 1}\dots w_{\ell k-1}v_{mj} : \ j \in [k-1]\}\,. \end{aligned}$$

For each of $L_\ell^{(k)}$, $Z_\ell^{(k)}$, and $LZ^{(k)}(m,\ell)$, we sometimes refer to the set $\{v_{11},\dots,v_{1k-1}\}$ as the *starting set*.

If, in addition, $s \in \mathbb{N}$ with $s > k$, we consider the k-graph obtained from an s-set X by adding, for each pair $e \in X^{(2)}$, a copy of $L_{\ell_e}^{(k)}$ such that these copies only intersect in vertices of X, where the length ℓ_e may depend on the pair e. More formally, given a set $X = \{x_1,\dots,x_s\}$, let $\{e_1,\dots,e_{\binom{s}{2}}\} = X^{(2)}$ be an enumeration of the pairs of elements in X. Furthermore, let $\ell_1,\dots,\ell_{\binom{s}{2}} \in \mathbb{N}$ with $\ell_i \geqslant \ell_{i+1}$ for all $i \in [\binom{s}{2}-1]$. For $x_ix_j = e \in X^{(2)}$ with $i < j$, we write $v^e_{1k-2} = x_i$ and $v^e_{1k-1} = x_j$. For all $e \in X^{(2)}$ and $j \in [k-3]$, let v^e_{1j} be pairwise distinct vertices which are also distinct from any vertex in X. Finally, for all $e_r \in X^{(2)}$, $i \in [\ell_r] \setminus \{1\}$, and $j \in [k-1]$, let $v^{e_r}_{ij}$ be distinct vertices which are also distinct from any previously chosen vertices. Then we define $GL^{(k)}(s;\ell_1,\dots,\ell_{\binom{s}{2}})$ to be the k-graph with vertex set

$$\begin{aligned} V(GL^{(k)}(s;\ell_1,\dots,\ell_{\binom{s}{2}})) = &\{v^{e_r}_{ij} : \ e_r \in X^{(2)},\ i \in [\ell_r],\ j \in [k-1]\} \\ &\dot\cup \{t^e : e \in X^{(2)}\} \end{aligned} \tag{3.1}$$

and edge set

$$\begin{aligned} E(GL^{(k)}(s;\ell_1,\dots,\ell_{\binom{s}{2}})) = &\{v^{e_r}_{i1}\dots v^{e_r}_{ik-1}v^{e_r}_{i+1j} : \ e_r \in X^{(2)},\ i \in [\ell_r-1],\ j \in [k-1]\} \\ &\cup \{v^{e_r}_{\ell_r 1}\dots v^{e_r}_{\ell_r k-1}t^{e_r} : \ e_r \in X^{(2)}\}\,. \end{aligned}$$

If $\ell_i = \ell$ for all $i \in [\binom{s}{2}]$, we simply write $GL^{(k)}(s,\ell)$ for $GL^{(k)}(s;\ell,\dots,\ell)$. We also note that $GL^{(k)}(s,1) = G_s^{(k)}$, as defined in Section 2.

Lastly, we need one more type of k-graph. Roughly speaking, it is obtained from $GL^{(k)}(s,\ell)$ by closing the ends of the ladders that come out of the set X into zycles. More formally, for $k, s, m, \ell \in \mathbb{N}$ with $s > k$ and $\ell \geqslant 2$, we define the k-graph $GLZ^{(k)}(s, m, \ell)$ to be the k-graph with vertex set

$$V(GLZ^{(k)}(s,m,\ell)) = \{v^e_{ij} : e \in X^{(2)},\, i \in [m],\, j \in [k-1]\} \cup \{w^e_{ij} : e \in X^{(2)},\, i \in [\ell] \setminus \{1\},\, j \in [k-1]\}$$

and edge set

$$\begin{aligned} E(GLZ^{(k)}(s,m,\ell)) = &\{v^e_{i1} \dots v^e_{ik-1} v^e_{i+1j} : e \in X^{(2)},\, i \in [m-1],\, j \in [k-1]\} \\ &\cup \{v^e_{m1} \dots v^e_{mk-1} w^e_{2j} : e \in X^{(2)},\, j \in [k-1]\} \\ &\cup \{w^e_{i1} \dots w^e_{ik-1} w^e_{i+1j} : e \in X^{(2)},\, i \in [\ell-1] \setminus \{1\},\, j \in [k-1]\} \\ &\cup \{w^e_{\ell 1} \dots w^e_{\ell k-1} v^e_{mj} : e \in X^{(2)},\, j \in [k-1]\}\,. \end{aligned}$$

From now on, we suppress the uniformity in the notation if it is clear from context, for instance, writing $GL(s, \ell)$ instead of $GL^{(k)}(s, \ell)$. In outline, the proof of Theorem 1.3 will proceed as follows. First, we show that for every integer $s > k \geqslant 3$ there is some $\alpha_s \in [0, 1]$ such that $\lim_{\ell\to\infty} \pi(GL(s, \ell)) = \alpha_s$, but $\pi(GL(s, \ell)) < \alpha_s$ for all $\ell \in \mathbb{N}$. Because $GL(s, \ell) \subseteq GL(s + 1, \ell)$ for all $s > k$ and ℓ, we have $\alpha_s \leqslant \alpha_{s+1}$. We will argue that, more strongly, for every $s > k$ there is some s' such that $\alpha_s < \alpha_{s'}$. Together, these imply Part I of the theorem. We will then show that for every $s > k$ and $\varepsilon > 0$, there are $m, \ell \in \mathbb{N}$ such that $\alpha_s \leqslant \pi(GLZ(s, m, \ell)) \leqslant \alpha_s + \varepsilon$, which will complete the proof.

3.1. **Part I.** Let $s > k \geqslant 3$ be an integer. To show that there is some $\alpha_s \in [0, 1]$ such that $\lim_{\ell\to\infty} \pi(GL(s, \ell)) = \alpha_s$, but $\pi(GL(s, \ell)) < \alpha_s$ for all $\ell \in \mathbb{N}$, it is sufficient to show that $\pi(GL(s, \ell)) < \pi(GL(s, \ell + 1))$ for all $\ell \in \mathbb{N}$. We do this by induction on ℓ.

First, let $\ell = 1$ and, for a given $n \in \mathbb{N}$, let H be the k-graph that is obtained from a balanced complete $(s - 1)$-partite k-graph with partition $V(H) = [n] = V_1 \cup \dots \cup V_{s-1}$ by adding a balanced complete k-partite k-graph inside each partition class (with partition $V_i = W_i^1 \cup \dots \cup W_i^k$ for each $i \in [s - 1]$). If there were a copy of $GL(s, 2)$ in H, there would have to be at least two vertices $x, x' \in X$ that lie in the same partition class V_i. By the constructions of $GL(s, 2)$ and H, the vertices $v^{xx'}_{21}$ and $v^{xx'}_{22}$, say, must lie in the same W_i^j. But then in H there is no edge containing both $v^{xx'}_{21}$ and $v^{xx'}_{22}$ (which exists in $GL(s, 2)$), meaning that, in fact, H has to be $GL(s, 2)$-free. This implies that $\pi(GL(s, 2)) > k!\binom{s-1}{k}\frac{1}{(s-1)^k} = \pi(GL(s, 1))$, where the last inequality comes from Theorem 2.2.

Now assume that $\ell > 1$ and that $\pi(GL(s, i)) < \pi(GL(s, i + 1))$ holds for all $i \in [\ell - 1]$. We will show that $\pi(GL(s, \ell)) < \pi(GL(s, \ell + 1))$.

By induction, we know that $\pi(GL(s,\ell)) > \pi(GL(s,\ell-1))$. Thus, there is some maximum $r \in [\binom{s}{2}]$ such that, setting $\ell_i = \ell$ for $i \in [r-1]$ and $\ell_i = \ell - 1$ for $i \in [r, \binom{s}{2}]$, we have $\pi(GL(s;\ell_1,\dots,\ell_{\binom{s}{2}})) < \pi(GL(s,\ell))$. Let $\ell'_i = \ell$ for $i \in [r]$ and $\ell'_i = \ell - 1$ for $i \in [r+1, \binom{s}{2}]$. Denote by $\mathcal{GL}$ the (finite) family of k-graphs F whose vertex set is a subset of $V(GL(s;\ell'_1,\dots,\ell'_{\binom{s}{2}}))$ and for which there exists a homomorphism $\varphi : GL(s;\ell'_1,\dots,\ell'_{\binom{s}{2}}) \to F$. By supersaturation (Theorem 2.1 (3)) (and the choice of r), we know that $\pi(\mathcal{GL}) = \pi(GL(s,\ell))$ and thus it suffices to show that $\pi(\mathcal{GL}) < \pi(GL(s,\ell+1))$.

Set $\pi_0 = \pi(GL(s;\ell_1,\dots,\ell_{\binom{s}{2}}))$ and note that $\pi(\mathcal{GL}) > \pi_0$. Therefore, setting $\eta = \pi(\mathcal{GL}) - \pi_0$, we have $\eta > 0$. Furthermore, by supersaturation (Theorem 2.1 (4)), there is some $\varepsilon_1 > 0$ such that, for n sufficiently large, every k-graph H on n vertices with at least $(\pi_0 + \eta/2)\binom{n}{k}$ edges contains a copy of the k-graph G that is obtained from $GL(s;\ell_1,\dots,\ell_{\binom{s}{2}})$ by blowing up the vertex t^{e_r} to a set T of size $\varepsilon_1 n$. Finally, let $\varepsilon_2 \ll \varepsilon_1, \eta$ with $\varepsilon_2 > 0$ and let $n \in \mathbb{N}$ be sufficiently large that[1]

$$\frac{\mathrm{ex}(n, GL(s;\ell_1,\dots,\ell_{\binom{s}{2}}))}{\binom{n}{k}} - \pi_0 < \varepsilon_2\,,$$
$$\frac{\mathrm{ex}(n,\mathcal{GL})}{\binom{n}{k}} - \pi(\mathcal{GL}) < \varepsilon_2\,, \text{ and}$$
$$\frac{\mathrm{ex}(n, GL(s,\ell+1))}{\binom{n}{k}} - \pi(GL(s,\ell+1)) < \varepsilon_2\,. \tag{3.2}$$

Now consider an extremal example H for $\mathcal{GL}$ on n vertices. By our choice of constants, we know that H contains a copy of G. If any $(k-1)$-subset of T is contained in an edge of H, then H would contain a (possibly) degenerate copy of $GL(s;\ell'_1,\dots,\ell'_{\binom{s}{2}})$, i.e., a copy of an element of $\mathcal{GL}$. Thus, no $(k-1)$-subset of T is contained in an edge of H.

Next we add to H a complete balanced k-partite k-graph on $T = T_1 \cup \dots \cup T_k$ and call the resulting k-graph H'. We claim that H' is $GL(s,\ell+1)$-free. Assume, for the sake of contradiction, that H' contains a copy of $GL(s,\ell+1)$ with vertex set as in (3.1).[2] Since this copy of $GL(s,\ell+1)$ is not contained in H, one of its edges must be an edge $z_1 \dots z_k \in E(H') \setminus E(H)$, so we also have $z_1,\dots,z_k \in T$. In fact, since H is (in particular) $GL(s,\ell)$-free, there must be $e \in X^{(2)}$, $i \in [\ell]$, and $j \in [k-1]$ such that $z_1 \dots z_k$ is one of the edges $v^e_{i1} \dots v^e_{ik-1} v^e_{i+1j}$. Without loss of generality, assume that $z_1 = v^e_{i1}, \dots, z_{k-1} = v^e_{ik-1}$ for some $e \in X^{(2)}$, $i \in [\ell]$, and $j \in [k-1]$ with $v^e_{i1} \in T_1, \dots, v^e_{ik-1} \in T_{k-1}$. Recall that, by the construction of H' and the discussion in the previous paragraph, any edge of H' containing

[1] By the monotonicity argument mentioned in the introduction, all of the terms on the left-hand side are non-negative.

[2] To avoid making the notation messier, we will not give the vertices new names. We do not mean the vertices of this copy of $GL(s,\ell+1)$ to be necessarily the same as some of the vertices of the copy of G.

a $(k-1)$-subset of T must be in $E(H') \setminus E(H)$ and must therefore contain exactly one vertex from each of $T_1, \dots, T_k$. Thus, $v^e_{(i+1)1}, \dots, v^e_{(i+1)k-1} \in T_k$ and so these $k-1$ vertices cannot lie together in any edge of H', contradicting that there is a copy of $GL(s, \ell+1)$ in H'. Hence, H' is indeed a $GL(s, \ell+1)$-free k-graph on n vertices.

By monotonicity, we know that H has at least $\pi(\mathcal{GL})\binom{n}{k}$ edges. Therefore, H' has more than

$$\pi(\mathcal{GL})\binom{n}{k} + \left(\frac{\varepsilon_1 n}{k+1}\right)^k > (\pi(\mathcal{GL}) + \varepsilon_2)\binom{n}{k}$$

edges. By (3.2), this means that $\pi(GL(s, \ell+1)) > \pi(\mathcal{GL}) = \pi(GL(s,\ell))$. We have therefore proved that $\lim_{\ell\to\infty} \pi(GL(s,\ell)) = \alpha_s$ for some $\alpha_s \in [0,1]$ with $\pi(GL(s,\ell)) < \alpha_s$ for all $\ell \in \mathbb{N}$.

Next we argue that for every integer $s > k$, there is some integer $s' \gg s$ such that $\alpha_{s'} > \alpha_s$. Note that since $GL(s', \ell) \supseteq G_{s'}$ for every $\ell \in \mathbb{N}$, Theorem 2.2 implies that

$$\alpha_{s'} \geqslant \frac{(s'-1)\cdot(s'-2)\cdots(s'-k)}{(s'-1)^k}.$$

On the other hand, observe that $GL(s,\ell)$ is contained in a blow-up of $K^{(k)}_{s+\binom{s}{2}\cdot[(k-3)+(k-1)]}$ for every $\ell \in \mathbb{N}$. Therefore, by Theorem 2.1 (2), $\alpha_s \leqslant \pi(K^{(k)}_{s+\binom{s}{2}\cdot(2k-4)})$. Since

$$\frac{(s'-1)\cdot(s'-2)\cdots(s'-k)}{(s'-1)^k} \to 1$$

as $s' \to \infty$ and $\pi(K^{(k)}_{s+\binom{s}{2}\cdot(2k-4)}) < 1$, we indeed have $\alpha_{s'} > \alpha_s$ for $s' \gg s$.

3.2. **Part II.** Let $s > k$ be an integer and let $\varepsilon > 0$. Choose $t, n \in \mathbb{N}$ such that

$$\varepsilon, s^{-1} \gg t^{-1} \gg n^{-1}$$

and, for simplicity, assume that $t \mid n$. Now let H be a k-graph with vertex set $[t]$ and $e(H) \geqslant (\alpha_s + \varepsilon)\binom{t}{k}$. We will show that there is a homomorphism from $GLZ(s, \binom{t}{k-1} + 1, \binom{t}{k-1}!)$ into H. Let $H_* = B(H, n/t)$ be the k-graph obtained from H by replacing every vertex i of H by n/t copies of itself, the set of which we call V_i. For $v \in V(H_*)$, let $f(v)$ denote the index of the partition class of H_* that contains v, i.e., if v is one of the copies of the vertex $i \in V(H)$, then $f(v) = i$. Then H_* is a k-graph on n vertices with $e(H_*) \geqslant (\alpha_s + \varepsilon)\binom{t}{k}\left(\frac{n}{t}\right)^k \geqslant (\alpha_s + \varepsilon/2)\binom{n}{k}$.

Since $\pi(GL(s,\ell)) < \alpha_s$ for every $\ell \in \mathbb{N}$, we have that H_* contains a copy of $GL(s, \binom{t}{k-1} + 1)$ with vertex set as in (3.1). Fix $e \in X^{(2)}$. Note that for each $i \in [\binom{t}{k-1} + 1]$, the indices $f(v^e_{ij})$ with $j \in [k-1]$ are pairwise distinct, since $v^e_{i1}, \dots, v^e_{ik-1}$ are contained in an edge together. As H_* only has t distinct partition classes, we deduce from the pigeonhole principle that, for some $i, i' \in [\binom{t}{k-1}+1]$ with $i' < i$, we have

$$\{f(v^e_{i1}), \dots, f(v^e_{ik-1})\} = \{f(v^e_{i'1}), \dots, f(v^e_{i'k-1})\}\,.$$

Since H_* is a blow-up of H, this implies that there is a homomorphism of a zycle of length at most $\binom{t}{k-1}$ into H that maps the starting set to $\{f(v^e_{i1}), \dots, f(v^e_{ik-1})\}$. As described in [8], "cycling" through any such zycle the right number of times yields a homomorphism from $Z_{\binom{t}{k-1}!}$ to H that maps the starting set to $\{f(v^e_{i1}), \dots, f(v^e_{ik-1})\}$. Note that this means that there is a homomorphism from $LZ(i, \binom{t}{k-1}!)$ into H that maps the starting set to $\{f(v^e_{11}), \dots, f(v^e_{1k-1})\}$. Furthermore, observe that for any $j, j' \in \mathbb{N}$ with $j \leqslant j'$ there is a homomorphism from $LZ(j', \binom{t}{k-1}!)$ to $LZ(j, \binom{t}{k-1}!)$ that preserves the starting set. Therefore, there is a homomorphism from $LZ(\binom{t}{k-1}+1, \binom{t}{k-1}!)$ to H that maps the starting set to $\{f(v^e_{11}), \dots, f(v^e_{1k-1})\}$. Since the above holds for all $e \in X^{(2)}$, we obtain a homomorphism from $GLZ(s, \binom{t}{k-1}+1, \binom{t}{k-1}!)$ to H. Thus, $\mathrm{ex}_{\hom}(t, GLZ(s, \binom{t}{k-1}+1, \binom{t}{k-1}!)) \leqslant (\alpha_s + \varepsilon)\binom{t}{k}$. Since the sequence

$$\frac{\mathrm{ex}_{\hom}\Big(m, GLZ(s, \binom{t}{k-1}+1, \binom{t}{k-1}!)\Big)}{\binom{m}{k}}$$

is non-increasing in m, this implies that $\pi_{\hom}(GLZ(s, \binom{t}{k-1}+1, \binom{t}{k-1}!)) \leqslant \alpha_s + \varepsilon$. Thus, we have

$$\begin{aligned}\alpha_s &\leqslant \pi\Big(GLZ\big(s, \binom{t}{k-1}+1, \binom{t}{k-1}!\big)\Big)\\ &= \pi_{\hom}\Big(GLZ\big(s, \binom{t}{k-1}+1, \binom{t}{k-1}!\big)\Big) \leqslant \alpha_s + \varepsilon\,,\end{aligned}$$

where the first inequality holds since if H contains $GLZ(s, \binom{t}{k-1}+1, \binom{t}{k-1}!)$, then there exists a homomorphism from $GL(s, \ell)$ into H for all $\ell \in \mathbb{N}$. In fact, for all integers $m \geqslant 2$ and $i \geqslant 1$, there is a homomorphism from $GL(s, \ell)$ into $GLZ(s, i, m)$ for all $\ell \in \mathbb{N}$, so that $\alpha_s \leqslant \pi(\{GLZ(s, i, m) : \ i, m \in \mathbb{N}, m \geqslant 2\})$. Since in the above argument $\varepsilon > 0$ was arbitrary, $\pi(\{GLZ(s, i, m) : \ i, m \in \mathbb{N},\ m \geqslant 2\}) = \alpha_s$.

§4. Concluding remarks

Our earlier paper [1] showed that $\Pi^{(k)}$ contains a subset of order type $\omega 2$ when $k \geqslant 3$. This is already enough to distinguish it from $\Pi^{(2)}$, which has order type ω. In this paper, we went further, showing that $\Pi^{(k)}$ contains a subset of order type ω^2 when $k \geqslant 3$. This is still likely far from the truth and we conjecture that $\Pi^{(k)}$ contains subsets of any countable order type when $k \geqslant 3$. However, it would already be interesting to push our techniques to handle, say, ω^3, ω^ω, or ε_0. It may also be that $-\Pi^{(k)}$ contains subsets of any countable order type when $k \geqslant 3$. This might be difficult, as finding a subset of order type ω would already show that $\Pi^{(k)}$ is not well-ordered, itself an interesting open problem. Finally, we note that a result of Pikhurko [9] saying that $\Pi_\infty^{(k)}$ has the cardinality of the continuum for $k \geqslant 3$ implies that $\Pi_\infty^{(k)}$ and, therefore, $\Pi_{\text{fin}}^{(k)}$ has uncountably many accumulation points. The same may well be true of $\Pi^{(k)}$.

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