HYPERGRAPHS ACCUMULATE

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Abstract. We show that for every integer $k \geqslant 3$ the set of Turán densities of k-uniform hypergraphs has an accumulation point in $[0, 1)$. In particular, $1/2$ is an accumulation point for the set of Turán densities of 3-uniform hypergraphs.

§1. Introduction

For $k \in \mathbb{N}$, a k-uniform hypergraph (or k-graph) $H = (V, E)$ consists of a vertex set V and an edge set $E \subseteq V^{(k)} = \{e \subseteq V : |e| = k\}$. Given $n \in \mathbb{N}$ and a k-graph F, the extremal number $\mathrm{ex}(n, F)$ is the maximum number of edges in a k-graph H with n vertices that does not contain a copy of F. The Turán density of F is then given by

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{k}},$$

where the limit is known, by a simple monotonicity argument [9], to be well-defined. The problem of determining these Turán densities is one of the oldest and most fundamental questions in extremal combinatorics.

When $k = 2$, that is, if F is a graph, $\pi(F)$ is essentially completely understood, with the final result, the culmination of work by Turán [14], Erdős and Stone [5] and Erdős and Simonovits [4], saying that $\pi(F) = \frac{\chi(F)-2}{\chi(F)-1}$, where $\chi(F)$ is the chromatic number of F. In contrast, very little is known about Turán densities for $k \geqslant 3$, with even the problem of determining the Turán density of the complete 3-graph on four vertices, a question first raised by Turán in 1941 [14], remaining wide open. For more on what is known about hypergraph Turán densities, we refer the interested reader to the many surveys on the topic [8, 10, 13].

Given the difficulty of determining the Turán density of specific 3-graphs, one might instead try to study the distribution of the set $\Pi^{(k)} = \{\pi(F) : F \text{ is a } k\text{-graph}\}$ of Turán densities of k-graphs. For example, by a result of Erdős [3] saying that $\pi(F) = 0$ if and only if F is a k-partite k-graph, we know that there is no k-graph F with $\pi(F) \in (0, k!/k^k)$. However, this direction turned out to be just as difficult and, beyond that simple result and the identification of some specific points in the set, very little is known about $\Pi^{(k)}$.

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If instead one considers the set $\Pi_\infty^{(k)} = \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a family of } k\text{-graphs}\}$, more is known. Of particular note here is the result of Frankl and Rödl [7] showing that $\Pi_\infty^{(k)}$ is not well-ordered, thereby disproving the jumping conjecture, for which Erdős had offered \$1000, saying that there is a non-trivial gap or jump between every two elements of $\Pi_\infty^{(k)}$. For more on the existence and non-existence of jumps, see, for instance, [1, 6].

A more systematic study of $\Pi_\infty^{(k)}$ and $\Pi_{\text{fin}}^{(k)} = \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a finite family of } k\text{-graphs}\}$ was undertaken by Pikhurko [11], who, roughly speaking, showed that for every iterative blow-up construction H there is some finite family $\mathcal{F}$ of k-graphs for which H is an extremal example. This then allowed him to show that $\Pi_{\text{fin}}^{(k)}$ contains irrational numbers and that $\Pi_\infty^{(k)}$ has the cardinality of the continuum. In particular, his results imply that $\Pi_{\text{fin}}^{(k)}$ has accumulation points in $[0,1)$ for all $k \geqslant 3$, though we remark that the finite families given by his construction are huge. In this note, we show that the same is true of $\Pi^{(k)}$, that is, that the set of Turán densities of single k-uniform hypergraphs has at least one accumulation point in $[0,1)$ for all $k \geqslant 3$.

Theorem 1.1. *For every integer $k \geqslant 3$, the set $\Pi^{(k)}$ has an accumulation point in $[0,1)$. Moreover, $1/2$ is an accumulation point for $\Pi^{(3)}$.*

This is a consequence of the following more general result.

Theorem 1.2. *For every integer $k \geqslant 3$, there is some $\alpha^{(k)} \in [0,1)$ such that all of the following hold:*

(1) There is a sequence of k-graphs $\{F_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \pi(F_n) = \alpha^{(k)}$ and $\pi(F_n) < \alpha^{(k)}$ for all $n \in \mathbb{N}$.
(2) For every $\varepsilon > 0$, there is a k-graph G_ε with $\alpha^{(k)} \leqslant \pi(G_\varepsilon) \leqslant \alpha^{(k)} + \varepsilon$. Moreover, $\pi(\{G_{\frac{1}{n}}\}_{n\in\mathbb{N}}) = \alpha^{(k)}$.
(3) $\alpha^{(k)} \leqslant \frac{k-2}{k-1}$ and $\alpha^{(3)} = 1/2$.

Perhaps surprisingly, the proofs of the first two points are abstract in the sense that they work without pinpointing $\alpha^{(k)}$. Regarding these values, it would be interesting to determine $\alpha^{(k)}$ for $k \geqslant 4$ or to show that $\alpha^{(k)}$ is itself in $\Pi^{(k)}$ for $k \geqslant 3$. In particular, highlighting the depth of our ignorance about Turán densities, it is not known if $1/2 \in \Pi^{(3)}$.

§2. Preliminaries

Given an integer t and a k-graph F, let $B(F,t)$ be the t-blow-up of F, the k-graph obtained from F by replacing every vertex by t copies of itself. The following phenomenon, which we make extensive use of, is well-known (see, for instance, Lemma 2.1 and Theorem 2.2 in [10], as well as the subsequent discussion).

Theorem 2.1 (Supersaturation). *(1) For every k-graph F and $\delta > 0$, there are $\varepsilon > 0$ and n_0 such that every k-graph on $n \geqslant n_0$ vertices with at least $(\pi(F) + \delta)\binom{n}{k}$ edges contains at least $\varepsilon n^{|V(F)|}$ copies of F.*

(2) For every integer t and k-graph F, $\pi(B(F,t)) = \pi(F)$.

(3) Let F be a k-graph and let $\mathcal{F}$ be the (finite) family of k-graphs F' whose vertex set is a subset of $V(F)$ and for which there exists a homomorphism $\varphi : F \to F'$. Then $\pi(\mathcal{F}) = \pi(F)$.

(4) For every k-graph F and $\delta > 0$, there are $\varepsilon > 0$ and n_0 such that, for all $v \in V(F)$, every k-graph on $n \geqslant n_0$ vertices with at least $(\pi(F) + \delta)\binom{n}{k}$ edges contains the k-graph obtained from F by replacing v by εn copies of v.

§3. Proof of Theorem 1.2

For $k, \ell \in \mathbb{N}$, we define the k-uniform *ladder of length ℓ* to be the k-graph $L_\ell^{(k)}$ with vertex set

$$V(L_\ell^{(k)}) = \{v_{ij} : i \in [\ell], j \in [k-1]\} \cup \{t\}$$

and edge set

$$E(L_\ell^{(k)}) = \{v_{i1} \dots v_{ik-1} v_{i+1j} : i \in [\ell - 1], j \in [k-1]\} \cup \{v_{\ell 1} \dots v_{\ell k-1} t\}\,.$$

For $m \in \mathbb{N}$, we further define the k-graph $L_\ell^{(k)}(m)$ to be the k-graph with vertex set

$$V(L_\ell^{(k)}(m)) = \{v_{ij} : i \in [\ell], j \in [k-1]\} \cup T\,,$$

where T is some set of size m, and edge set

$$E(L_\ell^{(k)}(m)) = \{v_{i1} \dots v_{ik-1} v_{i+1j} : i \in [\ell - 1], j \in [k-1]\} \cup \{v_{\ell 1} \dots v_{\ell k-1} t : t \in T\}\,.$$

Lastly, following [12], for an integer $\ell \geqslant 2$, we define the k-uniform *zycle of length ℓ* to be the k-graph $Z_\ell^{(k)}$ with vertex set $V(Z_\ell^{(k)}) = \{v_{ij} : i \in \mathbb{Z}/\ell\mathbb{Z}, j \in [k-1]\}$ and edge set

$$E(Z_\ell) = \{v_{i1} \dots v_{ik-1} v_{i+1j} : i \in \mathbb{Z}/\ell\mathbb{Z}, j \in [k-1]\}\,.$$

Let k be an integer with $k \geqslant 3$. We first show that there is some $\alpha^{(k)} \in (0,1)$ such that $\lim_{\ell\to\infty} \pi(L_\ell^{(k)}) = \alpha^{(k)}$ while $\pi(L_\ell^{(k)}) < \alpha^{(k)}$ for all $\ell \in \mathbb{N}$. Then we show that for every $\varepsilon > 0$ there is some $M \in \mathbb{N}$ such that $\pi(Z_M^{(k)}) \leqslant \alpha^{(k)} + \varepsilon$. Lastly, we will argue that $\alpha^{(k)} \leqslant \frac{k-2}{k-1}$ with equality for $k = 3$.

3.1. **Part I.** Note that, for every $\ell \in \mathbb{N}$, $L_\ell^{(k)}$ is contained in a sufficiently large blow-up of $K_{2(k-1)}^{(k)}$. Hence, by supersaturation (Theorem 2.1 (2)), we have that for all $\ell \in \mathbb{N}$, $\pi(L_\ell^{(k)}) \leqslant \pi(K_{2(k-1)}^{(k)}) < 1$. It is thus sufficient to show that $\pi(L_\ell^{(k)}) < \pi(L_{\ell+1}^{(k)})$ for all $\ell \in \mathbb{N}$. We do this by induction on ℓ. For $\ell = 1$, this follows from the result of Erdős [3] mentioned in the introduction, since $L_1^{(k)}$ is k-partite, but $L_2^{(k)}$ is not. Now assume that $\ell > 1$ and we know that $\pi(L_i^{(k)}) < \pi(L_{i+1}^{(k)})$ holds for all $i \in [\ell-1]$. Denote by $\mathcal{L}_\ell^{(k)}$ the (finite) family of k-graphs F whose vertex set is a subset of $V(L_\ell^{(k)})$ and for which there exists a homomorphism $\varphi : L_\ell^{(k)} \to F$. By supersaturation (Theorem 2.1 (3)), we know that $\pi(\mathcal{L}_\ell^{(k)}) = \pi(L_\ell^{(k)})$. Therefore, it remains to show that $\pi(\mathcal{L}_\ell^{(k)}) < \pi(L_{\ell+1}^{(k)})$.

Let $\eta = \pi(L_\ell^{(k)}) - \pi(L_{\ell-1}^{(k)})$ and note, by supersaturation (Theorem 2.1 (4)), that there is some ε_1 such that, for n sufficiently large, every k-graph H on n vertices with at least $(\pi(L_{\ell-1}^{(k)}) + \eta/2)\binom{n}{k}$ edges contains a copy of the k-graph $L_{\ell-1}^{(k)}(\varepsilon_1 n)$. Finally, let $\varepsilon_2 \ll \varepsilon_1, \eta$ and let n also be large enough that

$$\max\left(\left\{\left|\pi(L_i^{(k)}) - \frac{\mathrm{ex}(n, L_i^{(k)})}{\binom{n}{k}}\right| i \in \{\ell-1, \ell, \ell+1\}\right\} \cup \left\{\left|\pi(\mathcal{L}_\ell^{(k)}) - \frac{\mathrm{ex}(n, \mathcal{L}_\ell^{(k)})}{\binom{n}{k}}\right|\right\}\right) < \varepsilon_2 \,. \tag{3.1}$$

Now consider an extremal example H for $\mathcal{L}_\ell^{(k)}$ on n vertices. By our choice of constants, we know that H contains a copy of $L_{\ell-1}^{(k)}(\varepsilon_1 n)$ (say with vertex set $\{v_{ij} : i \in [\ell-1], j \in [k-1]\} \,\dot\cup\, T \subseteq V(H)$). If any $(k-1)$-subset of T is contained in an edge of H, then H would contain a (possibly) degenerate copy of $L_\ell^{(k)}$, i.e., a copy of an element in $\mathcal{L}_\ell^{(k)}$. Thus, no $(k-1)$-subset of T is contained in an edge of H.

Next we add to H a complete nearly balanced k-partite k-graph on $T = T_1 \,\dot\cup \ldots \dot\cup\, T_k$ and call the resulting k-graph H'. We claim that H' is $L_{\ell+1}^{(k)}$-free. Assume, for the sake of contradiction, that it contains a copy of $L_{\ell+1}^{(k)}$ (say with vertex set $\{u_{ij} : i \in [\ell+1], j \in [k-1]\} \,\dot\cup\, \{t\} \subseteq V(H)$). Since this copy is not contained in H, one of its edges must be an edge $x_1 \ldots x_k \in E(H') \smallsetminus E(H)$, whence we also have $x_1, \ldots, x_k \in T$. In fact, since H is (in particular) $L_\ell^{(k)}$-free, we know that, without loss of generality, there is some $i \in [\ell]$ with $x_1 = u_{i1} \in T_1, \ldots, x_{k-1} = u_{ik-1} \in T_{k-1}$. Recall that, by the construction of H' and the discussion in the previous paragraph, any edge of H' containing a $(k-1)$-subset of T must be in $E(H') \smallsetminus E(H)$ and must therefore contain exactly one vertex from each of $T_1, \ldots, T_k$. Thus, $u_{(i+1)1}, \ldots, u_{(i+1)k-1} \in T_k$ and so these $k-1$ vertices do not lie together in any edge of H', contradicting that $\{u_{ij} : i \in [\ell+1], j \in [k-1]\} \,\dot\cup\, \{t\}$ is the vertex set of a copy of $L_{\ell+1}^{(k)}$ in H'. Hence, H' is indeed an $L_{\ell+1}^{(k)}$-free k-graph on n vertices.

By (3.1), we know that H has at least $(\pi(\mathcal{L}_\ell^{(k)}) - \varepsilon_2)\binom{n}{k}$ edges. Therefore, H' has more than $(\pi(\mathcal{L}_\ell^{(k)}) - \varepsilon_2)\binom{n}{k} + \left(\frac{\varepsilon_1 n}{k+1}\right)^k > (\pi(\mathcal{L}_\ell^{(k)}) + \varepsilon_2)\binom{n}{k}$ edges. Again by (3.1), this means that $\pi(L^{(k)}_{\ell+1}) > \pi(\mathcal{L}^{(k)}_{\ell}) = \pi(L^{(k)}_{\ell})$. We have thus proved that $\lim_{\ell\to\infty}\pi(L^{(k)}_{\ell}) = \alpha^{(k)}$ for some $\alpha^{(k)} \in (0,1)$ with $\pi(L^{(k)}_{\ell}) < \alpha^{(k)}$ for all $\ell \in \mathbb{N}$.

3.2. **Part II.** Let $\varepsilon > 0$ and pick $t, n \in \mathbb{N}$ such that $\varepsilon, k^{-1} \gg t^{-1} \gg n^{-1}$ and, for simplicity, assume that $t \mid n$. Now let H be a k-graph with vertex set $[t]$ and $e(H) \geqslant (\alpha^{(k)} + \varepsilon)\binom{t}{k}$. We will show that there is a homomorphism from $Z^{(k)}_{\binom{t}{k-1}!}$ into H. Let $H_* = B(H, n/t)$ be the k-graph obtained from H by replacing every vertex i of H by n/t copies of itself, the set of which we call V_i. For $v \in V(H_*)$, let $f(v)$ denote the index of the partition class of H_* that contains v, i.e., if v is one of the copies of a vertex $i \in V(H)$, then $f(v) = i$. Then H_* is a k-graph on n vertices with $e(H) \geqslant (\alpha^{(k)} + \varepsilon)\binom{t}{k}\left(\frac{n}{t}\right)^k \geqslant (\alpha^{(k)} + \varepsilon/2)\binom{n}{k}$.

Since $\pi(L^{(k)}_{\ell}) < \alpha^{(k)}$ for every ℓ, we have that H_* contains a copy L of $L^{(k)}_{\binom{t}{k-1}+1}$. As above, let $V(L) = \{v_{ij} : i \in [\binom{t}{k-1}+1], j \in [k-1]\} \cup \{t\}$. Note that for each $i \in [\binom{t}{k-1}+1]$, the indices $f(v_{ij})$ with $j \in [k-1]$ are pairwise distinct, since $v_{i1}, \ldots, v_{ik-1}$ are contained in an edge together. As H_* only has t distinct partition classes, we deduce from the pigeonhole principle that, for some $i, i' \in [\binom{t}{k-1}+1]$ with $i < i'$, we have $\{f(v_{i1}), \ldots, f(v_{ik-1})\} = \{f(v_{i'1}), \ldots, f(v_{i'k-1})\}$. Since H_* is a blow-up of H, this implies that there is a homomorphism of a zycle of length at most $\binom{t}{k-1}$ into H. As described in [12], "cycling" through any such zycle the right number of times yields a homomorphism from $Z^{(k)}_{\binom{t}{k-1}!}$ into H, whence $\mathrm{ex}_{\mathrm{hom}}(t, Z^{(k)}_{\binom{t}{k-1}!}) \leqslant (\alpha^{(k)} + \varepsilon)\binom{t}{k}$. Since the sequence $\mathrm{ex}_{\mathrm{hom}}(m, Z^{(k)}_{\binom{t}{k-1}!})/\binom{m}{k}$ is non-increasing in m, this implies that $\pi_{\mathrm{hom}}(Z^{(k)}_{\binom{t}{k-1}!}) \leqslant \alpha^{(k)} + \varepsilon$. Thus, we have

$$\alpha^{(k)} \leqslant \pi(Z^{(k)}_{\binom{t}{k-1}!}) = \pi_{\mathrm{hom}}(Z^{(k)}_{\binom{t}{k-1}!}) \leqslant \alpha^{(k)} + \varepsilon, \tag{3.2}$$

where the first inequality holds since if H contains $Z^{(k)}_{\binom{t}{k-1}!}$, then there exists a homomorphism from $L^{(k)}_{\ell}$ into H for all $\ell \in \mathbb{N}$. In fact, for every integer $m \geqslant 2$, there is a homomorphism from $L^{(k)}_{\ell}$ into $Z^{(k)}_{m}$ for all $\ell \in \mathbb{N}$, so that $\alpha^{(k)} \leqslant \pi(\{Z^{(k)}_{m} : m \in \mathbb{N}, m \geqslant 2\})$. Since in the above argument $\varepsilon > 0$ was arbitrary, the Turán density of the family of all k-uniform zycles is $\alpha^{(k)}$.

3.3. **Part III.** In this subsection, we prove that $\alpha^{(k)} \leqslant \frac{k-2}{k-1}$ with equality for $k = 3$. In [2], DeBiasio and Jiang gave the following construction showing that $\pi(Z^{(3)}_{\ell}) \geqslant 1/2$ for all $\ell \geqslant 2$. Let $A = \{a_1, \ldots, a_{\lfloor \frac{n}{2} \rfloor}\}$ and $B = \{b_1, \ldots, b_{\lceil \frac{n}{2} \rceil}\}$ and consider the 3-graph on $A \cup B$ whose edges are given by all triples $a_i b_j a_k$ and $a_i b_j b_k$, with $i, j < k$. Thus, by Part II, we have $1/2 \leqslant \alpha^{(3)}$. A result by DeBiasio and Jiang combined with an argument from [12] also shows that for every $\varepsilon > 0$ there is an ℓ such that $\pi(Z^{(3)}_{\ell}) \leqslant 1/2 + \varepsilon$. This will also follow from Part II if we can show that $\alpha^{(k)} \leqslant \frac{k-2}{k-1}$. By Parts I and II, it is therefore left to argue that $\pi(L^{(k)}_{\ell}) \leqslant \frac{k-2}{k-1}$ for every $\ell \in \mathbb{N}$. Given $\ell \in \mathbb{N}$ and $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $\varepsilon, k^{-1}, \ell^{-1} \gg n^{-1}$

and let H be a k-graph on n vertices with $\delta(H) \geqslant (\frac{k-2}{k-1} + \varepsilon)\binom{n}{k-1}$ (a standard induction argument – see Proposition 4.2 in [10] – shows that to prove an upper bound on the Turán density, we may assume such a minimum degree condition). Let $v_{\ell 1} v_{\ell 2} \dots v_{\ell k-1} t$ be an edge in H. Then the minimum degree condition on H implies that the links of $v_{\ell 1}, \dots, v_{\ell k-1}$ have at least $\varepsilon\binom{n}{k-1}$ common edges. Now let $v_{\ell-1 1}, \dots, v_{\ell-1 k-1}$ be $k-1$ vertices other than t forming one of these edges. Using that $\varepsilon, k^{-1}, \ell^{-1} \gg n^{-1}$, for each $i \in [\ell-2]$ we can continue choosing vertices $v_{i1}, \dots, v_{ik-1}$ such that $v_{i1} \dots v_{ik-1}$ is an edge in the common intersection of the links of $v_{(i+1)1}, \dots, v_{(i+1)(k-1)}$ and such that $v_{i1}, \dots, v_{i(k-1)}$ are distinct from all previously chosen vertices. Indeed, when choosing $v_{i1}, \dots, v_{ik-1}$, at most $(k-1)(\ell-1)+1 \leqslant k\ell$ vertices have been chosen before. Hence, the links of $v_{(i+1)1}, \dots, v_{(i+1)(k-1)}$ have at least $\varepsilon\binom{n}{k-1} - k\ell n^{k-2} \geqslant \frac{\varepsilon}{2}\binom{n}{k-1}$ common edges which do not contain any previously chosen vertex. Eventually $v_{11}, \dots, v_{1k-1}, \dots, v_{\ell 1}, v_{\ell k-1}, t$ form the vertex set of $L_\ell^{(k)}$. Thus, $\pi(L_\ell^{(k)}) \leqslant \frac{k-2}{k-1}$, so that, by Part I, we have $\alpha^{(k)} \leqslant \frac{k-2}{k-1}$.

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