Lecture 9

A graph $H$ is said to be $p$-arrangeable if there is an ordering $v_1, \ldots, v_n$ of the vertices of $H$ such that, for every vertex $v_i$, the set of left neighbours of the set of right neighbours of $v_i$ has size at most $p$. The following theorem, generalising the result about graphs of bounded maximum degree, is due to Chen and Schelp.

**Theorem 1** For every $p$, there exists $c(p) > 0$ such that, if $H$ is a $p$-arrangeable graph with $n$ vertices,

$$r(H) \leq c(p)n.$$  

The best known value of $c(p)$, due to Graham, Rödl and Ruciński, is $2^{cp \log^2 p}$. One corollary is that planar graphs have linear Ramsey numbers. This follows because planar graphs are known to be 10-arrangeable.

All of these results on graphs of bounded maximum degree and bounded arrangeability stem from an important conjecture of Burr and Erdős. A graph $H$ is said to be $d$-degenerate if there is an ordering $v_1, \ldots, v_n$ of the vertices of $H$ such that, for every $1 \leq i \leq n$, the vertex $v_i$ has at most $d$ left neighbours. Equivalently, every subgraph of $H$ has a vertex of degree at most $d$. The Burr-Erdős conjecture states that for every $d$, there should exist $c(d) > 0$ such that, for every $d$-degenerate graph on $n$ vertices, $r(H) \leq c(d)n$. This remains open.

The best result, due to Fox and Sudakov, is that, for each $d$, there exists $c(d) > 0$ such that, if $H$ is a $d$-degenerate graph with $n$ vertices, then $r(H) \leq 2^{c(d)\sqrt{\log n}}$. So there is an $n^{1+o(1)}$ bound. Here we prove such a bound when $H$ is bipartite.

**Lemma 1** Let $t, r \geq 2$ and let $G$ be a bipartite graph with $N$ vertices on either side and at least $2N^{2-1/(t^2r)}$ edges. Then $G$ contains two subsets $U_1$ and $U_2$ such that, for $k = 1, 2$, every $r$-tuple in $U_k$ has at least $m = N^{1-1.8/t}$ common neighbours in $U_{3-k}$.

**Proof:** Let $q = \frac{7}{4}rt$. Note that the density of $G$ is at least $\alpha = 2N^{-1/(t^2r)}$ and apply the dependent random choice lemma with $\alpha$, $\beta = N^{-1.8/t^2r}$, $s = t^2r$ and $q$. Then we get a set of size at least $2N^{1-1/t}$ with fewer than

$$4\beta^q \alpha^{-s} \left( \begin{array}{c} N \\ q \end{array} \right) \leq 4\beta^q(N-1)\frac{Nq}{q!} < 1$$

bad $q$-tuples, where a $q$-tuple is bad if it has fewer than $\beta^q N = N^{1-1.8/t}$ common neighbours. For each bad $q$-tuple remove a vertex. This leaves us with a set of size at least $N^{1-1/t}$ where every $q$-tuple is good.

Choose a random subset $T \subset U_1$ consisting of $q-r$ (not necessarily distinct) uniformly chosen vertices of $U_1$. Since $t \geq 2$, we have $q-r = \frac{7}{4}rt - r \geq \frac{5}{4}rt$. Let $U_2$ be the set of common neighbours of $T$. The probability that $U_2$ contains a subset of size $r$ with at most $m$ common neighbours in $U_1$ is at most

$$\left( \begin{array}{c} N \\ r \end{array} \right) \left( \frac{m}{|U_1|} \right)^{q-r} \leq \frac{N^r}{r!} N^{-0.8(q-r)/t} \leq 1/r! < 1,$$

where we use that $m = N^{1-1.8/t}$ and $|U_1| \geq N^{1-1/t}$.

Therefore, there is a choice of $T$ such that every subset of $U_2$ of size $r$ has at least $m$ common neighbours in $U_1$. Consider now an arbitrary subset $S$ of $U_1$ of size at most $r$. Since $S \cup T$ is a subset of $U_1$ of
size at most \( q \), this set has at least \( m \) common neighbours in \( G \). By the definition of \( U_2 \) all common neighbours of \( T \) in \( G \) lie in \( U_2 \). Therefore, \( N(S \cup T) \subset N(T) \subset U_2 \). Hence \( S \) has at least \( m \) common neighbours in \( U_2 \), implying the result. 

The usefulness of this lemma is that the criteria on \( U_1 \) and \( U_2 \) is easily enough to embed an \( r \)-degenerate graph.

**Lemma 2** Let \( G \) be a graph with vertex subsets \( U_1 \) and \( U_2 \) such that, for \( k = 1, 2 \), every subset of at most \( r \) vertices in \( U_k \) have at least \( n \) common neighbors in \( U_{3-k} \). Then \( G \) contains every \( r \)-degenerate bipartite graph \( H \) with \( n \) vertices.

**Proof:** Let \( v_1, \ldots, v_n \) be an ordering of the vertices of \( H \) such that, for \( 1 \leq i \leq n \), vertex \( v_i \) has at most \( r \) neighbors \( v_j \) with \( j < i \). Let \( A_1 \) and \( A_2 \) be the two parts of \( H \). We find an embedding \( f : V(H) \to V(G) \) of \( H \) in \( G \) such that the image of the vertices in \( A_k \) belongs to \( U_k \) for \( k = 1, 2 \). We embed the vertices of \( H \) one by one, in the above order. Without loss of generality, suppose that the vertex \( v_i \) we want to embed is in \( A_1 \). Consider the set \( \{f(v_j) : j < i, (v_j, v_i) \in E(H)\} \) of images of neighbors of \( v_i \) which are already embedded. Note that this set belongs to \( U_2 \), has cardinality at most \( r \) and therefore has at least \( n \) common neighbors in \( U_1 \). All these neighbors can be used to embed \( v_i \) and at least one of them is yet not occupied, since so far we embedded less than \( n \) vertices. Pick such a neighbor \( w \) and set \( f(v_i) = w \).

Putting Lemmas 1 and 2 together gives the following density result.

**Corollary 1** If \( r, t \geq 2 \) and \( G \) is a bipartite graph with \( N \) vertices on each side and at least \( 2N^2 - 1/(t^3r) \) edges, then \( G \) contains every \( r \)-degenerate bipartite graph with at most \( N^{1-1.8/t} \) vertices.

The required Ramsey statement is a simple corollary of this result.

**Corollary 2** The Ramsey number of every \( r \)-degenerate bipartite graph \( H \) with \( n \) vertices, \( n \) sufficiently large, satisfies

\[
r(H) \leq 2^{8r^{1/3}(\log n)^{2/3}} n.
\]

**Proof:** In every two-colouring of the edges of the complete bipartite graph \( K_{N/2, N/2} \), one of the color classes contains at least half of the edges. Let \( N = 2^{8r^{1/3}(\log n)^{2/3}} n \) and let \( t = \frac{1}{2} \left( \frac{r}{1} \log n \right)^{1/3} \). Then \( 2N^2 - 1/(t^3r) \leq \frac{1}{2} \left( \frac{N}{2} \right)^2 \) and and \( N^{1-1.8/t} \geq n \). By Corollary 2, the majority color contains a copy of \( H \). \( \square \)