Lecture 8

In this lecture we will give a proof that, for bipartite graphs H with n vertices and maximum degree Δ, the Ramsey number satisfies $r(H) \leq 2^{c\Delta} n$. This was solved independently by the author and by Fox and Sudakov. We will begin by proving what is known as the dependent random choice lemma. This shows that every dense graph contains a large set of vertices U such that all (or almost all) small subsets of U have many common neighbours. This property is very useful because it allows one to greedily embed sparse bipartite graphs.

Lemma 1 *Let $G = (A, B; E)$ be a bipartite graph with $|A| = |B| = N$. Suppose that the graph has density α, that is that there are αN^2 edges. Then, for any $\beta < \alpha$ and any $r, s \in \mathbb{N}$, there exists a set $A' \subset A$ of size greater than $\frac{\alpha^s}{2} N$ which contains at most $\frac{4\beta^{rs}}{\alpha^s}\binom{N}{r}$ r-tuples with less than $\beta^r N$ common neighbours.*

Proof: For each vertex v, let $d(v)$ be its degree. For any given vertices $b_1, \cdots, b_s \in B$, let I be the set of common neighbours. What is the expectation of $|I|$ over all possible random choices of $b_1, \cdots, b_s$ (allowing repetitions)? If we apply Jensen's inequality, we see that

$$\begin{aligned} \mathbb{E}(|I|) &= \sum_{v \in A} \mathbb{P}(v \in I) = \sum_{v \in A} \left(\frac{d(v)}{N}\right)^s \\ &\geq \frac{N\left(\frac{\sum_{v \in A} d(v)}{N}\right)^s}{N^s} = \frac{N(\alpha N)^s}{N^s} = \alpha^s N. \end{aligned}$$

Therefore we see that with probability at least $\alpha^s/2$ we have $|I| \geq \frac{\alpha^s}{2} N$.

We also have that the expected number of bad r-tuples, that is r-tuples in I which have less than $\beta^r N$ common neighbours, is at most

$$\beta^{rs}\binom{N}{r}.$$

To see this note simply that any such bad r-tuple has at most $\beta^r N$ common neighbours in B and therefore the probability that such an r-tuple be chosen is β^{rs}. Thus, by Markov's inequality, the probability that the number of bad r-tuples is larger than $\frac{4\beta^{rs}}{\alpha^s}\binom{N}{r}$ is at most $\alpha^s/4$.

We therefore see that with positive probability we may choose a set A' of size at least $\frac{\alpha^s}{2} N$ which contains at most $\frac{4\beta^{rs}}{\alpha^s}\binom{N}{r}$ bad r-tuples. $\square$

Now we need an embedding lemma. The following, taken from the paper of Fox and Sudakov, is more than sufficient. To state the lemma we need to note that a hypergraph $\mathcal{H}$ is said to be down-closed if, whenever $e_1 \subset e_2$ and e_2 in an edge of $\mathcal{H}$, then e_1 is also an edge.

Lemma 2 *Let $\mathcal{H}$ be an n-vertex hypergraph with maximum degree d such that each edge of $\mathcal{H}$ has size at most h. If $\mathcal{F} = (V, E)$ is a down-closed hypergraph with $N = 4n$ vertices and more than $(1 - (4d)^{-h})\binom{N}{h}$ edges of cardinality h, then there are at least $(N/2)^n$ labeled copies of $\mathcal{H}$ in $\mathcal{F}$.*

Proof: Call a subset $S \subset V$ of size $|S| \leq h$ good if S is contained in more than $(1 - (4d)^{|S|-h})\binom{N}{h-|S|}$ edges of $\mathcal{F}$ of cardinality h. For a good set S with $|S| < h$ and a vertex $j \in V \backslash S$, call j bad with respect to S if $S \cup \{j\}$ is not good. Let B_S denote the set of vertices $j \in V \backslash S$ that are bad with respect to S. The key observation is that if S is good with $|S| < h$, then $|B_S| \leq N/(4d)$. Indeed, suppose $|B_S| > N/(4d)$, then the number of h-sets containing S that are not edges of $\mathcal{F}$ is at least

$$\frac{|B_S|}{h - |S|}(4d)^{|S|+1-h}\binom{N}{h-|S|-1} > (4d)^{|S|-h}\binom{N}{h-|S|},$$

which contradicts the fact that S is good.

Fix a labeling $\{v_1, \ldots, v_n\}$ of the vertices of $\mathcal{H}$. Since the maximum degree of $\mathcal{H}$ is d, for every vertex v_i there are at most d subsets $S \subset L_i = \{v_1, \ldots, v_i\}$ containing v_i such that $S = e \cap L_i$ for some edge e of $\mathcal{H}$. We use induction on i to find many embeddings f of $\mathcal{H}$ in $\mathcal{F}$ such that for each edge e of $\mathcal{H}$, the set $f(e \cap L_i)$ is good.

By our definition, the empty set is good. Assume at step i, for all edges the sets $f(e \cap L_i)$ are good. There are at most d subsets S of L_{i+1} that are of the form $S = e \cap L_{i+1}$ where e is an edge of $\mathcal{H}$ containing v_{i+1}. By the induction hypothesis, for each such subset S, the set $f(S \backslash \{v_{i+1}\})$ is good and therefore there are at most $\frac{N}{4d}$ bad vertices in $\mathcal{F}$ with respect to it. In total this gives at most $d\frac{N}{4d} = N/4$ vertices. The remaining at least $3N/4 - i$ vertices in $\mathcal{F} \backslash f(L_i)$ are good with respect to all the above sets $f(S \backslash \{v_{i+1}\})$ and we can pick any of them to be $f(v_{i+1})$. Notice that this construction guarantees that $f(e \cap L_{i+1})$ is good for every edge e in $\mathcal{H}$. In the end of the process we obtain a mapping f such that $f(e \cap L_n) = f(e)$ is good for every e in $\mathcal{H}$. In particular, $f(e)$ is contained in at least one edge of $\mathcal{F}$ of cardinality h and therefore $f(e)$ itself is an edge of $\mathcal{F}$ since $\mathcal{F}$ is down-closed. This shows that f is indeed an embedding of $\mathcal{H}$ in $\mathcal{F}$. Since at step i we have at least $3N/4 - i$ choices for vertex v_{i+1} and since $N = 4n$, we get at least $\prod_{i=0}^{n-1}\left(\frac{3}{4}N - i\right) = (N/2)^n$ labeled copies of $\mathcal{H}$. $\square$

We can now put these two lemmas together to prove the required Ramsey result. A density result also holds by a similar method, but we'll just prove the Ramsey version. We refer the interested reader to the excellent paper of Fox and Sudakov.

Theorem 1 *Let H be a bipartite graph with n vertices and maximum degree at most $\Delta \geq \Delta_0$. Then*

$$r(H) \leq 2^8 \Delta^4 2^{\Delta} n.$$

Proof: Associate to the graph $H = (U, V; F)$, with $|U|, |V| \leq n$, the Δ-uniform hypergraph $\mathcal{H}$ whose edge set is the set of neighbours $N(v)$ of any vertex $v \in V$.

One of the colours, red say, must have density at least $1/2$. Therefore, if we apply Lemma 1 to the red graph, with $r = \Delta$, $s = \Delta$, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}\left(1 - \frac{3\log 4\Delta}{\Delta}\right)$, we find a subset A' of A of size at least $2^{-(\Delta+1)}N$ which contains at most

$$2^{\Delta+2}\beta^{\Delta^2}\binom{N}{\Delta} \leq 2^{\Delta+3}\beta^{\Delta^2}\alpha^{-\Delta^2}2^{\Delta}\binom{|A'|}{\Delta} \leq 2^{2\Delta+3}e^{-3\Delta\log 4\Delta}\binom{|A'|}{\Delta} \leq (4\Delta)^{-\Delta}\binom{|A'|}{\Delta}$$

bad Δ-tuples, that is Δ-tuples with less than $\beta^{\Delta}N$ common neighbours in B. Let $\mathcal{F}$ be the down-closed hypergraph consisting of all good Δ-tuples and their subsets.

By Lemma 2, $\mathcal{F}$ must contain at least $\frac{|A'|}{2}^n$ copies of $\mathcal{H}$. Fix such an embedding. Then, if Δ is sufficiently large, for each hyperedge corresponding to a vertex $v \in V$ there are at least $2^{-\Delta}(4\Delta)^{-4}N$ vertices to which they are connected. Taking $N \geq 2^8\Delta^4 2^{\Delta} n$, we see that there are always n vertices to choose from, so even if we have already chosen some of these vertices there will always be some left. We therefore see that we can find a copy of H in the red graph, as required. $\square$

One simple corollary of this is the following estimate for the Ramsey number of the hypercube. The hypercube Q_n is the graph on vertex set $\{0,1\}^n$, where two vertices are adjacent if they differ in exactly one coordinate.

Corollary 1

$$r(Q_n) \leq 2^{(2+o(1))n}.$$

An old conjecture of Burr and Erdös says that this should be $O(2^n)$.