## Lecture 7

In this lecture, we will prove a lower bound, due to Graham, Rödl and Ruciński, for the Ramsey number of graphs with $n$ vertices and maximum degree $\Delta$, showing that there is some such graph $H$ for which $r(H) > 2^{c\Delta}n$. The graph $H$ in question will essentially be a random graph on $n$ vertices with maximum degree $\Delta$.

The proof is based on two lemmas. The first says that if one chooses such a random graph, then the edges must be quite well distributed. In particular, no matter how we split our vertex set into pieces, more than half of the bipartite chunks contain edges of the graph.

**Lemma 1** *There are fixed constants $c > 1 > \epsilon$ with $c\epsilon > 1$ and $\Delta_0$ such that, for each $\Delta \geq \Delta_0$ and $n \geq k^2$, where $k = c^\Delta$, there exists a graph $H$ with $n$ vertices and maximum degree $\Delta$ with the following property. For all partitions $V = V_1 \cup \cdots \cup V_k$ of the vertex set $V$ with $|V_i| \leq \epsilon^\Delta n$,*

$$\sum_{i<j:\ e_H(V_i,V_j)>0} |V_i||V_j| > 0.55\binom{n}{2}.$$

**Proof:** Choose $c < (0.7)^{-1/202}$ and $\epsilon$ so that $c\epsilon > 1$, but $c\epsilon^2 < 1$. Finally, choose $\Delta_0$ so that $(c\epsilon^2)^{\Delta_0} < 0.1$ and $\left(c(0.7)^{1/202}\right)^{\Delta_0} < 1/4$ . Let $d = \Delta/202$. Consider the random graph $G(m, dm)$ on $m$ vertices with $dm$ edges, where $m = 1.01n$. The number of vertices of degree larger than $\Delta$ is at most $\frac{2dm}{\Delta+1} < \frac{m}{101}$. We form a graph $H$ by deleting the $n/100$ vertices of largest degree. Note that $H$ is a graph with $n$ vertices and maximum degree less than or equal to $\Delta$.

We claim that, with positive probability, $G(m, dm)$ has the following property. For every partition $[m] = V_1 \cup \cdots \cup V_k \cup D$, $k = c^\Delta$, with $|V_i| \leq \epsilon^\Delta n$ and $|D| = n/100$,

$$\sum_{i<j:\ e_G(V_i,V_j)>0} |V_i||V_j| > 0.55\binom{n}{2}. \tag{1}$$

Note that the total number of pairs within the sets $V_i$ is at most

$$\sum_i \binom{|V_i|}{2} \leq (c\epsilon^2)^{\Delta_0}\binom{n}{2}.$$

Hence, since $(c\epsilon^2)^{\Delta_0} < 0.1$, this contribution is at most $0.1\binom{n}{2}$. Therefore, if a partition of the vertices of $G$ violates (1), the partition must satisfy

$$\sum_{i<j:\ e_G(V_i,V_j)=0} |V_i||V_j| \geq 0.35\binom{n}{2} \geq 0.3\binom{m}{2}. \tag{2}$$

However, the expected number of partitions $(V_1, \ldots, V_k, D)$ of $G(m, dm)$ which satisfy (2) is smaller than

$$(k+1)^m 2^{\binom{k}{2}} \frac{\binom{0.7\binom{m}{2}}{dm}}{\binom{\binom{m}{2}}{dm}} < 2^{\binom{k}{2}}(2k(0.7)^d)^m < 4^m \left((0.7)^{1/202}c\right)^{\Delta_0 m} < 1.$$

Note that the term $(k+1)^m$ bounds the number of partitions, the $2^{\binom{k}{2}}$ bounds the number of choices of pairs $(V_i, V_j)$ with no edges between them and the fractions is an upper bound on the probability that there is no edge between these pairs.

Hence, there exists a graph $G \in G(m, dm)$ satisfying (1). Setting $D$ to be the $n/100$ vertices in $G$ of largest degree, the graph $H = G \backslash D$ has the required properties. $\square$

The second lemma says that with high probability every large subset of vertices of the random graph $G(k, \frac{1}{2})$ spans about the expected number of edges. In actual fact, we need a more precise weighted version of this lemma.

**Lemma 2** *For every $k \geq 4$, there exists a graph $R$ on $[k]$ such that, for all functions $w : [k] \to [0, 1]$ with $\sum_{i=1}^{k} w(i) = x > (10^7 + 2)\log k$,*

$$W = \sum_{ij \in R} w(i)w(j) < 0.51\binom{x}{2} \text{ and } \overline{W} = \sum_{ij \notin R} w(i)w(j) < 0.51\binom{x}{2}.$$

**Proof:** To begin, note that, for any graph $R$ and any fixed $x$, $W$ is maximised by a choice of weight function where the set $K = \{i : 0 < w(i) < 1\}$ is a clique in $R$ or $K = \emptyset$. Suppose otherwise, and that there is some $ij \notin R$ for which $0 < w(i), w(j) < 1$ and that the sum of weights assigned to the neighbors of $i$ is at least the sum of weights of the neighbors of $j$. Let $w'(i) = w(i) + \epsilon$ and $w'(j) = w(j) - \epsilon$, where $\epsilon = \min\{1 - w(i), w(j)\}$. Then $W' \geq W$ and $K$ has one vertex less. Continuing this process yields that $K$ is a clique or empty. Similarly, $\overline{W}$ is maximal when $K$ is an independent set in $R$ or empty.

Now, let $R$ be a randomly chosen graph on $[k]$, where each edge is chosen with probability $\frac{1}{2}$. Then, with probability at least $\frac{1}{2}$, the largest clique or independent set has size at most $2\log_2 k$. To see this, suppose $s = 2\log_2 k + 1 \geq 5$. Then the expected number of cliques or independent sets of size $s$ is

$$2\binom{k}{s}2^{-\binom{s}{2}} < 2\left(\frac{e}{s}\right)^s < \frac{1}{2},$$

which implies the required estimate.

Moreover, by Chernoff's inequality, the probability that there is a set $S \subset [k]$ with $s = |S| \geq 10^7 \log_2 k$ and

$$|G(k, \frac{1}{2}) \cap [S]^2| > 0.501\binom{s}{2} \text{ or } |G(k, \frac{1}{2}) \cap [S]^2| < 0.499\binom{s}{2} \tag{3}$$

is smaller than

$$2\sum_{s=10^7 \log_2 k}^{k} \binom{k}{s} \exp\left\{-10^{-6}\binom{s}{2}/3\right\} < 2\sum_s \left(\frac{ek}{s}e^{-10^{-7}s}\right)^s = 2\sum_s (e/s)^s < \frac{1}{2}.$$

Therefore, there exists some $R$ such that (3) doesn't hold for any set $S$ of size at least $10^7 \log_2 k$ and there are no cliques or independent sets of size more than $2\log_2 k$.

Let $T = \{i : w(i) = 1\}$. Then $x \geq t = |T| \geq x - 2\log_2 k$, since the largest clique has size $2\log_2 k$. Therefore, $t \geq 10^7 \log_2 k$ and so

$$W \leq \sum_{ij \in R \cap [T]^2} 1 + (2\log_2 k)x < 0.501\binom{t}{2} + (2\log_2 k)x < 0.501\binom{x}{2}.$$

A similar argument applies to establish the bound for $\overline{W}$. $\square$

Putting these two lemmas together now gives the required bound.

**Theorem 1** *There exists a constant $c'$ such that there is a graph $H$ with $n$ vertices and maximum degree $\Delta$ such that*

$$r(H) \geq 2^{c'\Delta} n.$$

**Proof:** Suppose that $c$, $\epsilon$ and $\Delta_0$ are given as in Lemma 1. Suppose that $\Delta \geq \Delta_0$ and $n \geq c^{2\Delta}$. Choose $H$ as given by Lemma 1 and $R$ as given by Lemma 2. Use $R$ to 2-colour the edges of $K_N$, where $N = (c\epsilon)^{\Delta} n$, as follows. Partition $[N] = U_1 \cup \cdots \cup U_k$, $|U_i| = N/k$, where $k = c^{\Delta}$. Then, let an edge of the graph be red if it lies between two different sets $U_i$ and $U_j$, where $ij \in R$. Colour it blue if it lies between two different sets $U_i$ and $U_j$, where $ij \notin R$. If it lies within one of the sets, then colour it as you please.

We want to show that this colouring has no monochromatic $H$. Suppose otherwise, and that there is a red copy $H_0$ of $H$. Set $V_i = V(H_0) \cap U_i$. Lemma 1 tells us that

$$\sum_{ij \in R} |V_i||V_j| \geq \sum_{i<j:\ e_{H_0}(V_i,V_j)>0} |V_i||V_j| > 0.55\binom{n}{2}.$$

On the other hand, let $|V_i| = w(i)N/k$. Note that $n = \sum_i |V_i| = N/k \sum_i w(i)$ and, therefore,

$$x = \sum_i w(i) = \frac{kn}{N} = \frac{c^{\Delta} n}{(c\epsilon)^{\Delta} n} > (10^7 + 2)\log_2 k,$$

for $\Delta \geq \Delta_1$ and $\Delta_1$ chosen sufficiently large. By Lemma 2, this implies that

$$\sum_{ij \in R} |V_i||V_j| = \frac{N}{k^2} \sum_{ij \in R} w(i)w(j) < \frac{N^2}{k^2}(0.51)\binom{x}{2} \leq 0.51\binom{n}{2}.$$

This is a contradiction, so we're done in the case when $\Delta \geq \Delta' = \max\{\Delta_0, \Delta_1\}$ and $n \geq c^{2\Delta}$.

For $n < c^{2\Delta}$, note that the complete graph $K_{\Delta+1}$ is an example since

$$r(K_{\Delta+1}) \geq 2^{\Delta/2} \geq c^{3\Delta} > c^{\Delta} n.$$

For $\Delta < \Delta'$, let $H$ be a matching with $n$ or $n-1$ vertices. Then, since a matching with $n$ vertices has Ramsey number $\frac{3}{2}n + O(1)$ and $\Delta'$ is bounded, we can choose a (very small) $c_1$ for which $r(H) \geq c_1^{\Delta} n$. $\square$

This result can be strengthened to show that there are bipartite graphs $H$ with $n$ vertices and maximum degree $\Delta$ for which $r(H) > 2^{c'\Delta} n$. In the next lecture, we shall prove a matching upper bound for bipartite graphs, $r(H) \leq 2^{c\Delta} n$.