Lecture 5

We will now begin to turn our attentions towards Ramsey's theorem for more general classes of graphs. To this end, we need the following definition.

Given two graphs G and H, the Ramsey number $r(G,H)$ is the smallest natural number n such that, in any 2-colouring, in red and blue, of the edges of K_n, there is guaranteed to be a red copy of G or a blue copy of H. When $G = H$, we just write $r(G)$.

A tree is a graph which contains no cycles. We begin by proving a theorem, due to Chvátal, about the Ramsey number of trees versus cliques.

Theorem 1 *Let s, t be positive integers and let T be a tree of order t. Then*

$$r(T, K_s) = (s-1)(t-1) + 1.$$

Proof: To prove that $r(T, K_s) > (s-1)(t-1)$, simply note that $s-1$ red copies of T, all joined by blue edges cannot contain either a red copy of T or a blue copy of K_s.

Suppose now that there is a red/blue-edge-colouring of K_n, where $n = (s-1)(t-1)+1$, which contains no blue copy of K_s. Let G be the red graph. Then, since G has no independent sets of size s, its chromatic number $\chi(G) \geq \lceil n/(s-1) \rceil = t$. This implies that there is a subgraph of minimum degree $t-1$, as $\chi(G) \leq \max\{\delta(H) : H \subset G\} + 1$. But it is easy to embed any tree T with t vertices into a graph with minimum degree $t-1$. $\square$

For trees themselves, the famous conjecture of Erdős and Sós says that any graph with n vertices and more than $(t-2)n/2$ edges contains every tree on t vertices. Therefore, if $\frac{1}{2}\binom{n}{2} > (t-2)n/2$ or, rewriting, $n > 2t-3$, the Erdős-Sós conjecture implies that any 2-colouring of K_n contains a monochromatic copy of every tree T on t vertices, that is, $r(T) \leq 2t-2$. The example of the star on t vertices shows that this result would be sharp.

The cycle C_t is the graph with t vertices $v_1, v_2, \ldots, v_t$ whose edges are $v_1v_2, v_2v_3, \ldots, v_nv_1$. The following theorem is due to Bondy and Erdős, Faudree and Schelp, and Rosta.

Theorem 2 *If $t \geq 6$ is even,*

$$r(C_t) = \frac{3t}{2} - 1,$$

and, if $t \geq 5$ is odd,

$$r(C_t) = 2t - 1.$$

The proof of this theorem is surprisingly involved (the parity dependence is some indication of this) and so we do not give it here. A theorem similar to Theorem 1 but with cycles was obtained by Bondy and Erdős when t is substantially larger than s. The best current dependence of t on s is due to Nikiforov.

Theorem 3 *For $t \geq 4s + 2$,*

$$r(C_t, K_s) = (s-1)(t-1) + 1.$$

Given a graph G, the Ramsey number $r(G;k)$ is the smallest number n such that, in any q-colouring of the edges of K_n, there is a monochromatic copy of G.

To get a closer look at the parity problem, we shall examine the bounds for $r(G;k)$ when $G = C_3$ and $G = C_4$.

Theorem 4

$$2^q \leq r(C_3;q) \leq 3q!$$

Proof: The graph on 2 vertices has no triangle, so colouring the edge in colour 1 gives us a colouring without triangles. Suppose, therefore, that we have a $(q-1)$-colouring of the graph on 2^{q-1} vertices which contains no triangle. To produce a q-colouring of the complete graph on 2^q vertices without a triangle, we simply take two disjoint copies of the $(q-1)$-colouring of the complete graph on 2^{q-1} vertices and let all edges between the two copies have colour q.

For the upper bound, assume that $r(C_3;q-1) \leq 3(q-1)!$ (this is true for $q=2$) and suppose that we have a q-colouring of K_n, with $n = 3q!$. Let v be a vertex in K_n. By the pigeonhole principle, v must have neighborhood of size at least $3(q-1)!$ in one of the colours, colour i, say. This neighborhood cannot have an edge with colour i or we would have a triangle. Therefore, it is coloured in $q-1$ colours. Hence, by induction, since the neighborhood has size at least $r(C_3;q-1)$, there is a triangle in one of these colours. □

Both sides of this theorem may be improved, but the state-of-the-art is still of the form $c^q \leq r(C_3;q) \leq c'q!$. This differs sharply from what happens for C_4.

Theorem 5

$$r(C_4;q) \leq q^2 + q + 1,$$

and, for prime powers q,

$$r(C_4;q) > q^2 - q + 1.$$

We will not prove this theorem here. However, at least approximately, we can explain why $r(C_4;q)$ has the value it does. It is well-known that any graph on n vertices with more than roughly $\frac{1}{2}n^{3/2}$ edges contains a copy of C_4. Therefore, if $\frac{1}{q}\binom{n}{2} > \frac{1}{2}n^{3/2}$, we get a monochromatic C_4. This reduces to saying that n is greater than q^2 or thereabouts. Working this out more precisely gives the bound.

Why such a sharp difference between the behaviour of C_3 and C_4? In this case, it can be completely explained by the fact that C_4 is bipartite while C_3 is not. In the case of Theorem 2, the reason is a slightly more subtle one.