Lecture 4

We are now going to turn our attention to the topic of hypergraph Ramsey numbers. A hypergraph on vertex set V is just a collection of vertex subsets of V. A k-uniform hypergraph consists of subsets each of which have size k. In particular, an ordinary graph is a 2-uniform hypergraph. The complete k-uniform graph on n vertices will be denoted by $K_n^{(k)}$.

The Ramsey number $r_k(t)$ is the smallest natural number n such that, in any 2-colouring of the edges of $K_n^{(k)}$, there is a monochromatic copy of $K_t^{(k)}$. More generally, the Ramsey number $r_k(s,t)$ is the smallest natural number n such that, in any 2-colouring, in red and blue, of the edges of $K_n^{(k)}$, there is a red copy of $K_s^{(k)}$ or a blue copy of $K_t^{(k)}$.

In this lecture, we will focus on the study of 3-uniform hypergraphs, though we will state the equivalent results for higher uniformities. We begin with a theorem of Erdős and Rado.

Theorem 1

$$r_3(s,t) \leq 2^{\binom{r(s-1,t-1)}{2}}.$$

Proof: Let $N = 2^{\binom{r(s-1,t-1)}{2}}$ and let χ be a red-blue colouring of the triples of $[N]$. We will greedily construct a set of vertices $\{v_1, \ldots, v_{r(s-1,t-1)+1}\}$ such that for any given pair $1 \leq i < j \leq r(s-1,t-1)$, all triples $\{v_i, v_j, v_k\}$ with $k > j$ are of the same colour, which we denote by $\chi'(v_i, v_j)$. By definition of the Ramsey number, there is either a red clique of size $s-1$ or a blue clique of size $t-1$ in colouring χ', and this clique together with $v_{r(s-1,t-1)+1}$ forms a red set of size s or a blue set of size t in colouring χ.

The greedy construction of the set $\{v_1, \ldots, v_{r(s-1,t-1)+1}\}$ is as follows. First, pick an arbitrary vertex v_1 and set $S_1 = S \setminus \{v_1\}$. After having picked $\{v_1, \ldots, v_i\}$ we also have a subset S_i such that for any pair a, b with $1 \leq a < b \leq i$, all triples $\{v_a, v_b, w\}$ with $w \in S_i$ are the same colour. Let v_{i+1} be an arbitrary vertex in S_i and set $S_{i,0} = S_i \setminus \{v_{i+1}\}$. Suppose we already constructed $S_{i,j} \subset S_{i,0}$ such that, for every $h \leq j$ and $w \in S_{i,j}$, all triples $\{v_h, v_{i+1}, w\}$ have the same colour. If the number of edges $\{v_{j+1}, v_{i+1}, w\}$ with $w \in S_{i,j}$ that are red is at least $|S_{i,j}|/2$, then we let

$$S_{i,j+1} = \{w : \{v_{j+1}, v_{i+1}, w\} \text{ is red and } w \in S_{i,j}\}$$

and set $\chi'(i+1, j+1) = \text{red}$, otherwise we let

$$S_{i,j+1} = \{w : \{v_{j+1}, v_{i+1}, w\} \text{ is blue and } w \in S_{i,j}\}$$

and set $\chi'(i+1, j+1) = \text{blue}$. Finally, we let $S_{i+1} = S_{i,i}$. Notice that $\{v_1, \ldots, v_{i+1}\}$ and S_{i+1} have the desired properties to continue the greedy algorithm. Also, for each edge $v_{i+1}v_{j+1}$ that we colour by χ', the set $S_{i,j}$ is at most halved. So we lose a factor of at most two for each of the $\binom{r(s-1,t-1)}{2}$ edges coloured by χ'. This completes the proof. □

In particular, since $r(t) \leq 2^{2t}$ and $r(s,n) \leq c_s \frac{n^{s-1}}{(\log n)^{s-2}}$, this result has the following corollaries.

Corollary 1

$$r_3(t) \leq 2^{2^{4t}}.$$

Corollary 2

$$r_3(s,n) \leq 2^{c_s \frac{n^{2s-4}}{(\log n)^{2s-6}}}.$$

These results were recently improved by the author together with Fox and Sudakov. In particular, for $r_3(s,n)$, these authors proved that

$$r_3(s,n) \leq 2^{c_s n^{s-2} \log n}.$$

For higher uniformities, the Erdős-Rado theorem says that $r_k(s,t) \leq 2^{\binom{r_{k-1}(s,t)}{k-1}}$. In particular, the diagonal Ramsey number $r_k(t)$ is at most a tower of height k. That is, let $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$. Then $r_k(t) \leq t_k(ct)$. Moreover, applying the result above, we see that the off-diagonal Ramsey number $r_k(s,n)$ is bounded above by $r_k(s,n) \leq t_{k-1}(c_s n^{s-2} \log n)$.

What about lower bounds? Let's start with the off-diagonal case and give a simple proof that $r_3(6,n)$ is exponential.

Theorem 2

$$r_3(6,n) \geq 2^{n/2}.$$

Proof: Let $N = 2^{n/2}$. By the standard lower bound for Ramsey numbers, there is a 2-colouring of the edges of K_N which does not contain a monochromatic K_n. Now colour a 3-edge in $K_N^{(3)}$ blue if the triangle formed by the edge is monochromatic in the underlying graph. Otherwise colour it red. By construction, there is no monochromatic clique of size n. Therefore, there is no blue clique of size n in the 3-graph. On the other hand, there cannot be a red K_6 since that would imply that there was no monochromatic triangle in a particular graph on 6 vertices. □

This result can be improved to show that $r_3(4,n)$ is exponential. All one has to do is use a random tournament rather than a colouring. One then colours an edge red if and only if its edges form a directed cycle. We leave the details to the reader. A more difficult construction, due to the author, Fox and Sudakov shows that $r_3(4,n) \geq n^{cn}$.

What about the diagonal case $r_3(t)$? A simple probabilistic argument, akin to the standard lower bound for graphs, gives the following.

Theorem 3

$$r_3(t) \geq 2^{t^2/6}.$$

Rather annoyingly, this remains the state of the art. So, for 3-uniform hypergraphs, we only know that

$$2^{c't^2} \leq r_3(t) \leq 2^{2^{ct}}.$$

Erdős has offered a \$500 reward for a proof that the function is actually a double exponential. Some evidence that this is the case already exists. Let $r_k(t;q)$ be the smallest natural number n such that, in any q-colouring of the edges of $K_n^{(k)}$, there is a monochromatic copy of $K_t^{(k)}$. A simple extension of the Erdős-Rado argument allows us to show that $r_3(t;q) \leq 2^{2^{c_q t}}$. Moreover, an ingenious construction due to Erdős and Hajnal, known as the stepping-up lemma, allows us to show that, for $q \geq 4$, this is essentially sharp.

Theorem 4

$$r_3(t;4) \geq 2^{r(t-1)-1}.$$

Proof: Let C be a two-colouring, in red and blue, of a graph on $m = r(t-1)-1$ vertices which does not contain a monochromatic clique of size $t-1$. We are going to consider the complete 3-uniform hypergraph H on the set

$$T = \{(\gamma_1, \cdots, \gamma_m) : \gamma_i = 0 \text{ or } 1\}.$$

If $\epsilon = (\gamma_1, \cdots, \gamma_m)$, $\epsilon' = (\gamma'_1, \cdots, \gamma'_m)$ and $\epsilon \neq \epsilon'$, define

$$\delta(\epsilon, \epsilon') = \max\{i : \gamma_i \neq \gamma'_i\},$$

that is, $\delta(\epsilon, \epsilon')$ is the largest coordinate at which they differ. Given this, we can define an ordering on T, saying that

$$\epsilon < \epsilon' \text{ if } \gamma_i = 0, \gamma'_i = 1,$$
$$\epsilon' < \epsilon \text{ if } \gamma_i = 1, \gamma'_i = 0,$$

where $i = \delta(\epsilon, \epsilon')$. Equivalently, associate to any ϵ the number $b(\epsilon) = \sum_{i=1}^{m} \gamma_i 2^{i-1}$. The ordering then says simply that $\epsilon < \epsilon'$ iff $b(\epsilon) < b(\epsilon')$.

We will further need the following two properties of the function δ which one can easily prove.

(a) If $\epsilon_1 < \epsilon_2 < \epsilon_3$, then $\delta(\epsilon_1, \epsilon_2) \neq \delta(\epsilon_2, \epsilon_3)$ and

(b) if $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_p$, then $\delta(\epsilon_1, \epsilon_p) = \max_{1 \leq i \leq p-1} \delta(\epsilon_i, \epsilon_{i+1})$.

In particular, these properties imply that there is a unique index i which achieves the maximum of $\delta(\epsilon_i, \epsilon_{i+1})$. Indeed, suppose that there are indices $i < i'$ such that

$$\ell = \delta(\epsilon_i, \epsilon_{i+1}) = \delta(\epsilon_{i'}, \epsilon_{i'+1}) = \max_{1 \leq j \leq p-1} \delta(\epsilon_j, \epsilon_{j+1}).$$

Then, by property (b) we also have that $\ell = \delta(\epsilon_i, \epsilon_{i'}) = \delta(\epsilon_{i'}, \epsilon_{i'+1})$. This contradicts property (a) since $\epsilon_i < \epsilon_{i'} < \epsilon_{i'+1}$.

We are now ready to colour the complete 3-uniform hypergraph H on the set T. If $\epsilon_1 < \epsilon_2 < \epsilon_3$, let $\delta_1 = \delta(\epsilon_1, \epsilon_2)$ and $\delta_2 = \delta(\epsilon_2, \epsilon_3)$. Note that, by property (a) above, δ_1 and δ_2 are not equal. Colour the edge $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ as follows:

C_1, if (δ_1, δ_2) is red and $\delta_1 < \delta_2$;

C_2, if (δ_1, δ_2) is red and $\delta_1 > \delta_2$;

C_3, if (δ_1, δ_2) is blue and $\delta_1 < \delta_2$;

C_4, if (δ_1, δ_2) is blue and $\delta_1 > \delta_2$.

Suppose that C_1 contains a clique $\{\epsilon_1, \cdots, \epsilon_t\}_<$ of size t. For $1 \leq i \leq t-1$, let $\delta_i = \delta(\epsilon_i, \epsilon_{i+1})$. Note that the δ_i form a monotonically increasing sequence, that is $\delta_1 < \delta_2 < \cdots < \delta_{t-1}$. Also, note that since, for any $1 \leq i < j \leq t-1$, $\{\epsilon_i, \epsilon_{i+1}, \epsilon_{j+1}\} \in C_1$, we have, by property (b) above, that $\delta(\epsilon_{i+1}, \epsilon_{j+1}) = \delta_j$, and thus $\{\delta_i, \delta_j\}$ is red. Therefore, the set $\{\delta_1, \cdots, \delta_{t-1}\}$ must form a red clique of size $t-1$. But we have chosen the colouring so as not to contain such a clique, so we have a contradiction. A similar argument shows that none of the other colours can contain a clique of size t. $\square$

A second stepping-up lemma proved by Erdős and Hajnal allows one to exponentiate lower bounds for k-uniform Ramsey numbers to get lower bounds for $(k+1)$-uniform Ramsey numbers while keeping the number of colours the same. Unfortunately, this construction only works for $k \geq 3$. This allows one to show that $r_k(t) \geq t_{k-1}(ct^2)$, which is always one exponential smaller than the upper bound. For four or more colours, it allows one to show that $r_k(t;q) \geq t_k(c_q t)$, thus achieving a result which is sharp up to the constant. We do not go into the details, but refer the reader to the book of Graham, Rothschild and Spencer on Ramsey Theory.