Lecture 3

The aim of this lecture is to prove the Ajtai, Komlós and Szemerédi bound $r(3,n) \leq c \frac{n^2}{\log n}$. We will actually prove quite a bit more - the AKS bound follows as a simple corollary of our main result - but to state it we will need some definitions.

Given a graph $G$, let $\beta(G)$ be the size of the largest independent set in $G$, that is the largest set of vertices such that no two vertices are joined by an edge. Let $d(G)$ be the average degree of $G$, noting the standard convention that a graph with no vertices has average degree 0.

We will also need to define a very specific function. This function $f$, defined on and mapping to the positive integers, is given by

$$f(0) = 1, \quad f(1) = \frac{1}{2}, \quad f(d) = \frac{(d \log d - d + 1)/(d-1)^2, d \neq 0, 1.}$$

Note that $f$ is continuous, convex and decreases strictly from 1 to 0. Note also that $f'$ and, therefore,

$$1 - (d+1)f(d) + (d-d^2)f'(d) = 0.$$

We now prove a precise version of the AKS result, in a form due to Shearer.

**Theorem 1** Let $G$ be a triangle-free graph of order $N$ and average degree $d$. Then

$$\beta(G) \geq f(d)N.$$

**Proof:** We will prove the bound by induction on $N$. The result is plainly true for $N = 0$, so we may assume that $N > 0$ and the result is true for all smaller values. Since the result is vacuously true for $d = 0$, we may also assume that $d > 0$.

For each $x \in V$, let $d(x)$ be the degree of $x$ and $\bar{d}(x)$ the average degree of the neighbours of $x$. Moreover, let $D(x)$ be the set of neighbours of $x$. Then $G_x$ is the induced subgraph of $G$ on the subset $V \setminus \{x\} \cup D(x)$ and $d_x = d(G_x)$, the average degree of the graph $G_x$. Note that $|G_x| = N - d(x) - 1$.

Moreover, since the graph $G$ is triangle-free, no two neighbours of $x$ are joined by an edge. Therefore,

$$e(G_x) = e(G) - \sum_{y \in D(x)} d(y) = \frac{Nd}{2} - d(x)\bar{d}(x).$$

By the convexity of $f$,

$$f(d_x) \geq f(d) + (d_x - d)f'(d).$$

Hence, by induction,

$$\beta(G) = 1 + \frac{1}{N} \sum_{x \in V} \beta(G_x) \geq 1 + \frac{1}{N} \sum_{x \in V} f(d_x)(N - d(x) - 1)$$

$$\geq 1 + \frac{1}{N} \sum_{x \in V} (f(d) + (d_x - d)f'(d))(N - d(x) - 1)$$

$$= 1 + f(d)N - f(d)(d + 1) + (d^2 + d)f'(d) + \frac{f(d)}{N} \sum_{x \in V} (d_x(N - d(x) - 1) - Nd),$$

which completes the proof.
where, in the last line, we used $\sum_{x \in V} d(x) = N d$. Now $d(x)(N - d(x) - 1) = 2e(G_x)$ and, therefore,

$$d(x)(N - d(x) - 1) = Nd - 2d(x)d(x).$$

Hence,

$$\beta(G) \geq 1 + f(d)N - f(d)(d + 1) + (d + d^2)f'(d) + \frac{f'(d)}{N} \sum_{x \in V} (-2d(x)d(x)).$$

Therefore, since

$$\sum_{x \in V} d(x)d(x) = \sum_{x \in V} \sum_{y \in D(x)} d(y) = \sum_{y \in V} d^2(y) \geq N d^2$$

and $f' < 0$,

$$\beta(G) \geq f(d)N + 1 - f(d)(d + 1) + (d - d^2)f'(d).$$

However, as we observed before the proof, $1 - f(d)(d + 1) + (d - d^2)f'(d) = 0$. This completes the proof. \qed

For the sake of the next proof, note that, for $d \geq 3$,

$$f(d) = \frac{(d \log d - d + 1)}{(d - 1)^2} \geq \frac{\log d}{d}.$$

**Theorem 2**

$$r(3, n) \leq \frac{n^2}{\log n}.$$

**Proof:** Suppose that there is a colouring of $K_N$ without a red triangle or a blue $K_n$. Let $G$ be the red graph. Since it is triangle-free, the set of neighbours of every vertex of $G$ are independent. Therefore, $d(G) \leq n$. Thus,

$$n \geq \beta(G) \geq \frac{\log n}{n} N,$$

implying the result. \qed

Finally, note that, as we mentioned in the last lecture, a variation on the AKS proof may be used to prove that $r(s, n) \leq c_s \frac{n^{s-1}}{(\log n)^2}$. This proof uses a robust version of Theorem 1 where one knows that $G$ has few triangles rather than no triangles. This remains the state of the art.