Lecture 3

The aim of this lecture is to prove the Ajtai, Komlós and Szemerédi bound $r(3,n) \leq c\frac{n^2}{\log n}$. We will actually prove quite a bit more - the AKS bound follows as a simple corollary of our main result - but to state it we will need some definitions.

Given a graph G, let $\beta(G)$ be the size of the largest independent set in G, that is the largest set of vertices such that no two vertices are joined by an edge. Let $d(G)$ be the average degree of G, noting the standard convention that a graph with no vertices has average degree 0.

We will also need to define a very specific function. This function f, defined on and mapping to the positive integers, is given by $f(0) = 1$, $f(1) = \frac{1}{2}$ and

$$f(d) = (d \log d - d + 1)/(d-1)^2, d \neq 0, 1.$$

Note that f is continuous, convex and decreases strictly from 1 to 0. Note also that

$$f' = \frac{(-(d+1)\log d + (d^2-1)/d + (d-1)^2/d}{(d-1)^3}$$

and, therefore,

$$1 - (d+1)f(d) + (d-d^2)f'(d) = 0.$$

We now prove a precise version of the AKS result, in a form due to Shearer.

Theorem 1 *Let G be a triangle-free graph of order N and average degree d. Then*

$$\beta(G) \geq f(d)N.$$

Proof: We will prove the bound by induction on N. The result is plainly true for $N = 0$, so we may assume that $N > 0$ and the result is true for all smaller values. Since the result is vacuously true for $d = 0$, we may also assume that $d > 0$.

For each $x \in V$, let $d(x)$ be the degree of x and $\tilde{d}(x)$ the average degree of the neighbours of x. Moreover, let $D(x)$ be the set of neighbours of x. Then G_x is the induced subgraph of G on the subset $V\backslash\{x\} \cup D(x)$ and $d_x = d(G_x)$, the average degree of the graph G_x. Note that $|G_x| = N - d(x) - 1$. Moreover, since the graph G is triangle-free, no two neighbours of x are joined by an edge. Therefore,

$$e(G_x) = e(G) - \sum_{y \in D(x)} d(y) = \frac{Nd}{2} - d(x)\tilde{d}(x).$$

By the convexity of f,

$$f(d_x) \geq f(d) + (d_x - d)f'(d).$$

Hence, by induction,

$$\begin{aligned}
\beta(G) &= 1 + \frac{1}{N}\sum_{x\in V}\beta(G_x) \geq 1 + \frac{1}{N}\sum_{x \in V} f(d_x)(N - d(x) - 1) \\
&\geq 1 + \frac{1}{N}\sum_{x\in V}\left(f(d) + (d_x - d)f'(d)\right)(N - d(x) - 1) \\
&= 1 + f(d)N - f(d)(d+1) + (d^2+d)f'(d) + \frac{f'(d)}{N}\sum_{x\in V}\left(d_x(N - d(x) - 1) - Nd\right),
\end{aligned}$$

where, in the last line, we used $\sum_{x \in V} d(x) = Nd$. Now $d_x(N - d(x) - 1) = 2e(G_x)$ and, therefore,

$$d_x(N - d(x) - 1) = Nd - 2d(x)\tilde{d}(x).$$

Hence,

$$\beta(G) \geq 1 + f(d)N - f(d)(d+1) + (d + d^2)f'(d) + \frac{f'(d)}{N} \sum_{x \in V} \left(-2d(x)\tilde{d}(x)\right).$$

Therefore, since

$$\sum_{x \in V} d(x)\tilde{d}(x) = \sum_{x \in V} \sum_{y \in D(x)} d(y) = \sum_{y \in V} d^2(y) \geq Nd^2$$

and $f' < 0$,

$$\beta(G) \geq f(d)N + 1 - f(d)(d+1) + (d - d^2)f'(d).$$

However, as we observed before the proof, $1 - f(d)(d+1) + (d - d^2)f'(d) = 0$. This completes the proof. □

For the sake of the next proof, note that, for $d \geq 3$,

$$f(d) = \frac{(d \log d - d + 1)}{(d-1)^2} \geq \frac{\log d}{d}.$$

Theorem 2

$$r(3, n) \leq \frac{n^2}{\log n}.$$

Proof: Suppose that there is a colouring of K_N without a red triangle or a blue K_n. Let G be the red graph. Since it is triangle-free, the set of neighbours of every vertex of G are independent. Therefore, $d(G) \leq n$. Thus,

$$n \geq \beta(G) \geq \frac{\log n}{n} N,$$

implying the result. □

Finally, note that, as we mentioned in the last lecture, a variation on the AKS proof may be used to prove that $r(s, n) \leq c_s \frac{n^{s-1}}{(\log n)^{s-2}}$. This proof uses a robust version of Theorem 1 where one knows that G has few triangles rather than no triangles. This remains the state of the art.