Lecture 2

We would also like to have a lower bound for the Ramsey number $r(t)$. The first construction one might think of is to take $t-1$ red cliques, each of size $t-1$, where every edge between cliques is blue. This implies a lower bound $r(t) > (t-1)^2$. For a brief period, it was believed that this or something close to it might be sharp. However, as the next theorem shows, this is wildly false.

Theorem 1

$$r(t) > \frac{t}{\sqrt{2}e}\sqrt{2}^t.$$

Proof: If we two-colour the edges of K_n at random, each colour being chosen with probability $1/2$, the expected number of monochromatic cliques K_t is $\frac{2}{2^{\binom{t}{2}}}\binom{n}{t}$. If we take $n = \frac{t}{\sqrt{2}e}\sqrt{2}^t$, we see that the expectation satisfies

$$\begin{aligned}\frac{2}{2^{\binom{t}{2}}}\binom{n}{t} &< 2^{-\binom{t}{2}+1}\frac{n^t}{t!} \\ &\leq 2^{-\binom{t}{2}+1}\left(\frac{t2^{(t-1)/2}}{e}\right)^t \frac{1}{2}\left(\frac{e}{t}\right)^t = 1.\end{aligned}$$

Here we used the bound $t! \geq 2\left(\frac{t}{e}\right)^t$. Therefore, since the expectation is smaller than 1, there must be some colouring of K_n which contains no monochromatic copy of K_t. Choosing this graph completes the proof. □

In the sixty years since Erdős proved this bound, it has only been improved by a factor of 2, by Spencer. His proof uses the Lovász local lemma, an important probabilistic tool, which often allows one to show that unlikely events do occur with positive probability.

The bounds given by explicit constructions are much worse. The best-known example, due to Frankl and Wilson, gives a lower bound of $r(t) > t^{c\frac{\log t}{\log\log t}}$. Recently, there has been some work improving these bounds, but the constructions aren't as obviously explicit. Instead, they take polynomial time in t to describe.

We are now going to focus on off-diagonal Ramsey numbers $r(s,n)$, where we fix the value of s and allow n to grow. An easy upper bound, which follows from the Erdős-Szekeres formula, is

$$r(s,n) \leq \binom{n+s-2}{s-1} \leq \frac{(n+s-2)^{s-1}}{(s-1)!},$$

or, roughly speaking, $c_s n^{s-1}$. Ajtai, Komlós and Szemerédi have improved this to

$$r(s,n) \leq c_s \frac{n^{s-1}}{(\log n)^{s-2}}.$$

We shall prove the special case $r(3,n) \leq c\frac{n^2}{\log n}$ in the next lecture. For the rest of this lecture, we will focus on lower bounds.

Theorem 2 *For all $s \geq 3$, there exists a constant c_s such that*

$$r(s,n) \geq c_s\left(\frac{n}{\log n}\right)^{(s-1)/2}.$$

Proof: Let $N = c\left(\frac{n}{\log n}\right)^{(s-1)/2}$. Two-colour the edges of K_N at random, colouring edges red with probability $p = \frac{1}{2N^{2/(s-1)}}$ and blue with probability $1-p$. Note that $p \geq \frac{4s\log n}{n}$ for c sufficiently small. The expected number of red K_s is at most $p^{\binom{s}{2}}N^s$ while the expected number of blue K_n is at most $(1-p)^{\binom{n}{2}}N^n$. Adding the two, we see that the expected number of red K_s and blue K_n is at most

$$\begin{aligned} p^{\binom{s}{2}}N^s + (1-p)^{\binom{n}{2}}N^n &\leq \left(\frac{1}{2N^{2/(s-1)}}\right)^{\binom{s}{2}} N^s + \left(1 - \frac{4s\log n}{n}\right)^{\binom{n}{2}} N^n \\ &\leq 2^{-\binom{s}{2}}N^{-s}N^s + n^{-2s(n-1)}N^n \\ &\leq 2^{-\binom{s}{2}} + n^{-sn} < 1. \end{aligned}$$

Here we used the inequality $1 - x \leq e^{-x}$. Therefore, there must be some colouring of K_N which contains neither a red K_s or a blue K_n. □

Another application of the Lovász local lemma allowed Spencer to show that

$$r(s,n) \geq c_s\left(\frac{n}{\log n}\right)^{(s+1)/2}.$$

For $s = 3$, this gives $r(3,n) \geq c(\frac{n}{\log n})^2$. This was improved, by Kim, to give a sharp bound of the form $r(3,n) \geq c\frac{n^2}{\log n}$ using the semi-random method. Recently, Bohman and Keevash have showed how to improve the lower bound for all $s \geq 4$ using the so-called H-free process. Their result says that

$$r(s,n) \geq c_s(\log n)^{1/(s-2)}\left(\frac{n}{\log n}\right)^{(s+1)/2}.$$

Rather than pursuing any of these routes, we will discuss a more elementary proof, due to Erdős, that $r(3,n) \geq c(\frac{n}{\log n})^2$.

Theorem 3 *There exists a constant c such that*

$$r(3,n) \geq c\left(\frac{n}{\log n}\right)^2.$$

Proof: Let $N = c\left(\frac{n}{\log n}\right)^2$. Let $p = \frac{a\log n}{n}$, where a will be chosen later. We colour the edges of K_N red with probability p and blue with probability $1-p$. This graph may have many red triangles. However, let E be a minimal set of red edges which, if recoloured blue, would give a triangle-free red graph. We will show that with high probability this recoloured graph contains no blue K_n.

By standard large deviation inequalities, we may assume that there are no vertices of degree greater than $2pN$. For the remainder of the proof, we shall assume this is the case. This means that all our probabilities should be calculated conditional upon this event. However, since it makes very little difference to the probabilities, we have chosen to ignore this complication.

For any given subset W of V, the vertex set of K_N, let C_W be the event that the induced red graph on W has an edge xy which is not contained in any red triangle xyz with $z \in V \backslash W$. The critical thing to notice is that if a graph satisfies C_W, then any maximal triangle-free subgraph H of the red graph formed by recolouring edges has blue complement which is not monochromatic on W. To see this, suppose that xy has been recoloured blue. Since H is maximal, the graph $H + xy$ must contain a red triangle xyz. But then, by property C_W, z must be in W. But xz and yz are red. Hence, if we can prove that the event $\cap_W C_W$, where the intersection is taken over all W of size n, occurs with high probability, we will be done.

We will try and estimate the probability $\mathbb{P}(\overline{C}_W)$, where W is a subset of V of size n. If we can show that $\mathbb{P}(\overline{C}_W) \leq n^{-n}$, we will be done, since there are only $\binom{N}{n} < \left(\frac{eN}{n}\right) \ll n^n$ sets W of size n. We will prove the required inequality in two steps. First, we will show that with high probability, most pairs in W have no common neighbours outside W. Then we shall prove that any given large set of pairs of vertices from W must contain an edge.

Let $d_i = e^{2i}pn/i$ and $N_i = n/e^{2i}$. Let P_i be the probability that at least N_i vertices in $V \backslash W$ have at least d_i vertices in W.

Claim 1 *For all $1 \leq i \leq \log n$, the probability $P_i \leq n^{-2n-1}$.*

Therefore, adding over all $1 \leq i \leq \log n$, we see that with probability at least $1 - n^{-2n}$, there are at most N_i vertices which have d_i neighbours in W. Moreover, note that, for $i_0 = (\log n - \log\log n)/2$, $d_{i_0} > 2pN$. Our assumption that all vertices have degree at most $2pN$ therefore implies that there are no vertices with degree d_{i_0} in W. Hence, the number of pairs of vertices in W which share a neighbour in $V \backslash W$ is at most

$$\begin{aligned} N\binom{d_1}{2} + \sum_{i=2}^{i_0-1} N_{i-1}\binom{d_i}{2} &\leq c\left(\frac{n}{\log n}\right)^2 50a^2\log^2 n + 10a^2 n\log^2 n \sum_{i=2}^{i_0-1} e^{2i}/i^2 \\ &\leq 50a^2cn^2 + 20a^2n\log^2 n\left(\frac{n}{\log n}\right)4(\log n)^{-2} \\ &\leq 50a^2cn^2 + 80a^2\frac{n^2}{\log n}, \end{aligned}$$

which may be made as small as any δn^2, for c sufficiently small depending on a and δ. Therefore, for c small, at least $(1-\delta)\binom{n}{2}$ of the edges in W do not have common neighbours in $V \backslash W$. But, for $\delta = 1/2$ and $a = 12$, the probability that this set, which has at least $n^2/6$ edges, doesn't contain an edge of the random graph is at most

$$(1-p)^{n^2/6} \leq e^{-pn^2/6} = e^{-2n\log n} = n^{-2n}.$$

Note that, since the edges within W and the edges between $V \backslash W$ and W are independent, this latter probability is independent of each of the P_i. Therefore,

$$\mathbb{P}(\overline{C}_W) \leq n^{-2n} + n^{-2n} < n^{-n},$$

completing the proof. □

The proof of the claim is conditional upon Chernoff's inequality, which we give without proof. Given a set X, let X_p be the set formed by choosing each element randomly with probability p. Chernoff's inequality gives us a good tail estimate for the size of the set $|X_p|$.

Lemma 1 (Chernoff's inequality) *For $u \geq 4$ and $0 < p \leq \frac{1}{2}$, the binomial random variable X_p satisfies*

$$\mathbb{P}(|X_p| \geq up|X|) < (e/u)^{up|X|}.$$

Proof of Claim 1: Let $d_W(z)$ be the degree of the vertex $z \in V \backslash W$ in W. Each of the random variables $d_W(z)$ is independent. Applying Chernoff implies

$$\mathbb{P}(d_W(z) \geq e^{2i}pn/i) \leq e^{-e^{2i}pn}.$$

Therefore,

$$P_i \leq \binom{N}{N_i} e^{-(e^{2i}a\log n)N_i} < n^{3ne^{-2i}-an} < n^{-2n-1}.$$

This completes the proof. □